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The Emergence of the Law of Value in a Dynamic Simple Commodity Economy

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ABSTRACT *A dynamic computational model of a simple commodity economy is examined and a theory of the relationship between commodity values, market prices and the efficient division of social labour is developed. The main conclusions are: (i) the labour value of a commodity is an attractor for its market price; (ii) market prices are error signals that function to allocate the available social labour between sectors of production; and (iii) the tendency of prices to approach labour values is the monetary expression of the tendency of a simple commodity economy to allocate social labour efficiently. The model demonstrates that, in the special case of simple commodity production, Marx's law of value can naturally emerge from multiple local exchanges and operate 'behind the backs' of actors solely via money flows that place budget constraints on their local evaluations of commodity prices, which are otherwise subjective and unconstrained.*

1. Introduction

Marx, following Ricardo, held a labour theory of the economic value of reproducible commodities. The value of a commodity is determined by the prevailing technical conditions of production and measured by the socially necessary labour-time required to produce it (Marx, 1867). The value of a commodity is to be distinguished from its price, which is the amount of money it fetches in the market.

According to Marx, although individual economic actors may differ in their subjective evaluations of the worth or 'value' of commodities, market prices are nevertheless determined by labour values due to the operation of the 'law of value', an objective economic law that emerges as an unintended consequence of local and distributed market exchanges.

In a theoretical simplification of an economy often referred to as the 'simple commodity economy' (Rubin, 1928) capitalist investment and profit income are

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assumed to be absent. In this case, Marx proposed that prices would tend to 'correspond' to labour values, given that a few simple conditions are met:

> For prices at which commodities are exchanged to approximately correspond to their values, nothing more is necessary than 1) for the exchange of the various commodities to cease being purely accidental or only occasional; 2) so far as direct exchange of commodities is concerned, for these commodities to be produced on both sides in approximately sufficient quantities to meet mutual requirements, something learned from mutual experience in trading and therefore a natural outgrowth of continued trading; and 3) so far as selling is concerned, for no natural or artificial monopoly to enable either of the contracting sides to sell commodities above their value or to compel them to undersell. By accidental monopoly we mean a monopoly which a buyer or seller acquires through an accidental state of supply or demand.
>
> The assumption that the commodities of the various spheres of production are sold at their value merely implies, of course, that their value is the centre of gravity around which their prices fluctuate, and their continual rises and drops tend to equalise. (Marx, 1894, p. 178)

The aim of this paper is to check Marx's claim that prices gravitate toward labour values under conditions of simple commodity production. It is important to check Marx's claim for a number of reasons.

The law of value is a fundamental building block of Marx's theory of economic value. If it doesn't hold in this simple case, it is unlikely to hold in more general cases. The emergence of the law of value under conditions of simple commodity production is a necessary, but not sufficient, condition for the theoretical integrity of Marx's theory of value.

Marx argues that under capitalist conditions prices tend to gravitate toward profit-equalising prices of production that are constrained by aggregate labour values although not proportional to them (Marx, 1894). The tendency of prices to gravitate toward costs of production is a fundamental building block of all schools in the objective cost of production tradition, including the Marxist, neo-Ricardian and Post Keynesian schools. Marx's theory is unique, however, in maintaining that labour-cost is the 'substance of value' and ultimate determinant of price even under conditions of competitive prices. Hence, for Marx, the theory of economic value is ultimately an objective theory of labour costs.

This aspect of Marx's theory has caused great controversy. This paper is not intended to contribute directly to the debate between Marxist, neo-Ricardian and neoclassical authors regarding the status of the 'transformation problem' in Marx's labour theory of value (e.g. see Samuelson, 1971; Steedman, 1981; Steedman *et al.*, 1981; Wright, 2007). However, all sides in the debate agree that prices are proportional to labour values in the context of simple commodity production. Yet the value controversy has been almost exclusively conducted in terms of static linear production models in which the mechanism of 'gravitation' is assumed to exist and to have operated to completion. An analysis of the dynamics of simple commodity production, and a demonstration of the assumed gravitational properties, can help to clarify the economic significance of this debate (Keen, 1998).

The dynamics of simple commodity production also has implications for Pasinetti's analysis of the structural dynamics of a pure labour economy. Pasinetti (1993) examines structural dynamics under conditions of technical change but he abstracts from the 'institutional problem' of matching effective demand to productive capacity. Pasinetti discusses possible institutional arrangements that might satisfy the Keynesian principle of effective demand but does not integrate adjustment mechanisms into his formal analysis. The model developed in this paper examines to what extent the institution of a competitive market can match labour supply with labour demand in a pure labour economy without technical change. The analysis therefore fills a gap in Pasinetti's analysis.

To check Marx's claim, I have used the technique of agent-based computational modelling. The advantage of a computational approach is that features of the model need not be restricted to those directly amenable to mathematical analysis. Nonetheless, some stringent simplifications have been made, such as production of commodities by means of labour alone. The results obtained therefore only hold in the special case of simple commodity production, or a pure labour economy where each sector is interpreted as producing a vertically integrated composite commodity. The model is therefore only a preliminary step toward examining the more general case of dynamics on input–output graphs in which commodities are produced by means of other commodities (e.g. Sraffa, 1960).

The main result is that Marx's law of value does emerge as an unintended consequence of uncoordinated market activity. The computational model and formal analysis yield new and particularly satisfying dynamic relationships between values, prices, the allocation of social labour-time and money.

2. The Computational Model

The law of value is a theory of how the total labour of a society of commodity producers, who freely exchange their products in a marketplace, is divided and allocated to different branches of production via the market mechanism. The exchange of commodities at prices that deviate from values is the mechanism by which social labour-time is transferred from one sector of production to another. When prices equal values the division of labour has reached an equilibrium that satisfies social demand: 'the law of value is the law of equilibrium of the commodity economy' (Rubin, 1928, p. 67). For,

> it is only through the 'value' of commodities that the working activity of separate independent producers leads to the productive unity which is called a social economy, to the interconnections and mutual conditioning of the labour of individual members of society. Value is the transmission belt which transfers the movement of working processes from one part of society to another, making that society a functioning whole. (Rubin, 1928, p. 81)

The law of value is the process by which a simple commodity economy (i) reaches an equilibrium, in which (ii) prices correspond to labour values, and (iii) social labour is allocated to different branches of production according to social demand (where 'social demand' is understood to mean consumption requirements constrained by income).

The model therefore consists of a set of N economic actors (labelled $1 \ldots N$) that produce, consume and exchange a set of L commodity types (labelled $1 \ldots L$); a fixed amount of paper money M, which is distributed amongst the actors; a market mechanism that mediates commodity and money exchange; a set of rules to control the dynamics; and a top-level simulation rule that processes events and increments time, which is measured in update steps.

2.1. Actor Production

Every actor specialises in the production of a single commodity at any one time. The current specialisation of actor i is given by $A(i)$. All commodities are simple, and do not require other commodities for their manufacture. Each commodity requires the work of a single actor for its production. Constant returns to scale prevail and consequently there is no rationale for the existence of firms. Actors never cease production. A production column vector, $\mathbf{l} = (1/l_1, \ldots, 1/l_L)$, where $l_j > 0$, defines the rate at which an actor can produce each commodity type. For example, an actor that specialises in commodity type j produces at a rate of $1/l_j$ units per time step. Each l_j is the labour value of commodity j. The production vector is identical and fixed for all actors. Labour in the economy is therefore homogeneous and is not subject to changes in technique. Once a commodity is produced it remains the property of the actor until consumed or exchanged. Each actor has an associated endowment vector that represents how much of each commodity is currently held.

Actors produce according to the following rule.

Production update rule P_1: (Deterministic).

At the start of the simulation, initialise the endowment vector for actor i to zero: $\mathbf{e}_i = 0$. Actor i subsequently generates one unit of commodity $A(i)$ every $l_{A(i)}$ time steps, and the appropriate element of the endowment vector, $\mathbf{e}_i[A(i)]$, is incremented by one.

