CLASSICAL MACRODYNAMICS AND THE LABOR THEORY OF VALUE

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This paper outlines a multisector dynamic model of the convergence of market prices to natural prices in conditions of fixed technology and composition of demand. Prices and quantities adjust in real-time in response to excess supplies and differential profit-rates. Finance capitalists earn interest income by supplying money-capital to fund production. Industrial capitalists, as the owners of firms, are liable for profits and losses. Market prices stabilize to profit-equalizing prices of production proportional to the total coexisting labor required to reproduce commodities. This result resolves the classical problem of the incommensurability between money and labor-value accounts in conditions of 'profits on stock', i.e. Marx's 'transformation problem'.

This paper is primarily a philosophical analysis of some of the conceptual foundations of the classical labor theory of value. My starting premise is that dynamic multisector models of capitalist competition, in which out-of-equilibrium adjustments occur in real, historical time are a necessary prerequisite to the investigation of issues in the theory of economic value (see also Fisher [1, pg. 16]). To this end I outline a macrodynamic model of a hypothetical or 'ideal' capitalist economy along classical lines paying particular attention to Marx's depiction of the formation of profit-equalizing prices of production in Volume 3 of Capital, including his distinction between interest income and profit of enterprise. The model is intentionally simple yet nonetheless attempts to include all the essential phenomena required to construct a formal framework in which key issues in the classical theory of value can be posed. Although my focus is value theory the macrodynamic model may be of independent interest and could be extended and generalized in various directions.

First, I describe the model and briefly outline some of its properties including an example of convergence to 'long period' equilibrium. Then I apply the model to examine what kind of relationship, if any, holds between monetary phenomena and objective costs, specifically the labor-time required to produce commodities.

I. A MULTISECTOR MODEL OF CAPITALIST MACRODYNAMICS

Reproducible commodities as those 'that may be multiplied ... almost without any assignable limit, if we are disposed to bestow the labor necessary to obtain them' [2]. Smith, Ricardo, and Marx argued that market prices of reproducible commodities tend to gravitate toward or around their natural prices (e.g., Smith [3] , Book 1, Chapter VII or Marx [4]). Natural prices are robust to 'accidental and temporary deviations' [2] between supply and demand and manifest when quantities supplied equal quantities demanded. In the classical framework market prices are short-term, out-of-equilibrium prices formed by imbalances between supply and demand that get driven by profit-seeking dynamics toward natural prices, which are long-term equilibrium prices determined by the objective conditions of production (e.g., see [5]).

A dynamic approach to the formation of natural prices requires modeling out-of-equilibrium adjustment. By construction Neoclassical tâtonnement or market-clearing assumptions cannot explain the outcome of trial-and-error processes that occur in real time. Instead, I follow the classical 'cross-dual' tradition of formulating 'general disequilibrium' models [6] of economic dynamics over aggregated, multi-firm sectors in which prices and quantities adjust in tandem and trading occurs at out-of-equilibrium prices (e.g., Duménil and Lévy [6, 7] and Flaschel *et al.* [8, ch. 2]).

Assume $n \in \mathbb{Z}^+$ sectors that consist of a collection of competing firms that specialize in the production of the same commodity type. The technique is a non-negative $n \times n$ input-output matrix of inter-sector coefficients, $\mathbf{A} = [a_{i,j}]$. Each $a_{i,j} \geq 0$ is the quantity of commodity i directly required to output 1 unit of commodity j. Assume that matrix $\mathbf{A}$ is fully connected and $\mathbf{I} - \mathbf{A}$ is of full rank. There exists a vector $\mathbf{x}^{\mathrm{T}} \in \mathbb{R}^n_+$ such that $\mathbf{x}^{\mathrm{T}} > \mathbf{A}\mathbf{x}^{\mathrm{T}}$; that is, the technique is productive. The direct labor coefficients are a $1 \times n$ vector, $\mathbf{l} = [l_i]$. Each $l_i > 0$ is the quantity of labor directly required to output 1 unit of commodity i. Assume constant returns to scale; $\mathbf{A}$ and $\mathbf{l}$ are therefore fixed throughout. Market prices are a $1 \times n$ vector $\mathbf{p}(t) = [p_i(t)]$ and quantities produced (or sectoral activity levels) are a $1 \times n$ vector $\mathbf{q}(t) = [q_i(t)]$. The constant L denotes the size of the potential labor force and the constant M denotes the total nominal value of the stock of base money. These are the only non-reproducible, fixed resources.

I.1. Workers' propensity to consume

Money stocks are the only form of savings. The aggregate savings of worker households is a stock of money m_w. Workers' propensity to consume is a constant fraction, $\alpha_w \in (0, 1]$, of this sum.[1] At any time the aggregate expenditure of worker households is therefore $\alpha_w m_w$.

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[1] This definition differs from the more familiar Keynesian propensity to consume, which is a flow ratio of consumption to income. In this model stocks of money-holdings, which fluctuate according to the difference between the flow rates of consumption and income, influence the level of consumption.

I.2. The real wage

Marx, in general, assumed a given, subsistence real wage determined by the cost of production of workers [9]. Instead we will assume a fluctuating real wage that is always sufficient to ensure the reproduction of the available labor force, L. Workers are flexible with regard to the scale of their consumption but not the commodity bundle they demand. The $1 \times n$ real wage vector, $\mathbf{w}' = [w_i']$, has a constant composition, denoted by the $1 \times n$ wage composition vector $\mathbf{w} = [w_i]$, but has variable scale; that is $\mathbf{w}' = k\mathbf{w}$ always for some scale factor k.