Although no producer is more efficient than another, a distinction between socially necessary labour-time and actual labour-time expended can be maintained. Overproduction of a commodity relative to the social demand implies that some of the labour-time expended was socially unnecessary.

2.2. Actor Consumption

Every actor desires to consume all commodity types. This behaviour can be interpreted as subsistence or aspirational. A consumption column vector, $\mathbf{c} = (1/c_1, \ldots, 1/c_L)$, where $c_j \geq 0$, defines the desired rate of consumption events for all actors. For example, every actor desires to consume commodity j at a rate of $1/c_j$ units per time step. The consumption vector is identical and fixed for all actors and represents an economy with homogeneous tastes that do not change. Note the asymmetry between production rates and consumption rates: an actor always meets its single production rate, but only conditionally meets its consumption rates. Actual consumption rates depend on the availability of commodities produced by other actors.

Actors consume according to the following rule.

Consumption update rule C_1: (Deterministic).

At the start of the simulation initialise the consumption deficit vector for actor i to zero: $\mathbf{d}_i = 0$. Actor i subsequently generates one unit of consumption deficit for each commodity $j = 1, \ldots, L$ every c_j time steps, and the appropriate element of the deficit vector, $\mathbf{d}_i[j]$, is incremented by one. Each time step actor i consumes $\mathbf{o}_i = \min(\mathbf{e}_i, \mathbf{d}_i)$ commodities from its endowment to satisfy its current consumption deficit. A new endowment vector, $\mathbf{e}'_i = \mathbf{e}_i - \mathbf{o}_i$, and a new deficit vector, $\mathbf{d}'_i = \mathbf{d}_i - \mathbf{o}_i$ are formed.

Note that in each time step more than one commodity may be consumed, although only one commodity can be in production. The assumption of universal and constant production and consumption vectors could be relaxed by introducing supply and demand noise due to heterogeneity of consumption tastes and production efficiency.

2.3. *The Reproduction Coefficient*

The reproduction coefficient, $\eta = \sum_{j=1}^{L} l_j / c_j$, measures whether, given the 'social facts' of the production and consumption vectors, the economy may realise an overall social surplus, deficit or balance. A value of $\eta = 1$ implies the economy can achieve a state of simple reproduction (where total production equals total consumption), $\eta > 1$ implies an economy permanently in overall deficit (unrealised consumption capacity) and $\eta < 0$ implies an economy permanently in overall surplus (redundant production capacity). The analysis is restricted to economies with $\eta = 1$ that can theoretically achieve a balance between supply and demand but may over- and under-produce commodities due to a suboptimal division of labour.

2.4. *Money*

Each actor i owns a sum of symbolic money $m_i \geq 0$, which is used to purchase commodities for consumption. The total amount of money in the economy $M = \sum_{i=1}^{N} m_i$ is conserved. The unit of measure of money is the 'coin', although it is an arbitrarily divisible unit. Coins are neither produced nor consumed by actors. Actors exchange money for commodities, and therefore gain money when they sell, and lose money when they buy. Complications due to changes in the money supply are ignored.

2.5. *Subjective Prices*

Actors form subjective evaluations of commodity prices during bilateral exchange. Two requirements are placed on the evaluations: (i) purchasers cannot offer more coins than they possess, and (ii) offer prices must not be fixed a priori. The second requirement is important because the law of value trivially does not hold in an economy of homogeneous, a priori evaluators. For example, if every actor evaluated commodity j at 0 coins for all time then prices cannot converge to labour values. The law of value operates 'behind the backs' of economic actors because they adapt to changing local circumstances that are not of their own choosing but the result of global system properties. A natural approach to satisfying the requirement for adaptivity is to employ an adaptive algorithm from the field of machine learning that has some psychological plausibility and functions to minimise the consumption error. But this is an unnecessary level of detail at present. Instead, actors form selling and buying prices for each commodity according to the following:

Price offer rule O_1: (Stochastic).

The price of commodity j according to actor i is $P_j^{(i)}$, and it is randomly selected from the discrete interval $[0, m_i]$ according to a uniform distribution. The price is random but bound by the number of coins currently held.

The actors are adaptive in a weak sense: if they have less (respectively, more) coins they probably will offer less (more). Their changing circumstances are defined solely by how many coins they hold. The law of value, if it is to function, must therefore do so only via money flows, not by directly influencing or changing individual cost evaluations.

O_1 is one of many possible adaptive rules, but it is the simplest, and represents the minimal theoretical commitment to the decision processes employed by actors in real economies. In addition, Gode & Sunder (1993) have shown that random traders with a budget constraint realise the same allocative efficiency as human actors under the same market discipline, so there is reason to believe that market structure plays a more important causal role than individual rationality. This is a not a new finding: the relative unimportance of micro-structure compared to macro constraints is an important feature of statistical mechanics, a successful physical theory of large ensembles of abstract particles in interaction (Wannier, 1987). In this model, effort is concentrated on the structural determinations of the conditions under which evaluations take place, rather than the process of evaluation itself. Rule O_1 assumes that, absent a decision theory, a range of possible decision outcomes is equally likely.

2.6. The Market

Periodically actors meet in the marketplace. Trading behaviour continues until the market is cleared, which occurs when for every commodity type there are either no buyers or no sellers. Commodities are bought and sold in single units. A cleared market does not imply that all needs are satisfied or all commodities sold.

Market clearing rule M_1: (Stochastic).
Initialise the set of uncleared commodities to $C = \{j: 1 \leq j \leq L\}$.

(1) Randomly select an uncleared commodity j from the set according to a uniform distribution.
(2) Form the set of candidate sellers S, which contains all actors with a desire to sell commodity j (i.e. $S = \{x: \mathbf{e}_x[j] > \mathbf{d}_x[j], 1 \leq x \leq N\}$). Select the seller s from S according to a uniform distribution.

(3) Form the set of candidate buyers B, which contains all actors with a desire to buy commodity j (i.e. $B = \{x: \mathbf{d}_x[j] > \mathbf{e}_x[j], 1 \leq x \leq N\}$). Select the buyer b from B according to a uniform distribution.
(4) If no seller or no buyer (i.e. $S = \phi \ \vee \ \mathbf{B} = \phi$ then remove commodity j from C; otherwise, invoke market exchange rule E1 (see below).
(5) Repeat until there are no remaining uncleared commodities (i.e. $C = \phi$).

Rule M1 matches buyers with sellers who then conditionally exchange coins for commodities according to the following rule:

Market exchange rule E1: (Stochastic).

Given a buyer b and seller s of commodity j with offer prices $p_j^{(b)}$ and $p_j^{(s)}$ respectively, determined by price offer rule O1, select the exchange price, x, from the discrete interval $[p_j^{(b)}, p_j^{(s)}]$ according to a uniform distribution. The exchange price is randomly selected to lie between the two offer prices. If the buyer has sufficient funds ($m_b \geq x$) then the transaction takes place. Actor b loses x coins and gains one unit of commodity j, and the appropriate element of its endowment vector, $\mathbf{e}_b[j]$, is incremented by 1. Actor c gains x coins and loses 1 unit of commodity j, and the appropriate element of its endowment vector, $\mathbf{e}_s[j]$, is decremented by 1.

Rules M_1 and E_1 do not represent a typical Walrasian market in which transactions take place at equilibrium after a process of extended price signalling or *tâtonnement*. Instead, transactions occur at disequilibrium prices, commodities may go unsold, and the same commodity type may exchange for many different prices in the same market period. Further, commodities in oversupply may initially fail to sell only to find willing buyers at a later time, and commodities in undersupply may not necessarily realise a higher price. In sum, although the rules do implement short-term price signalling due to disequilibrium between supply and demand, the detailed dynamics of this process are not straightforward, and can only be approximated by mathematical models that assume continuous and immediate price adjustment.