The real wage is a function of the aggregate expenditure of worker households, $\alpha_w m_w$, and the current price of workers' consumption goods, $\mathbf{p}\mathbf{w}^{\mathrm{T}}$. The fraction $\alpha_w m_w / \mathbf{p}\mathbf{w}^{\mathrm{T}}$ denotes the number of real wage bundles of composition $\mathbf{w}$ that can be purchased at money prices $\mathbf{p}$. The real wage is therefore

$$\mathbf{w}' = \frac{\alpha_w m_w}{\mathbf{p}\mathbf{w}^{\mathrm{T}}} \mathbf{w}^{\mathrm{T}},$$

where $k = \alpha_w m_w / \mathbf{p}\mathbf{w}^{\mathrm{T}}$ is the variable scale factor (composition vector $\mathbf{w}$ defines a ray in commodity space that the real wage traverses). Given a constant aggregate expenditure lower (resp. higher) prices imply higher (resp. lower) real consumption.

I.3. Workers' savings

The level of employment is $\mathbf{l}\mathbf{q}^{\mathrm{T}}$. Workers' savings, m_w, are increased by an inflow of wage payments, $\mathbf{l}\mathbf{q}^{\mathrm{T}} w$, where w is the money wage rate, and reduced by an outflow of consumption spending, which is the fraction $\alpha_w m_w$ spent on the real wage. The rate of change of total savings is the sum of deposits and withdrawals,

$$\frac{\mathrm{d}m_w}{\mathrm{d}t} = \mathbf{l}\mathbf{q}^{\mathrm{T}} w - \alpha_w m_w. \tag{1}$$

Hence $\frac{\mathrm{d}m_w}{\mathrm{d}t} > 0$ indicates active saving and $\frac{\mathrm{d}m_w}{\mathrm{d}t} < 0$ indicates dissaving.

I.4. The wage rate

Marx [9, pg. 5] states that the money wage rate fluctuates with the supply and demand of labor: 'the same general laws which regulate the price of commodities in general, naturally regulate wages, or the price of labour-power. Wages will now rise, now fall, according to the relation of supply and demand, according as competition shapes itself between the buyers of labour-power, the capitalists, and the sellers of labour-power, the workers'. To express this I adopt a Phillips-like (1958) description of the labor market such that the change in the wage rate depends both on the level of unemployment and the rate of change of unemployment. The wage rate, w, given a fixed working population L, varies with the demand for labor. So an increase (resp. decrease) in the level of unemployment, $-\mathbf{l}\frac{\mathrm{d}\mathbf{q}^{\mathrm{T}}}{\mathrm{d}t} > 0$ (resp. < 0) causes a relative wage decrease (resp. increase); that is $\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}t} \propto \mathbf{l}\frac{\mathrm{d}\mathbf{q}^{\mathrm{T}}}{\mathrm{d}t}$. In addition, as the level of employment rises, and the labor market tightens, the relative wage also rises, until it approaches ∞ at the hypothetical maximum of full employment; that is, $\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}t} \propto \frac{1}{L - \mathbf{l}\mathbf{q}^{\mathrm{T}}}$. Combining these two factors we get

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \eta_w \mathbf{l} \frac{\mathrm{d}\mathbf{q}^{\mathrm{T}}}{\mathrm{d}t} \frac{1}{L - \mathbf{l}\mathbf{q}^{\mathrm{T}}} w, \tag{2}$$

where $\eta_w > 0$ is a constant elasticity of the wage rate with respect to unemployment.

I.5. Capitalists' propensity to consume

The aggregate savings of capitalist households is a stock of money m_c. Capitalists' propensity to consume is defined as a constant fraction, $\alpha_c \in (0, 1]$, of this sum. The effective demand from capitalist households is therefore $\alpha_c m_c$. The total aggregate expenditure in the economy is the sum of worker and capitalist expenditure, $\alpha_w m_w + \alpha_c m_c$.

I.6. Capitalist consumption

Capitalist consumption is specified in a similar way to worker consumption. The fraction $\alpha_c m_c / \mathbf{p}\mathbf{c}^{\mathrm{T}}$ denotes the number of bundles of composition $\mathbf{c}$ that are purchased at prices $\mathbf{p}$. Capitalist consumption is therefore

$$\mathbf{c}' = \frac{\alpha_c m_c}{\mathbf{p}\mathbf{c}^{\mathrm{T}}} \mathbf{c}^{\mathrm{T}},$$

where $k = \alpha_c m_c / \mathbf{p}\mathbf{c}^{\mathrm{T}}$ is the variable scale factor.

I.7. A monetary production economy

Marx, in Volume 3 of *Capital*, described an abstract specification of the economic relations between capitalists and firms. He splits the capitalist class into two functional roles: finance capitalists or 'money-capitalists' who lend money at interest to fund production, and industrial capitalists who, as owners and managers of firms, borrow money to expand production in order to gain 'profit of enterprise'.

Money-capital is money lent at interest that creates a creditor-debtor relation. 'It is this use-value of money as capital – this faculty of producing an average profit – which the money-capitalist relinquishes to the industrial capitalist for the period, during which he places the loaned capital at the latter's disposal' [4, pg. 351]. Money-capital is not a 'factor of production' in a technical sense but its availability and cost act as a constraint since firms need purchasing power to complete their production plans. Industrial capitalists borrow from finance capitalists with the expectation of earning a return in excess of the cost of borrowing. So the general or total profit breaks down into two different kinds of profit income: interest and profit of enterprise (or simply 'profit'). This functional division includes cases where the same individual performs both roles either in a single firm or over multiple firms; for example Marx [4, pg. 373] writes that 'the capitalist operating on his own capital, like the

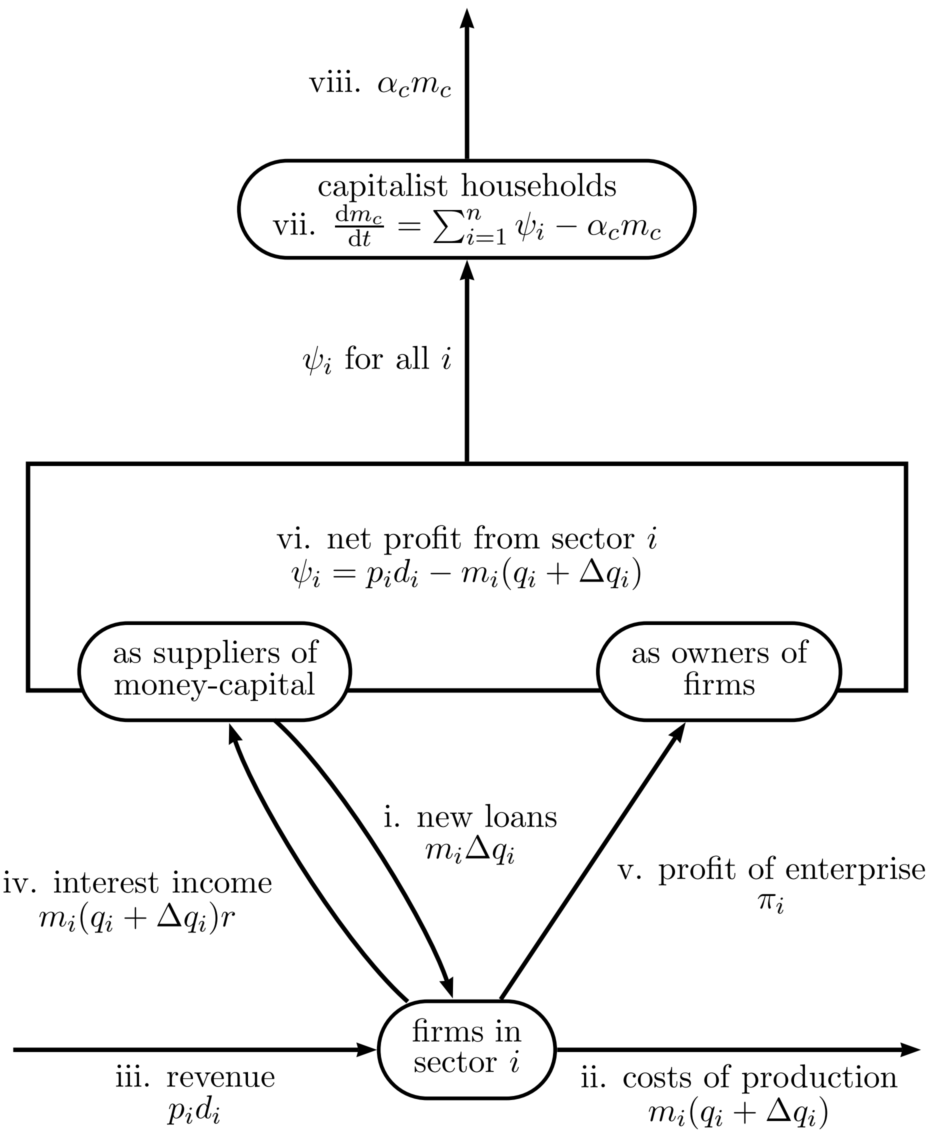


FIG. 1. **Flow of funds between capitalist households and the system of production.** Capitalists receive profit income in virtue of two property relations: as suppliers of money-capital they receive interest and as firm owners they are liable for profits and losses. For the sake of exposition consider a period δt during which sector i's level of borrowing alters due to a change in the scale of production, $\Delta q_i \ll q_i$. (i) If $\Delta q_i > 0$ then industrial capitalists borrow $m_i \Delta q_i$ new money from finance capitalists; if $\Delta q_i < 0$ then industrial capitalists repay $m_i \Delta q_i$ principal. The total money-capital invested in the sector is now $m_i(q_i + \Delta q_i)$. (ii) Firms spend $m_i(q_i + \Delta q_i)$ on commodity and labor inputs. (iii) Firms produce $q_i + \Delta q_i$ output and earn revenue $p_i d_i$ from d_i sales of commodity i. (iv) $m_i(q_i + \Delta q_i)r$ interest on the total outstanding debt is paid to finance capitalists. (v) The residual or net income is total revenue minus costs, $\pi_i = p_i d_i - m_i q_i(1 + r) - m_i \Delta q_i r$. Positive net income, $\pi_i > 0$, is distributed as profit of enterprise to industrial capitalists; but if net income is negative, $\pi_i < 0$, then industrial capitalists transfer money to firms to cover losses. (vi) Total profit income for the capitalist class as a whole (interest and industrial profit) from sector i is total firm revenue minus total firm costs excluding the cost of borrowing, $\psi_i = p_i d_i - m_i(q_i + \Delta q_i)$. (vii) The change in capitalist savings is the sum of profit income from all sectors minus (viii) capitalist expenditure on articles of consumption, $\alpha_c m_c$. Instantaneous net flows at activity level q_i, as discussed in the main text, are recovered as $\Delta q_i \to 0$.

one operating on borrowed capital, divides the gross profit into interest due to himself as owner, as his own lender, and into profit of enterprise due to him as to an active capitalist performing his function'.