2.7. Division of Labour

The set $A_j = \{i: 1 \leq i \leq N, A(i) = j\}$ contains those actors that specialise in the production of j. The set $D = \{A_j: j = 1, \ldots, L\}$ partitions the actors into production sectors and represents the total division of labour of the economy. The division of labour is dynamic because actors change what they produce. Actors attempt to meet their consumption requirements but do not explicitly maximise wealth. They switch from one production sector to another according to the following rule:

Sector-switching rule S_1: (Stochastic).

For actor i, at the end of every nth period of length T time steps, form the consumption error, defined as the Euclidean norm of the consumption deficit vector, $\|\mathbf{d}_i^{(n)}\|$. $\|\mathbf{d}_i^{(n)}\|$ is compared to the consumption error of the previous period $\|\mathbf{d}_i^{(n-1)}\|$. If $\|\mathbf{d}_i^{(n)}\| > \|\mathbf{d}_i^{(n-1)}\|$ then randomly select a new production sector from the available L according to a uniform distribution. In other words, if the consumption error has increased from the previous period then the actor will swap to a new sector. T is a constant multiple of the maximum consumption period, c_i, such that $c_i \geq c_j$ for all $j = 1, \ldots, L$. Hence, actors produce and have the opportunity to sell at least one commodity before sampling the consumption error and deciding whether to switch.

There are no switching costs. The result of all actors following rule S_1 is to perform a parallel search over possible social divisions of labour. Dissatisfied actors randomly switch to new sectors in search of sufficient income to meet their consumption requirements.

2.8. Simulation Rule

The cycle of production, consumption, exchange and reallocation of social labour proceeds according to the following rule.

Simulation rule R_1:
Randomly construct production (**l**) and consumption vectors (**c**) for the economy, such that the reproduction coefficient $\eta = 1$. Allocate M/N coins to each of the N actors (the initial distribution does not affect the final outcome).

(1) Increment the global time step.
(2) For each actor invoke production rule P_1.
(3) For each actor invoke consumption rule C_1.
(4) Invoke market clearing rule M_1.
(5) For each actor invoke sector-switching rule S_1.
(6) Repeat.

The rule set for the simple commodity economy

$$\text{SCE} = \{R_1, P_1, C_1, O_1, \{M_1, E_1\}, S_1\}$$

defines the computational model. The implementation has five parameters: (i) the number of actors N, (ii) the number of commodities L, (iii) the amount of coins in the economy M, (iv) an upper bound, R, on the maximum possible consumption period, which is used to constrain the random construction of production and consumption vectors during initialisation, and (v) a switching parameter C that is the constant multiple of the maximum consumption period required by sector-switching rule S_1.

3. Simulation Results

Computational models are suited to the detailed analysis of causal processes that are not amenable to straightforward mathematical treatment. The detailed supply and demand dynamics in this model are an example. But unlike mathematical proofs, which normally quantify over the whole parameter-space, the execution of a computational model is only a single sample of the parameter-space. It isn't

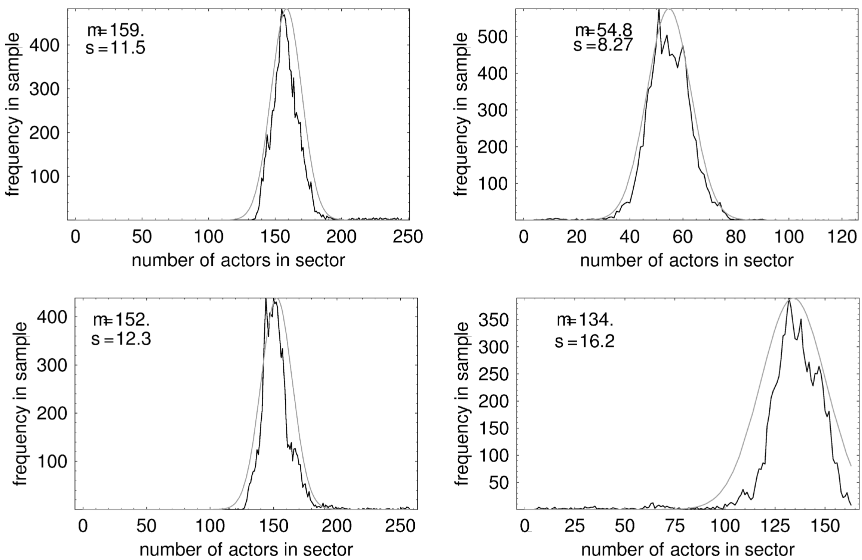

Figure 1. Stationary distributions of sector sizes with fitted normal distributions collected from a random sample of a four-commodity economy with parameter settings N: 500, L: 4, M: 2.5×10^5, R: 25, C: 2. The mean division of labour, (159, 54.8, 152, 134), is close to the theoretical efficient division of labour, $N\ (l_i/c_i) = (152, 56, 146, 146)$

practical to explore the entire parameter-space so the sampling procedure is biased toward subspaces that may be feasibly computed (for example, the time cost of the simulation rapidly increases with N), are realistic (for example, economies with a single coin are not considered) and conform to the requirements for the law of value to operate (for example, if the consumption period of a commodity j greatly exceeds the number of actors, i.e. if $R >> N$, then the probability that a seller of j will find a buyer in the marketplace is low; hence exchange becomes occasional, failing a requirement for the law of value to operate). All simulation runs follow a similar pattern of initial non-equilibrium activity prior to settling down to stable averages and stationary distributions (Appendix B contains further experimental details). Many variables of interest could be measured. Here we examine the stationary distributions of the division of labour and market prices.

3.1. Division of Labour

The distribution of actors in each sector of the economy settles to a normal distribution centred on a mean sector size. Figure 1 shows the stationary distributions of a typical sample. The equilibrium mean size of sector j is always approximately $N(l_i/c_i)$. Figure 2 reveals this relationship sampled over many runs.

Definition 1. A division of labour is *efficient* if for every commodity type the number of commodities produced equals the social demand.

Proposition 1. Let $a_j = |A_j|/N$ be the proportion of actors producing commodity j. Then $a_j = l_j/c_j$ ($j = 1, \ldots, L$) is an efficient division of labour.

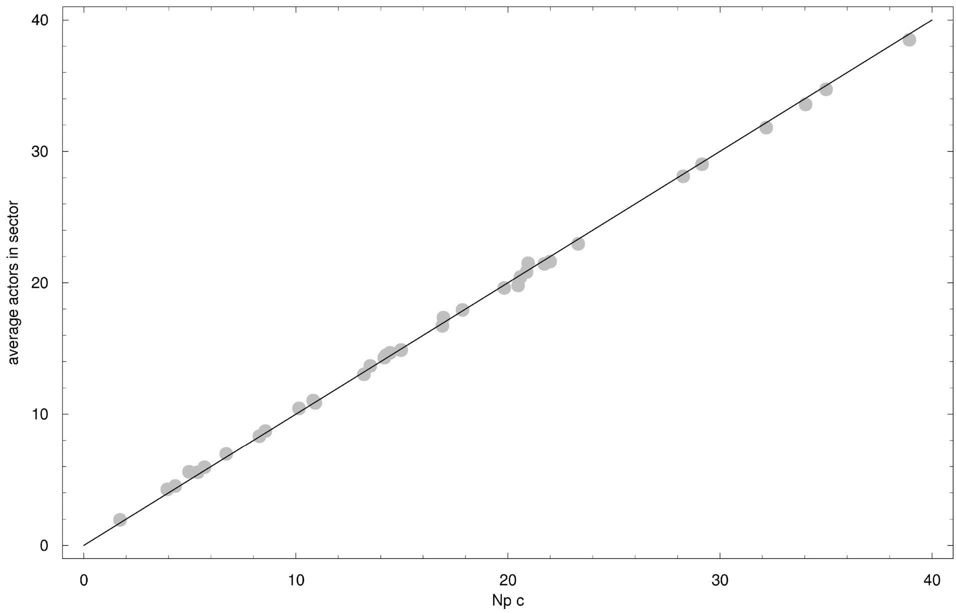

Figure 2. Relationship between mean sector size and $N(l_i/c_i)$ from 20 random samples of three-commodity economies with parameter settings *N*: 50, *L*: 3, *M*: 2500, *R*: 25, *C*: 2. The straight line represents the identity relationship $y = x$

Proof The social demand for commodity j is N/c_j units per unit time. When $a_j = l_j/c_j$ the number of units produced is $N(a_j/l_j) = N/c_j$ units per unit time, which equals the social demand.

On average, the division of labour is approximately efficient, but due to stochastic fluctuations perfect efficiency is never achieved. An efficient division of labour implies that the global consumption error is minimised and all actors meet their consumption requirements, absent market friction. Actual runs only approximate maximum consumption, and unsold commodities and unsatisfied demands can either stabilise or slowly accumulate over time. The results show that the SCE attains a (dynamic) equilibrium of the division of labour, and that the labour equilibrium is approximately efficient.

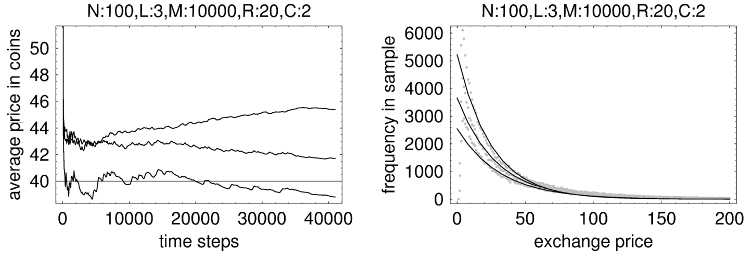

Figure 3. Evolution of mean commodity prices in a three-commodity economy (left figure) and stationary distribution of commodity prices with fitted exponential distributions (right figure)

3.2. Objective Prices

The stationary distributions of commodity prices can be approximately fitted by exponentials. Figure 3 shows the evolution of mean prices during a typical run and the associated stationary distributions. The price distributions have an exponential tail at the high end, but drop to zero at the low end, but the exponential distribution accurately models the price distributions over most of the price range. In equilibrium, a single commodity type does not have a single price, but has a range of prices that occur with differing but fixed probabilities.

The law of value states that, in equilibrium, market prices 'correspond' to labour values. The Pearson correlation coefficient, r, between two vectors, $\mathbf{x}$ and $\mathbf{y}$ measures the linear relationship between them ($-1 \leq r \leq 1$). A value of -1.0 is a perfect negative (inverse) correlation, 0.0 is no correlation, and 1.0 is a perfect positive correlation. $r = 1.0$ implies that there is a single scalar constant, λ, such that $\mathbf{x} = \lambda\mathbf{y}$. The correspondence between market prices and labour values is measured by their correlation. Denote the average price of commodity j as $\langle p_j \rangle$. Figure 4 graphs representative time series of the correlation between the market price column vector $\mathbf{p} = (\langle p_1 \rangle, \ldots, \langle p_L \rangle)$ and the labour values column vector $\mathbf{v} = (l_1, \ldots, l_L)$ (recall that l_j is the time period required to produce commodity j). The main simulation result of this paper may now be stated: the correlation between mean market prices and labour values approaches unity in equilibrium. Table B1 in Appendix B contains further experimental results that demonstrate the robustness of this result.

The results confirm that the SCE attains a dynamic equilibrium in which the mean equilibrium price of a commodity, measured over a sampling period, is proportional to the labour-time required to make it. Prices 'gravitate' around

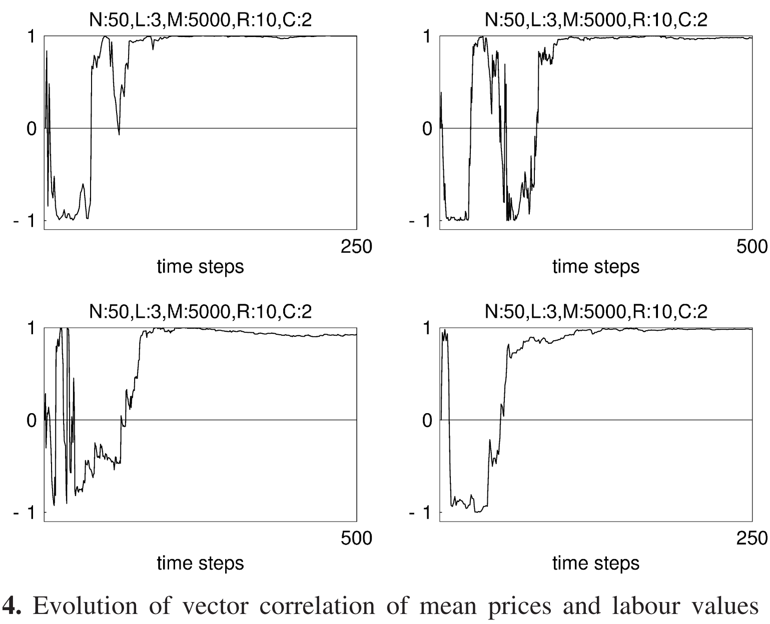

Figure 4. Evolution of vector correlation of mean prices and labour values over four samples of three-commodity economies

labour values and this equilibrium coexists with local and subjective pricing decisions constrained only by money endowments.

The equilibrium constant of proportionality λ, between mean prices and labour values, such that $\mathbf{p} \cong \lambda \mathbf{v}$, must have dimensions of coins per unit labour-time. λ summarises the causal relationship between expenditure of labour-time in production and the representation of that time in the market price of commodities. It measures how much labour-time money represents. Dumenil (1983) and Foley (1982) first re-emphasised the importance of this constant in Marxist economic theory and proposed a definition of it suitable for a capitalist economy.

Definition 2. The *Monetary Expression of Labour-time* (MELT) is the ratio of the net product at current prices relative to the productive labour expended in an economy over a given period of time.

In a pure labour economy there is no distinction between gross and net product and hence the MELT is the ratio of the product at current prices relative to the labour expended, which can be directly measured as:

$$\lambda = \frac{\gamma M}{\sum_{i=1}^{L} l_i v_i} \tag{1}$$

where γ is the proportion of the total money in the economy that on average changes per unit of time and hence γM is the mean money velocity and v_j is the mean exchange velocity of commodity j. The numerator in the definition is the rate of money exchange, the denominator is the rate of labour-time exchanged in the form of commodities, and the MELT is the ratio of the two, measured in coins per unit of labour-time. This definition translates into a computational rule to sample λ that executes per application of rule R_1. The mean velocities of commodities and coins are calculated as historical averages. For a detailed analysis of the dynamics of the MELT, not pursued here, either a moving average or an instantaneous velocity measure is more appropriate. Figure 5 plots equilibrium mean prices $<p_j>$ against labour values multiplied by the MELT, λl_j, for a typical run of a 10-commodity economy. It demonstrates that the MELT is the constant of proportionality implied by the correlation results. The role of money as a representation of labour-time is particularly clear in this relationship.

The definition of MELT does not represent a causal theory of how the MELT is determined. The value of MELT will vary under different institutional arrangements, such as how the market operates in detail, what kind of money and commodity throughput obtains, and so forth. Unlike the venerable quantity theory of money $MV = PT$ (where M is money, V is money velocity, P is the price level and T the level of transactions), which is an accounting identity between market phenomena, the MELT abstracts a non-obvious causal relationship between non-market phenomena (production times) and market phenomena (prices).