Although interest and profit of enterprise are both types of profit income they nonetheless derive from different kinds of property claims. A finance capitalist, as an owner of stocks of money-capital (i.e., outstanding loans), maintains a property claim on the principal plus interest payments. This property claim terminates when the loan is repaid. In contrast, the industrial capitalist, as owner of the firm, is the residual claimant of the firm's net income, and therefore is liable for both profit and loss after all costs are deducted from revenue, including the cost of borrowing money. This property claim terminates when ownership is transferred or the firm dissolves. In general, loans are contractually secured such that debt holders are paid before net income is distributed to firm owners.

In Marx's theory the interest rate is determined in capital markets whereas profit of enterprise is determined by the conditions of production and the state of the economy as a whole. The interest rate is 'assumed to be given beforehand, before the process of production begins, hence before its result, the gross profit, is achieved' [4, pg. 373]. Interest payments are an *ex ante* cost of production whereas profit (or loss) is an *ex post* residual. Marx [4, pg. 367] writes that 'the general rate of profit, therefore, derives actually from causes far different and far more complicated than the market rate of interest'.

In reality firms finance their production from a wide variety of funding sources, such as short-term overdrafts, loans of different duration with fixed and variable rates of interest, and longer-term sources of funds, such as bonds and equity. For analytical simplicity I ignore this complexity. Instead, I represent the aggregate financing of a large number of firms within a sector by a 'line of credit'. Industrial capitalists continually revise their borrowing requirements as economic conditions change, thereby altering the aggregate level of borrowing. All costs of production (i.e., monies required to pay for inputs prior to the receipt of revenue) are originally financed by borrowing. Finance capitalists receive interest payments on the money-capital they currently have 'tied-up' in production on a continuous ('daily') basis, at a varying, instantaneous rate of interest $r(t)$, which represents the current cost of borrowing in the economy.[2]

Money, at certain points in its circulation, enters the hands of finance capitalists and changes its function from means of payment to money-capital. But the amount of money-capital lent to firms is independent of the stock of money in circulation; for example, Marx [4, pg. 510] writes that '*Prima facie* loan capital always exists in the form of money, later as a claim to money, since the money in which it originally exists is now in the hands of the borrower in actual money-form. For the lender it has been transformed into a claim to money, into a title of ownership. The same mass of actual money can, therefore, represent very different masses of money-capital'. In this model, the stock of base money is an exogenous constant but the volume of outstanding loans to industrial capitalists is an endogenous variable. The phrase 'supply of money-capital' does not refer to an occurrent supply of money but to the provision of loan services, i.e. the maintenance of a creditor-debtor relationship between finance and industrial capitalists. At this level of abstraction banks and fractional reserve banking do not exist; hence the granting of a loan is an actual transfer of base money that creates new debt but does not create new commercial bank money.

Figure 1 specifies the flow of funds between capitalist households and the system of production. I now describe these relationships in further detail.

[2] In consequence, the circuit of money-capital in this model is perfectly smooth and lacks the 'lumpy' emergence of distributed stocks of forms of capital. See Foley [11] for a more concrete approach that includes distributed time lags in the circuit of capital.

I.8. Profit of enterprise

The revenue generated by sector i is the total product sold multiplied by the current price. Demand has two components: demand from other sectors and demand from households. The demand from other sectors, $\mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}}$, is a function of the technique and current activity levels. The demand from capitalist households is the ith component of capitalist consumption, $(\alpha_c m_c/\mathbf{p}\mathbf{c}^{\mathrm{T}})c_i$; and the demand from worker households is the ith component of the real wage, $(\alpha_w m_w/\mathbf{p}\mathbf{w}^{\mathrm{T}})w_i$. The total demand for commodity i is then

$$d_i = \mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}} + \frac{\alpha_c m_c}{\mathbf{p}\mathbf{c}^{\mathrm{T}}}c_i + \frac{\alpha_w m_w}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}w_i$$

and hence total revenue for sector i is $p_i d_i$.

The total costs incurred by sector i during the production of q_i is the quantity of inputs bought in the market multiplied by their respective prices. Cost also has two components: input costs and the interest charged on loans.

The cost of input commodities, $\mathbf{p}\mathbf{A}^{(i)}q_i$, is a function of the technique, commodity prices and the activity level. The wage cost, $l_i q_i w$, is a function of the direct labor coefficient, the wage rate and the activity level. The unit cost of production, excluding the cost of borrowing, is therefore

$$m_i = \mathbf{p}\mathbf{A}^{(i)} + l_i w.$$

Vickers [12] analyzes the capital structure of firms, in particular the partial financing of production by debt capital. He defines 'money capital requirement coefficients' as the amount of money-capital required to finance a unit of 'factor capacity'. In an economy with pure circulating capital and production entirely financed by borrowing the unit costs m_i are also 'money capital requirement coefficients', measured in units of nominal debt per unit output, Coefficients m_i therefore denote the amount of money-capital currently required to finance unit output of commodity i.