4. Analysis

The experimental results demonstrate that (i) a dynamic equilibrium is reached, in which (ii) mean prices are linearly related to labour values by a constant of

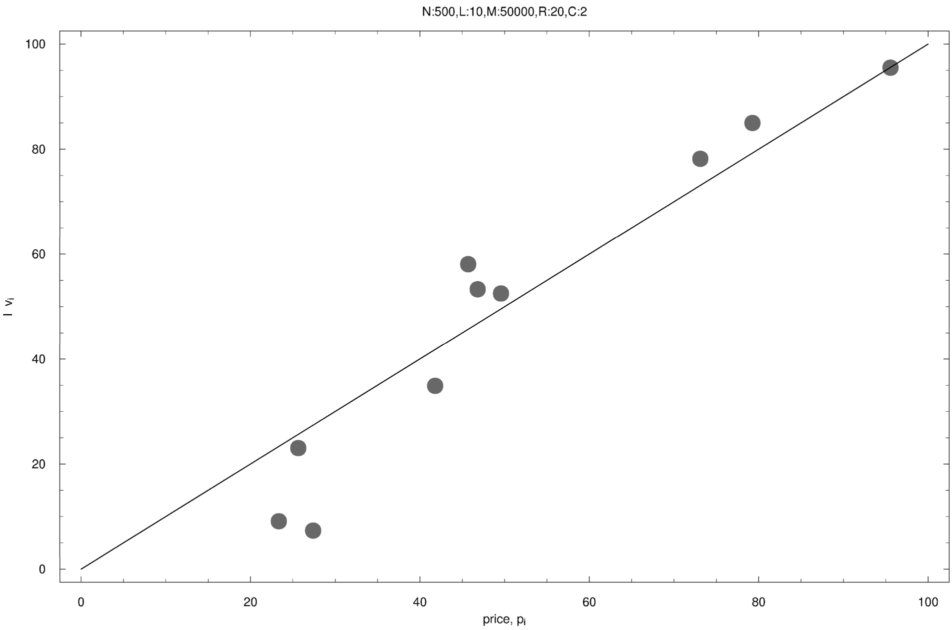

Figure 5. Stationary market prices and MELT transformed labour values in a 10-commodity economy with $r = 0.96$. The straight line represents the identity relationship $y = x$

proportionality called the monetary expression of labour-time (MELT), and (iii) social labour is allocated approximately efficiently. The computational model generates these regularities but it does not provide an adequate explanation of them. The law of value *emerges* from dynamic interactions of the constituent parts of the SCE, but a theory is required to explain this emergence.

The qualitative theory of the law of value was most fully developed by Isaak Rubin in his 1928 book, *Essays on Marx's Theory of Value*. In what follows, we model the SCE by a system of ordinary differential equations that refer to the means of the variables of interest, thereby extending Rubin's theory. The mathematical analysis aims to provide an intuitive explanation of the gross causal features of the computational model rather than provide definitive proofs of its properties or develop an accurate stochastic theory of the steady distributions. The mathematical model is a derivative and highly simplified analysis of the causal properties of the computational model. For example, discrete change is approximated by continuous change under the assumption that the size of discrete variables in the computational model is large compared to their change in magnitude per time step.

4.1. The Labour Equation

The rate money enters and leaves the market, or money velocity, is a proportion of the total money in the economy, denoted γM ($0 \leq \gamma \leq 1$). Assume that γ is fixed constant, which is an approximation. A money allocation column vector, $\mathbf{b}(t) = (b_1, \ldots, b_L)$, where $\sum_{j=1}^{L} b_j = 1$ and $0 \leq b_j \leq 1$, represents the instantaneous proportion of the total money flow received by each sector at time t. The sectoral income rate is therefore given by $b_j \gamma M$.

The labour allocation column vector, $\mathbf{a}(t) = (a_1, \ldots, a_L)$, where $a_j = |A_j|/N$ (see Section 2.7), $\sum_{j=1}^{L} a_j = 1$ and $0 \leq a_j \leq 1$, represents the proportions of actors 'employed' in each sector at time t.

Use the mean price of a commodity to approximate its price distribution. Recall that the average price of commodity j is $\langle p_j \rangle$. The average cost of the universal commodity bundle, given current prices, is then $\sum_{j=1}^{L} \langle p_j \rangle / c_j$.

Actors switch sectors based on the consumption error, which is a function of the quantities of commodities received. To simplify the analysis, price signals, in the form of the mismatch between income and the average cost of the commodity bundle, are used as a proxy for the consumption error. This simplifying assumption is used in the remainder of the analysis.

Each sector has an ideal expenditure rate that represents the money that would need to be spent in order for the constituent actors to meet their desired consumption rates. The rate is a function of the number of actors in the sector and current prices, and is given by: $a_j N \sum_{k=1}^{L} \langle p_k \rangle / c_j$.

The sectoral income error, denoted ϕ_j, measured in coins per unit time, is the difference between the actual income rate and the ideal expenditure rate:

$$\phi_j(t) = b_j \gamma M - a_j N \sum_{k=1}^{L} \frac{\langle p_k \rangle}{c_k}$$

A value of $\phi_j > 0$ implies a sectoral 'profit' (the sector receives more income than its constituent actors require to purchase the commodity bundle); $\phi_j < 0$ implies a sectoral deficit (there is insufficient income for the actors employed in the sector to purchase the commodity bundle); and $\phi_j = 0$ implies sectoral income equals ideal expenditure.

Approximate the switching behaviour of actors by assuming that the rate of change of labour allocation (or sector size) is proportional to the sectoral income error:

$$\frac{\mathrm{d}}{\mathrm{d}t} a_j = \Psi \phi_j(t) = \Psi \left(b_j \gamma M - a_j N \sum_{k=1}^{L} \frac{\langle p_k \rangle}{c_k} \right) \quad (2)$$

where $\Psi > 0$ is a reaction coefficient. It follows from the definition that $\phi_j < 0$ implies a net decrease in the sectoral population, and $\phi j > 0$ a net increase subject to the constraint $\sum_{j=1}^{L} a_j = 1$. Call equation (2) the labour equation because it defines how the allocation of labour to different sectors of production changes according to the money income received from the sale of commodities. The labour equation for the whole economy in vector notation is:

$$\dot{\mathbf{a}} = \Psi[\gamma M \mathbf{b} - N(\mathbf{p} \cdot \mathbf{c})\mathbf{a}] \quad (3)$$

where $\mathbf{p} \cdot \mathbf{c}$ is the dot product of the average price vector and the consumption vector.

The production rate for commodity j is given by $a_j N/l_j$. The average price of a commodity is defined as the current sectoral income rate divided by the sectoral production rate:

$$\langle p_j \rangle = \frac{\gamma M}{N} \frac{b_j}{a_j} l_j \tag{4}$$

Hence, each $\langle p_j \rangle$ is a function of a_j and b_j.

4.2. The Money Equation

The labour equation describes how the division of labour depends on incomes and prices, but as yet there is no model of how these change. A sector's income depends on the number of commodities produced. The maximum possible social consumption rate or 'social demand' for commodity j is N/c_j. The sectoral 'production error', denoted ε_j, measured in units of commodity j per unit time, is the difference between supply and demand:

$$\varepsilon_j(t) = \frac{a_j N}{l_j} - \frac{N}{c_j}$$

A value of $\varepsilon_j > 0$ implies over-production; $\varepsilon_j < 0$ implies under-production; and $\varepsilon_j = 0$ implies supply equals social demand. Assume that market rule M_1 operates such that it can be approximated by the expected relationship between supply, demand and price: commodities in over-supply have lower average prices than those in under-supply. This implies that the rate of change of sector income is negatively proportional to the production error:

$$\frac{\mathrm{d}}{\mathrm{d}t} b_j = -\omega \varepsilon_j(t) = -\omega N \left(\frac{a_j}{l_j} - \frac{1}{c_j} \right) \tag{5}$$

where $\omega > 0$ is a reaction coefficient. It follows from the definition that $\varepsilon_j < 0$ implies an increase in sectoral income, and $\varepsilon_j > 0$ a net decrease, subject to the constraint $\sum_{j=1}^{L} b_j = 1$. Call equation (5) the money equation because it defines how the allocation of money to different sectors of production changes according to the over- or under-production of commodities. The money equation for the whole economy in vector notation is:

$$\dot{\mathbf{b}} = -N\omega(\mathbf{A}\mathbf{l} - \mathbf{c}) \tag{6}$$

where $\mathbf{A}$ is the L by L diagonal matrix with element (i, i) equal to a_i and element (i, j) $(i \neq j)$ zero.