The current cost of borrowing 1 unit of money is r units of money, where the interest rate r is the 'price' of money-capital. The total money-capital required to produce at the current scale of production, or outstanding debt, is $m_i q_i$. The interest charged on $m_i q_i$ money-capital is simply this quantity multiplied by its price, $m_i q_i r$. So the total cost of production in sector i, including the cost of borrowing, is $m_i q_i(1 + r)$.

We can now construct a profit function. The current profit (or loss) in sector i is the difference between total revenue and total cost; that is

$$\pi_i = p_i d_i - m_i q_i(1 + r). \tag{3}$$

I.9. Capitalist savings

Capitalists' savings, m_c, which consist of the aggregate money holdings of finance and industrial capitalists, are augmented by an inflow of profit – consisting of total interest income, $\mathbf{m}\mathbf{q}^{\mathrm{T}}r$, and total entrepreneurial profits (or losses), $\sum_{i=1}^{n}\pi_i$ – and are reduced by an outflow of consumption spending, which is the fraction of savings, $\alpha_c m_c$ spent on the consumption bundle $\mathbf{c}'$. The change in savings is the sum of income minus expenditure; that is,

$$\frac{\mathrm{d}m_c}{\mathrm{d}t} = (\mathbf{pA} + \mathbf{l}w)\mathbf{q}^{\mathrm{T}} r - \alpha_c m_c + \sum_{i=1}^{n} \pi_i, \tag{4}$$

Profit of enterprise, $\sum_{i=1}^{n} \pi_i$, may vary in sign and therefore represent either a profit inflow (from firms to industrial capitalists) or a loss-covering outflow (from industrial capitalists to firms).

I.10. The interest rate

Marx, in his unfinished notes published as Volume 3 of *Capital*, sketches an incomplete loanable funds theory of the rate of interest. ‘As concerns the perpetually fluctuating market rate of interest, however, it exists at any moment as a fixed magnitude, just as the market-price of commodities, because in the money-market all loanable capital continually faces functioning capital as an aggregate mass, so that the relation between the supply of loanable capital on one side, and the demand for it on the other, decides the market level of interest at any given time’ [4, pg. 366].

Money-capital is both like and unlike other commodities: ‘interest-bearing capital, although a category which differs absolutely from a commodity, becomes a commodity *sui generis*, so that interest becomes its price, fixed at all times by supply and demand like the market-price of an ordinary commodity.’ [4, pg. 366]. Money-capital is not produced and hence its price is regulated solely by supply and demand and not by a cost of production. In consequence, ‘there is rather no law of division [between finance capitalists and industrial capitalists] except that enforced by competition, because ... no such thing as a “natural” rate of interest exists’ [4, pg. 365]. In Marx’s view a natural price is a property of production and therefore independent of competition in the market. So although the interest rate is the market price of money-capital it simply lacks a corresponding natural price to gravitate toward.

In this model the total stock of loanable funds available to finance production is the stock of base money held by capitalists. Individual capitalists may both lend and borrow and therefore can function as their ‘own lenders’. I abstract from the labor and wages of money management and investment (i.e., financial costs of production). Individual capitalists manage subsets of the total stock of loanable funds, which are subject to fluctuations: for example industrial capitalists experience both profit and loss. Such micro-level fluctuations alter the distribution of loanable funds within the capitalist class. But in the aggregate the net change in the stock of loanable funds depends only on the class distribution of savings.

Assume that an individual finance capitalist’s willingness to lend depends on the current stock of loanable funds at their disposal. Finance capitalists therefore tend to raise the cost of borrowing when their stocks of funds decrease because industrial capitalists tend to outbid each other when competing to buy the reduced supply of loans; conversely, finance capitalists tend to lower the cost of borrowing when their stocks of funds increase because they tend to underbid each other when competing to sell the increased supply of loans to industrial capitalists. Given these assumptions the relative change in the interest rate is negatively proportional to the relative change in the total amount of loanable funds; that is,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\eta_c \frac{1}{m_c} \frac{\mathrm{d}m_c}{\mathrm{d}t} r, \tag{5}$$

where $\eta_c > 0$ is a constant elasticity of the interest rate with respect to the stock of loanable funds. Equation (5) has a cross-dual form: a change in the quantity of loanable funds causes a corresponding change in the price of money-capital. In consequence the interest rate varies with the scarcity (or abundance) of the total stock of loanable funds. But this behavioral relationship should not be taken at face value. Money-capital is not produced. The stock of funds turns over multiple times to support very different masses of money-capital. Money-capital need not be ‘saved up’ before it can be ‘used up’. Any level of demand for loans, at the given interest rate, can in principle be supplied regardless of the stock of funds. So money-capital is scarce in virtue of its use not its nature.

I.11. Inventory stocks

The supply of commodity i will not in general equal the real demand for it, that is $q_i \neq d_i$. Each sector stores a stock of unsold inventories, denoted s_i. When supply is greater than (resp. less than) demand then inventories increase (resp. decrease). The rate of change of inventories is therefore equal to the excess supply, $\frac{\mathrm{d}s_i}{\mathrm{d}t} = q_i - d_i$; or, in full,

$$\frac{\mathrm{d}s_i}{\mathrm{d}t} = q_i - (\mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}} + \frac{\alpha_c m_c}{\mathbf{pc}^{\mathrm{T}}} c_i + \frac{\alpha_w m_w}{\mathbf{pw}^{\mathrm{T}}} w_i). \tag{6}$$

Assume that commodities are imperishable so unsold inventories can be stored indefinitely.[3]

I.12. Cross-dual price adjustment

A sector’s overall price and quantity adjustment is the aggregate of the adjustments of the individual firms that comprise it. An excess or lack of demand for a commodity translates into a change in the size of inventories. For example, underproduction relative to demand means that inventories shrink, whereas overproduction means that inventories grow. Firms tend to raise prices when inventories shrink on the assumption that buyers will outbid each other to obtain the scarce product, whereas firms tend to lower prices when inventories grow on the assumption that other firms will underbid each other in order to sell to scarce buyers. The sector as a whole, therefore, adjusts the relative price of its commodity in proportion to excess demand,

[3] A more general model would allow inventories to be destroyed according to a per sector decay rate. Then the inventory held by service sectors could be interpreted as short-term excess capacity, for example due to the ability of service providers to store intermediate products and work with greater intensity.

that is $\frac{1}{p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \propto -\frac{\mathrm{d}s_i}{\mathrm{d}t}$. This has a cross-dual form: a quantity imbalance, represented by the change in inventory size, translates into a price adjustment.

Assume that the change in price approaches positive ∞ as inventory approaches zero and the commodity is completely scarce, that is $\frac{1}{p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \propto \frac{1}{s_i}$. Combining these two factors we get the price adjustment equation

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\eta_i \frac{\mathrm{d}s_i}{\mathrm{d}t}\frac{p_i}{s_i}, \tag{7}$$

where $\eta_i > 0$ is a constant elasticity of price with respect to excess supply. Sectors with small (resp. large) inventories will tend to adjust prices relatively quickly (resp. slowly). I assume that firms do not reduce prices to dump inventory and realize value but instead maintain an inventory buffer to manage any variance in excess demand.

I.13. Cross-dual quantity adjustment

Industrial capitalists, as owners and managers of firms, adjust their production plans based on profits and losses. A firm that returns a profit (resp. loss) borrows more (resp. less) money in order to increase (resp. decrease) supply with the expectation of earning greater profit (resp. reducing losses).

Industrial capitalists, as a whole, own a portfolio of firms grouped into sectors that, at any time, make different profits or losses. The costs of production in sector i, including payment of interest on money-capital, is $m_i q_i(1+r)$. The ratio of total profit to costs of production,

$$\frac{\pi_i}{m_i q_i (1+r)},$$

is the profit rate, which we can interpret as the expected increase of profit of enterprise from 1 unit of additional investment of money in sector i *ceteris paribus*. Capitalists aim to maximize their profit by differentially injecting or withdrawing money investments based on these profit-rate signals. The relative change in the scale of production is therefore proportional to the profit rate, that is $\frac{1}{q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} \propto \frac{\pi_i}{m_i q_i (1+r)}$. This has a cross-dual form: a price imbalance, represented by the profit rate, translates into a quantity adjustment. In consequence, we get the quantity adjustment equation

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \eta_{n+i}\frac{\pi_i}{m_i(1+r)}, \tag{8}$$

where $\eta_{n+i} > 0$ is a constant elasticity of supply with respect to profit. Sectors with a high (resp. low) profit rate increase (resp. decrease) their borrowing in order to increase (resp. decrease) the supply of goods to the market.

We can also interpret quantity adjustment equation (8) in terms of the rate of return,

$$r_i = \frac{p_i d_i - m_i q_i}{m_i q_i},$$

which is the expected return from 1 unit of additional investment of money in sector i prior to its distribution as interest income and profit of enterprise. An equivalent expression for quantity adjustment, in terms of the rate of return, is then

$$\begin{aligned}\frac{\mathrm{d}q_i}{\mathrm{d}t} &= \eta_{n+i}\frac{q_i}{1+r}(r_i - r) \\ &\propto (r_i - r).\end{aligned} \tag{9}$$

Industrial capitalists compare the expected return from investing additional money in production, r_i, with the interest rate, r. If $r_i > r$ then additional money is productively invested and supply expands; if $r_i < r$ then money is withdrawn from production and supply decreases. Industrial capitalists expand production if the rate of return is greater than the cost of borrowing. The demand for credit from industrial capitalists therefore varies with the rate of return.

This completes the phenomenological description of the model. Next we will examine some of its properties.

II. MARKET PRICES AND SCARCITY

Solve price adjustment equation (7) to obtain market prices as a function of inventory levels; that is,

$$p_i(t) = p_{i_0} s_{i_0}^{\eta_i}\frac{1}{s_i(t)^{\eta_i}} \tag{10}$$

for all i (the subscript 0 indicates the value of the variable at time zero). A high market price indicates low inventory. An obvious and natural interpretation of this relationship is that market prices measure the relative scarcity (or abundance) of a commodity.

Solve wage adjustment equation (2) to give the wage rate as a function of the level of employment,

$$w(t) = k_w \frac{1}{(L - \mathbf{l}\mathbf{q}^{\mathrm{T}})^{\eta_w}}, \tag{11}$$

where

$$k_w = w_0 (L - \mathbf{l}\mathbf{q}_0^{\mathrm{T}})^{\eta_w}$$

is a positive constant. The wage rate indicates the scarcity of unemployed labor available for hire.