4.3. Equilibrium

The $2L$ labour (equation (3)) and money (equation (6)) equations mutually interact and describe the evolution of the division of labour via the mechanism of market price changes. The causal schema is as follows: (i) an existing division of labour results in (ii) over- and under-production of commodities that causes (iii) error-correcting price changes on the market due to supply and demand, which

(iv) generate changes in sectoral incomes that (v) cause actors that cannot meet their consumption requirements to swap sectors, resulting in (vi) a new division of labour. Some results are now derived that show that the mutual interaction results in an equilibrium point at which prices equal labour values.

Definition 3. A *simple commodity system* is described by the following system of $2L$ coupled differential equations:

$$\dot{\mathbf{a}} = \Psi\,(\gamma M\mathbf{b} - N(\,\mathbf{p}\cdot\mathbf{c})\mathbf{a}) \tag{7}$$

$$\dot{\mathbf{b}} = -N\omega(\mathbf{A}\mathbf{l} - \mathbf{c}) \tag{8}$$

and

$$\langle p_j \rangle = \frac{\gamma M}{N}\frac{b_j}{a_j} l_j$$

subject to the constraints

$$\begin{array}{ll} \sum_{j=1}^{L} a_j = 1 & 0 \leq a_j \leq 1 \\ \sum_{j=1}^{L} b_j = 1 & 0 \leq b_j \leq 1 \\ \sum_{j=1}^{L} \frac{l_j}{c_j} = 1 = \eta & l_j, c_j > 0 \\ M, N > 0 & \omega,\ \Psi > 0 \\ 0 \leq \gamma \leq 1 & \end{array}$$

Note that the constraint of finite and constant amounts of labour and money entails that the system cannot be decomposed into independent subsystems.

Lemma 1 (Equilibrium point). The simple commodity system has the unique equilibrium point

$$\mathbf{a}^* = \left(\frac{l_1}{c_1}, \frac{l_2}{c_2}, \ \ldots, \frac{l_L}{c_L}\right) = \mathbf{b}^* \tag{9}$$

Proof. The proof is in Appendix A.

The lemma states that $\dot{a}_j = \dot{b}_j = 0$ (i.e. the system is at rest) when the proportion of actors employed in a sector equals the proportion of money received by the sector, and that proportion is l_j/c_j. This makes intuitive sense: every actor consumes the same consumption bundle; therefore, on average, they require the same income (otherwise actors move to different sectors and the system is not at rest). The lemma does not imply that every actor receives the same income in equilibrium, only that sectoral averages are equal. (In fact, the stationary income distribution in the SCE is highly unequal and approximately exponential.)

Lemma 2 (Global stability). The equilibrium point is globally asymptotically stable.

Proof. The proof is in Appendix A.

The lemma states that the system, regardless of its initial conditions, always approaches the equilibrium point. The simple commodity system is a feedback system that functions to minimise both income and production 'errors'. This formalises Rubin's assertion that

> [a] given level of market prices, regulated by the law of value, presupposes a given distribution of social labour among the individual branches of production. . . . Marx speaks of the 'barometrical fluctuations of the market prices.' This phenomenon must be supplemented. The fluctuations of market prices are in reality a barometer, an indicator of the process of distribution of social labour which takes place in the depths of the social economy. But it is a very unusual barometer; a barometer which not only indicates the weather, but also corrects it. (Rubin, 1928, p. 78)

Lemma 2 explains why simulation runs tend to equilibrium.

Corollary 1 (Efficient division of labour). The division of labour is efficient in equilibrium.

Proof. By Lemma 1 the proportion of actors in sector j at equilibrium is $a_j = l_j/c_j$, which by Proposition 1 is efficient.

Corollary 1 explains why simulation tends to an approximately efficient division of labour. The experimental results do not exhibit perfect efficiency because the SCE is non-deterministic and undergoes stochastic fluctuations in equilibrium.

Theorem 1 (The law of value). Labour values are global attractors for average market prices.

$$\lim_{t \to \infty} \mathbf{p}(t) = \lambda \mathbf{v} \tag{10}$$

Proof. Substituting the equilibrium point, $a_j = l_j/c_j = b_j$, into equation (4) yields $<p_j> = \lambda l_j$, which by Lemma 2 is the globally asymptotically stable market price.

At equilibrium, the average price of a commodity is proportional to the labour-time required to make it. The constant of proportionality, $\lambda = \gamma M/N$, represents the monetary value of one unit of labour-time. Theorem 1 accounts for the observed correlations between prices and labour values.

In equilibrium, actors receive equal mean incomes but are engaged in productive activity of unequal periods. Hence, commodities that take longer to produce sell for higher mean market prices. This is the fundamental reason why prices correspond to labour values at the equilibrium of the simple commodity economy.

4.4. Disequilibrium Deviation of Price from Value

A key insight of Marx's theory of the law of value is that prices refer to amounts of labour time and deviations of prices from values are *social error signals* that function to redistribute labour. Only in the hypothetical situation of balanced supply and demand in which labour is efficiently distributed are prices proportional to labour values. The deviation of price from value at disequilibria can be analysed by introducing the concept of labour commanded.

Definition 4 A commodity commands an amount of labour in exchange. The *labour commanded* by a commodity is its money price divided by the MELT, measured in units of labour-time. The mean labour commanded

$$\langle \kappa_j \rangle = \frac{\langle p_j \rangle}{\lambda} \tag{11}$$

represents how much social labour-time a commodity on average fetches in the marketplace.

If a commodity-type commands an amount of social labour $\kappa_j < l_j$ then it is *undervalued*; if it commands amount $\kappa_j > l_j$ it is *overvalued*. The labour commanded an objective property of the exchange, and is distinct from any subjective valuations of the utility of the transaction from the perspective of a particular actor. At equilibrium, labour embodied equals labour commanded, that is $<\kappa_j> = l_j$ for all $j = 1, \ldots, L$. But otherwise commodities sell below value or above value, in accordance with the laws of supply and demand.

An act of exchange involves more than swapping of a commodity for an amount of money. It is also an exchange of a representation of an amount of social labour-time, measured by labour commanded, for an amount of private labour-time actually expended in the production of the commodity. Normally this is not an exchange of equivalents.

If the global division of labour mismatches the social demand then labour associated with scarce commodities is rewarded with access to additional social labour-time, whereas labour associated with unwanted commodities is punished by a reduction of access. Out of equilibrium, not all private labours are mutually equalised and not all private labours are socially necessary. But if the reallocation of labour resources is based on these monetary reward signals then the feedback loop completes and a division of labour emerges in which unnecessary private production is minimised and prices approach labour values. This dynamic relationship between labour embodied and labour commanded as regulator of the division of labour is apparent in the following relationship

$$\dot{a}_j = a_j \left(\frac{\langle \kappa_j \rangle}{l_j} - 1 \right) \psi \gamma M \tag{12}$$

which is derived in Appendix A. The term in brackets is positive if the commodity type is overvalued (implying an increase in the sector size) and negative if the commodity type is undervalued (implying a decrease in the sector size). Equation (12) reveals the causal connection between labour allocation and prices that occurs under the surface of the simple commodity economy. It is a precise formulation of

Rubin's observation that 'value is the transmission belt that transfers the movement of working processes from one part of society to another, making that society a functioning whole' (Rubin, 1928, p. 81) and summarises how the interaction of private commodity producers, using a monetary representation of the total social labour-time, spontaneously allocates labour to different branches of production according to social demand.