Solve interest rate adjustment equation (5) to give the interest rate as a function of the quantity of loanable funds,

$$r(t) = r_0 m_{c0}^{\eta_c}\frac{1}{m_c^{\eta_c}}. \tag{12}$$

The interest rate is also a market price that indicates scarcity, in this case the scarcity of loanable funds. As the stock of loanable funds decreases the interest rate rises.

III. CONSERVATION OF THE MONEY STOCK

Aggregate expenditure, $\alpha_w m_w + \alpha_c m_c$, varies depending on the distribution of savings between workers and capitalists. Since firms do not hold money stocks the aggregate expenditure must return to households as income – in the form of wages, interest or profit. The sum of savings, $m_w + m_c$, is therefore always equal to the fixed stock of base money.

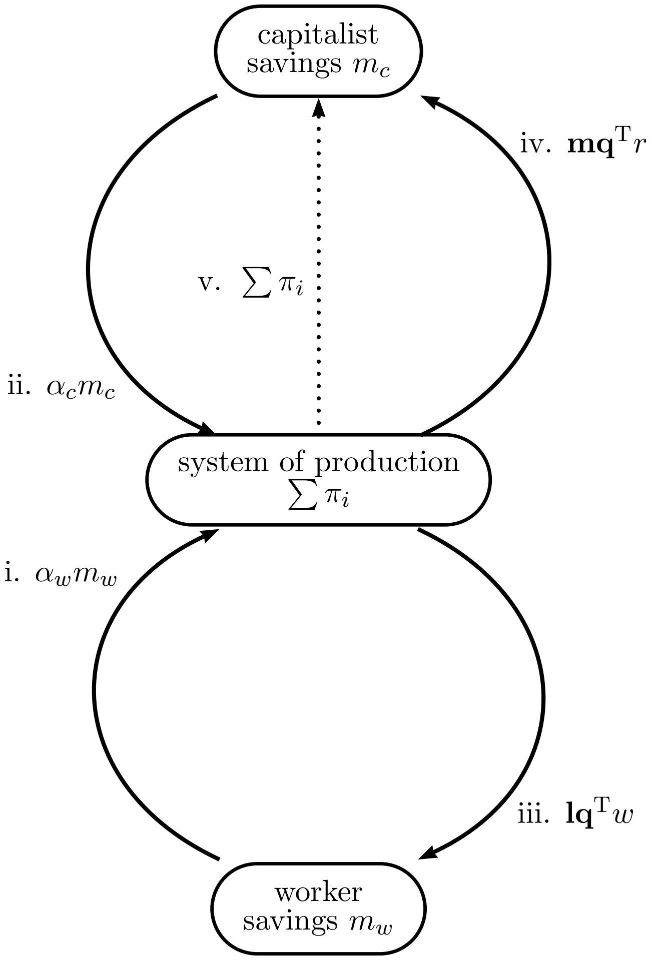


FIG. 2. **Aggregate profit of enterprise as an aggregate residual income**. The total aggregate expenditure in the economy consists of spending from (i) workers and (ii) capitalists. The aggregate expenditure is exchanged for goods and services supplied by firms. Firms payout (iii) total wages and (iv) total interest payments, which are *ex ante* liabilities. The mismatch between aggregate expenditure and firm liabilities is (v) the total profit of enterprise, which is an *ex post* residual distributed to firm owners. Total profit is therefore positive (resp. negative) only if aggregate expenditure is greater than (resp. less than) total wage and interest income; that is, $\sum \pi_i = (\alpha_w m_w + \alpha_c m_c) - (\mathbf{lq}^{\mathrm{T}}w + \mathbf{mq}^{\mathrm{T}}r)$.

Lemma 1. Total savings, M, are constant,

$$\begin{aligned} M &= m_w(t) + m_c(t) \\ &= m_{w_0} + m_{c_0}. \end{aligned}$$

Proof. Sum equations (3) to get

$$\sum_{i=1}^{n} \pi_i = \alpha_w m_w + \alpha_c m_c - \mathbf{lq}^{\mathrm{T}}w - (\mathbf{pA} + \mathbf{l}w)\mathbf{q}^{\mathrm{T}}r. \quad (13)$$

Sum equations (1) and (4) to get

$$\begin{aligned} \frac{\mathrm{d}m_w}{\mathrm{d}t} + \frac{\mathrm{d}m_c}{\mathrm{d}t} =& -\alpha_w m_w - \alpha_c m_c + \quad (14) \\ & \mathbf{lq}^{\mathrm{T}}w + (\mathbf{pA} + \mathbf{l}w)\mathbf{q}^{\mathrm{T}}r + \sum_{i=1}^{n} \pi_i. \end{aligned}$$

Substitute (13) into (14) to get $\frac{\mathrm{d}m_w}{\mathrm{d}t} + \frac{\mathrm{d}m_c}{\mathrm{d}t} = 0$. Hence $m_w(t) + m_c(t) = k$, where k is a constant of integration. At $t = 0$ we have $k = m_{w_0} + m_{c_0}$. □

A key conclusion is the existence of a direct trade-off between workers and capitalists over ownership of the stock of money wealth in the economy. What one class gains the other must lose. So although total household spending always returns as income it nonetheless transfers from one class to another during its circulation.