The actual price distribution will be sensitive to the particular price offer rule (or rules) employed by the actors. The more important point, therefore, is that in statistical equilibrium the same commodity type realises a range of different market prices, $p_k^{(1)}, p_k^{(2)}, \ldots,$ each of which represents different transfers of social labour-time between buyer and seller. The role of the mismatch between labour embodied and labour commanded in regulating the division of labour is apparent 'on average' and is a property of the price distributions, not a property of individual transactions. Hence, a commodity type may be correctly valued in equilibrium while, at the same time, particular transactions may represent under- or over-valuations of the commodity instance. The law of value states that, whatever the precise distribution of exchange prices, mean equilibrium prices are proportional to labour values. This view that the law of value manifests in probability distributions rather than individual transactions is consistent with the probabilistic approach to political economy initiated by Farjoun & Machover (1989).

5. Discussion

The choice of modelling symbolic money (e.g. paper or coins), which has nominal but no intrinsic value, rather than money in the form of commodity such as gold, which has intrinsic value by virtue of the labour required for its production, differs from Marx's presentation but has the advantage of separating two definitions that may be easily conflated in his analysis of money (for a discussion, see Foley, 1983): (i) the 'value of money', which is the inverse of the MELT and is the labour-time represented by the monetary unit (e.g. 1 hour of social labour-time is represented by 1 coin), and (ii) the 'value of the money commodity', which is the amount of social labour-time required for the production of a unit of the money commodity (e.g. 1 ounce of gold requires 1 hour of social labour-time for its production).

Roemer (1982, pp. 27–31) presents a static argument in which, in a simple commodity economy, the only prices capable of reproducing the system are those proportional to embodied labour times. The derived prices satisfy the constraints of the economic situation represented as a linear programming problem. The deduction abstracts from market interactions that occur in historical time and from disequilibrium supply and demand dynamics; hence, the mechanism by which such prices are reached is absent. The model is constraint-based rather than causal. The idea that labour values are *attractors* for prices in the simple commodity economy does not contradict this static result. A dynamic analysis, however, is a more stringent test of the conceptual integrity of the Marx–Rubin law of value, which is essentially concerned with how markets function to allocate social labour-time via error-correcting price signals. In static models, such as Roemer's, prices are nominal and lack a casual connection with the reallocation of labour. The mechanism of the law of value should not be reduced to its attractor.

Krause (1982) understands the importance of the dynamic coordination of concrete labours in market economies via the price mechanism. He contends that most modern formulations of the labour theory of value, from Sraffa onwards, assume that concrete labours of different types are equivalently valued, an assumption he labels 'the dogma of homogenous labour' (Krause, 1982, pp. 160–161). According to Krause (1982, p. 101), the 'supposition of homogenous labour supplants any analysis of the specific coordination of concrete labours.' The static methods employed by Krause, which represent the economic situation in terms of linear algebra, are sophisticated, and can quantify over-complex production structures, in particular the production of commodities by means of others. In contrast, the dynamic approach taken here is relatively unsophisticated and models a simple production structure. Unlike static approaches, however, the dynamic approach can model the coordination of concrete labours, and this reveals a new dynamic relationship between homogeneous and heterogeneous labour.

Following Krause, let α_{ij} be the reduction coefficient of concrete labour type i to j $(i \neq j)$, such that 1 hour of labour of type i is equivalent to α_{ij} hours of labour type j, where the equivalence relation is induced by market exchanges. The assumption of homogeneous labour is that $\alpha_{ij} = 1$ for all i and j. The reduction coefficients in the simple commodity system are:

$$\alpha_{ij} = \frac{\langle p_i \rangle / l_i}{\langle p_j \rangle / l_{ji}} = \frac{b_i / a_i}{b_j / a_j}$$

Note that the assumption of homogeneous labour is not made. Theorem 1 can be reformulated as

$$\lim_{t \to \infty} \frac{b_i}{a_i} = 1$$

and by the quotient rule for limits it follows that for all $i = 1, \ldots, L$ and $j = 1, \ldots, L$

$$\lim_{t \to \infty} \alpha_{ij} = 1 \tag{13}$$

The statement that labour values are attractors for prices is equivalent to the statement that homogeneous reduction coefficients are attractors for heterogeneous reduction coefficients. Krause (1982, p. 101) writes: 'It is conceivable that certain assumptions about the mechanism of coordination could *produce* equal reduction coefficients. But the classical/contemporary labour theory of value does not formulate such assumptions, so the homogeneity is mere dogma.' But it is inaccurate to state that the Marx–Rubin formulation of the law of value assumed homogeneous labour without justification. The law of value is a dynamic theory of labour allocation based on the tendency of heterogeneous labour to be homogenised via commodity exchange, and in this sense is very different from modern static formulations of it. The reduction coefficients are continuously calculated by a distributed computation that is implemented through the actions of the economic actors. Homogeneity emerges in the simple commodity economy under the assumption that economic actors have equal productive powers as members of the same species, strive for equal remuneration for their labour time (that is, they consider themselves equal), and are free to realise their equality through unconstrained economic activity. Rubin (1928, p. 87) states that the '*equalization of exchanged commodities* reflects the basic social characteristic of the commodity economy: *the equality of commodity producers*.' The SCE models this ideal situation by allowing identical actors to move freely between sectors of production in order to meet identical consumption requirements. In reality, things are not so simple, and in the context of tendencies to narrow the wage dispersion, Rubin (1928, ch. 15) discusses factors that prevent homogenisation.

The emergence of the law of value calls into question the idea that a detailed consideration of individual rationality is a necessary component of an explanation of equilibrium prices. The results of this paper confirm Gode & Sunder's (1993) results and extend them to the more general context of a simple model of production.

The model specification can be interpreted as a design for an economic experiment in which human subjects replace the computational agents in order to play a production and trading game defined by the computational rules. An investigation of the emergence of the law of value in a laboratory setting is therefore possible.

6. Conclusion

The law of value is a phenomenon that emerges from the dynamic interactions of private commodity producers. The results demonstrate for the case of simple commodity production that (i) labour values are global attractors for market prices, (ii) market prices are error signals that function to allocate the available social labour between sectors of production, and (iii) the tendency of prices to approach labour values is the monetary expression of the tendency to efficiently allocate social labour. The constant of proportionality of the linear relationship between labour values and market prices is the monetary expression of labour-time (MELT), which measures how many units of money represent one unit of social labour-time. The MELT summarises a non-obvious causal relationship between non-market phenomena (production times) and market phenomena (prices), and links the total available social labour-time to its monetary representation. The concept of labour commanded, which measures how much social labour-time a commodity fetches in the marketplace, is important for theorising how deviations of price from value are labour reallocation 'signals'. The labour commanded by a commodity normally mismatches the private labour-time expended in its construction, indirectly signalling whether the labour was socially necessary or not. The law of value operates 'behind the backs' of actors via money flows that place income constraints on their local evaluations of commodity prices. The equilibrium of the simple commodity economy is a statistical equilibrium, in which a single commodity type may realise many different prices. In consequence, the regulating role of exchange value is a property of price distributions, not individual transactions. Further, the law of value can only emerge in broad models of economic systems that complete the feedback loop between production, consumption, exchange *and* reallocation of labour resources.

An actor engaged in free exchange derives personal benefit from transactions and the immediate apprehension of this fact may motivate subjective theories of value. However, an exchange has causal consequences beyond the immediate moment and the satisfactions of mutual commerce that derive from its embodiment within a system of generalised commodity production. Actors do not normally think money into existence although they do decide to spend more or less of what they have. Their income is a local representation of a global resource constraint not under their subjective control. Although money *exchanges* according to demands for use-values, and is normally accompanied by the satisfaction of desires, it *refers* to amounts of social labour-time. Local flows are easier to apprehend than global reference, and this partially accounts for the relative neglect of objective theories of value.

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Appendix A: Proofs

Proof of Lemma 1. Substituting equation (4) into (7) and considering a single sector gives:

$$\dot{a}_j = \psi\gamma M\left(b_j - a_j\sum_{k=1}^{L}\frac{l_k b_k}{c_k a_k}\right) \tag{14}$$

which is coupled with:

$$\dot{b}_j = -\omega N\left(\frac{a_j}{l_j} - \frac{1}{c_j}\right) \tag{15}$$

Setting $\dot{\mathbf{a}} = \dot{\mathbf{b}} = \mathbf{0}$ yields the unique equilibrium point of the system. Equation (15) implies $a_j = l_i/c_j$ and equation (14) implies $a_j = b_j \sum_{k=1}^{L} b_k = 1$. This solution is valid and unique for economies with reproduction coefficient $\eta = 1$, such that the equalities $\sum_{k=1}^{L} a_k = \sum_{k=1}^{L} b_k = \sum_{k=1}^{L} l_k/c_k = 1 = \eta$ hold.

Proof of Lemma 2. The non-linear sum in equation (14) can be eliminated as follows. Summing over all sectors:

$$\sum_{j=1}^{L}\dot{a}_j = \sum_{j=1}^{L}\left[\psi\gamma M\left(b_j - a_j\sum_{k=1}^{L}\frac{l_k}{c_k}\frac{b_k}{a_k}\right)\right]$$

but given that

$$\sum_{j=1}^{L}\dot{a}_j = 1 \Longrightarrow \dot{a}_1 + \dot{a}_2 + \cdots + \dot{a}_L = 0$$

then

$$\sum_{j=1}^{L}\left[\psi\gamma M\left(b_j - a_j\sum_{k=1}^{L}\frac{l_k}{c_k}\frac{b_k}{a_k}\right)\right] = 0 \Longrightarrow \gamma M\sum_{j=1}^{L}\left[\psi b_j - a_j\sum_{k=1}^{L}\frac{l_k}{c_k}\frac{b_k}{a_k}\right] = 0$$

As $\gamma M \neq 0$, then

$$\sum_{j=1}^{L}\psi b_j = \sum_{j=1}^{L}a_j\sum_{k=1}^{L}\frac{l_k}{c_k}\frac{b_k}{a_k}$$

Recalling that $\sum_{j=1}^{L} a_j = 1$ and $\sum_{j=1}^{L} b_j = 1$, then

$$\sum_{j=1}^{L}\frac{l_j}{c_j}\frac{b_j}{a_j} = 1$$

Substitution into equation (14) yields a linear form of the labour equation:

$$\dot{a}_j = \psi\gamma M(b_j - a_j) \tag{16}$$

A change of variables, $x_j = a_j - (l_j/c_j)$ and $y_j = b_j - (l_j/c_j)$ translates the equilibrium point to the origin. Given that $\dot{\mathbf{x}} = \dot{\mathbf{a}}$ and $\dot{\mathbf{y}} = \dot{\mathbf{b}}$ the transformed linear system is:

$$\dot{\mathbf{x}} = \psi\gamma M(\mathbf{y} - \mathbf{x})$$
$$\dot{\mathbf{y}} = -\omega N\mathbf{X}\mathbf{l}$$

where $\mathbf{X}$ is the $L \times L$ diagonal matrix with (i, i) entry equal to x_i and the (i, j) $(i \neq j)$ entry zero.

The x_j and y_j represent production and income errors respectively. Consider the function

$$V: \Re^{2L} \to \Re$$

$$V(x_1, \ldots, x_L, y_1, \ldots, y_L) = \frac{1}{2\psi\gamma M}\sum_{j=1}^{L} x_j^2 + \frac{1}{2\omega N}\sum_{j=1}^{L} l_y y_j^2$$

that associates a scalar error measure with each possible state of the simple commodity system. In fact, V defines an error potential.

Global stability is now deduced by Lyapunov's direct method (see Brauer & Nohel, 1989). V is positive definite as $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{0}$. Hence, V is a Lyapunov function. V is now shown to be strictly decreasing on all state trajectories:

$$V^* = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \cdots + \frac{\partial V}{\partial x_L}\dot{x}_L + \frac{\partial V}{\partial y_1}\dot{y}_1 + \frac{\partial V}{\partial y_2}\dot{y}_2 + \cdots + \frac{\partial V}{\partial y_L}\dot{y}_L$$

$$V^* = \frac{1}{\psi\gamma M}\sum_{j=1}^{L} x_j\dot{x}_j + \frac{1}{\omega N}\sum_{j=1}^{L} l_j y_j \dot{y}_j$$

Substituting for $\dot{x}_i$ and $\dot{y}_i$ gives

$$V^* = \sum_{j=1}^{L} x_j(y_j - x_j) - \sum_{j=1}^{L} x_j y = \sum_{j=1}^{L} x_j y_j - \sum_{j=1}^{L} x_j^2 - \sum_{j=1}^{L} x_j y_j = -\sum_{j=1}^{L} x_j^2 \leq 0$$

with $V^* = 0$ only when $\mathbf{x}^* = \mathbf{0}$. Hence, as time progresses, the simple commodity system always follows an error-reducing trajectory that approaches the origin. By Lyapunov's Theorem, the equilibrium point is asymptotically stable. Stability properties for linear systems are global. Therefore, the equilibrium point is globally asymptotically stable.

Derivation of equation (12). Labour commanded is given by

$$\langle \kappa_j \rangle = \frac{\langle p_j \rangle}{\lambda} = \frac{\gamma M}{\lambda N}\frac{b_j}{a_j} l_j = \frac{b_j}{a_j} l_j$$

Rearrange to give $b_j = a_j < \kappa_j > l_j$ and substitute into equation (16):

$$\dot{a}_j = \psi\gamma M(b_j - a_j)$$
$$= a_j\left(\frac{\langle\kappa_j\rangle}{l_j} - 1\right)\psi\gamma M$$

Appendix B. Experimental Details

Table B1. Labour value/market price correlations from random samples of the SCE, with parameter settings N: 200, L: n ($n = 3,\ldots, 10$), M: 500, R: 20, C: 2. Each parameter setting is sampled 10 times. Results are rounded to two decimal places. The current implementation runs out of memory when the number of commodities exceeds 10 (and is also prohibitively slow). If $L \to N$ (i.e. the number of commodities approaches the number of actors) then the economy is unlikely to sustain production rates and correlations will decrease

L	3	4	5	6	7	8	9	10
corr.	1.0	0.99	0.98	0.96	0.98	0.88	0.96	0.96
	0.99	1.0	0.99	0.97	0.98	0.97	0.96	0.99
	0.98	0.94	0.99	0.99	0.97	0.94	0.96	0.96
	0.98	0.99	0.93	0.99	0.95	0.97	0.97	0.91
	0.99	0.99	0.99	0.99	0.94	0.91	0.86	0.92
	0.96	0.84	0.99	0.93	0.98	0.99	0.95	0.95
	1.0	0.95	0.99	0.99	0.96	0.93	0.95	0.98
	0.99	0.97	1.0	0.98	0.96	0.94	0.94	0.95
	1.0	0.96	0.96	0.95	0.95	0.94	0.93	0.99
	1.0	1.0	0.95	0.97	0.95	0.95	0.99	0.93
Mean	0.99	0.96	0.98	0.97	0.96	0.94	0.95	0.95

The SCE is defined to have reached a state of statistical equilibrium when the rate of change (sampled every 1000 time steps) of the labour value/market price vector correlation is lower than a small threshold. When this convergence condition is met the simulation continues for a further 5000 time steps in order to sample the stationary distributions. (An alternative convergence condition is to check when the rate of change of entropy of every commodity price distribution is lower than a small threshold, but this was not tried). An upper-limit of 200,000 time steps is set in case convergence is not achieved within a reasonable time period. In almost every case, convergence is reached before the upper-limit. Market clearing rule M_1 cycles until there are either no buyers or no sellers for every commodity. With a large number of actors, the clearing loop takes a prohibitively long time, therefore, in practice, an upper limit of the maximum number of transaction attempts per actor is set. Once the number of maximum transactions is reached the actor is neither a buyer nor seller for any commodity. This can be interpreted as a 'time limit' on the market period.