Equation (13) is a macroeconomic money conservation equation, which relates aggregate expenditure and total income. Figure 2 depicts the relationship. Total household spending and total wage and interest income are in general unequal. The difference is profit of enterprise.

Simplify further and assume zero interest income, i.e. $r_0 = 0$. Then (13) is $\sum_{i=1}^{n} \pi_i = \alpha_w m_w + \alpha_c m_c - \mathbf{lq}^{\mathrm{T}}w$. So total profit of enterprise is positive if total spending exceeds the total wage bill. If workers spend what they earn then total profit is realized entirely by capitalist consumption ('In point of fact, paradoxical as it may seem at the first glance, the capitalist class itself casts into circulation the money that serves towards the realisation of the surplus-value contained in its commodities' [13, Ch. 17] and see also Trigg [14]). In such circumstances sector-level losses represent transfers within the capitalist class.[4]

IV. A NUMERICAL EXAMPLE

What dynamics does this model generate?

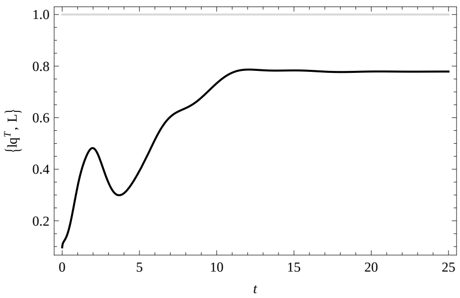
Consider the following example of a small, 3-sector economy that produces corn, iron and sugar, with parameters

$$\mathbf{A} = \begin{bmatrix} 0.2 & 0 & 0.4 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

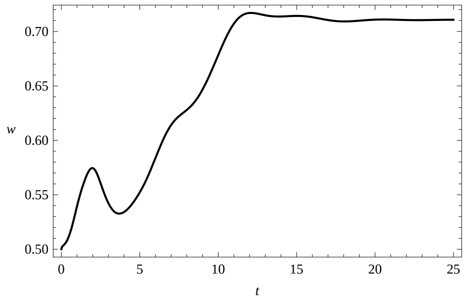
$\mathbf{l} = [0.7, 0.6, 0.3]$, $\mathbf{w} = [0.6, 0, 0.2]$ (workers consume corn and sugar but not iron), $\mathbf{c} = [0.2, 0, 0.4]$ (capitalists proportionally consume more sugar than corn compared to workers), $\mathbf{p}_0 = [1, 0.8, 0.5]$, $\mathbf{q}_0 = [0.01, 0.1, 0.1]$ (the initial supply of corn is relatively low), $\mathbf{s}_0 = [0.01, 0.1, 0.25]$ (the initial stock of corn is relatively low), $w_0 = 0.5$, $r_0 = 0.03$, $m_{w0} = m_{c0} = 0.5$ (worker and capitalist savings are initially equal and the total money stock in the economy is $M = 1$), $\alpha_w = 0.8$ and $\alpha_c = 0.7$ (workers have a higher propensity to consume), $L = 1$, the price elasticities are $\eta_1 = \eta_2 = \eta_3 = 2$, the quantity elasticities are $\eta_4 = \eta_5 = \eta_6 = 1$, the wage is relatively inelastic, $\eta_w = 0.25$, and the interest rate relatively elastic, $\eta_c = 2$. These parameters generate an economy that follows a growth trajectory until it reaches a real and monetary, self-replacing equilibrium.

The scale of real demand from households depends on aggregate expenditure and the current price structure. In this

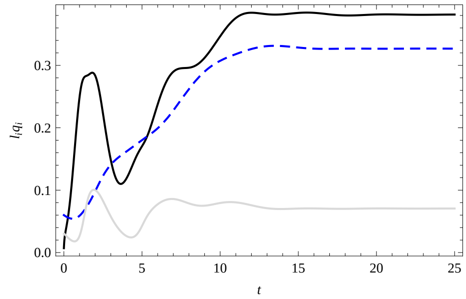
[4] This model supports Keen's point [15] that a fixed stock of base money turns over multiple times to support variable income flows in excess of that stock. The so-called 'paradox of monetary profit' in the Circuitist approach [16] disappears once sufficient attention is paid to the dynamic relationships between stocks and flows.



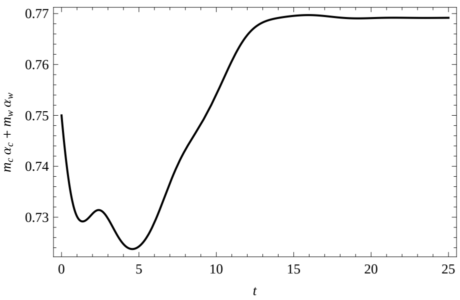
(a)Employment, $\mathbf{l}\mathbf{q}^{\mathrm{T}}$ and population, $L = 1$.



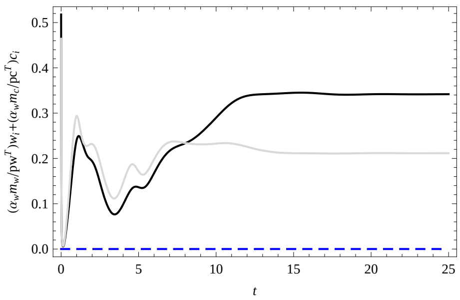
(b)Wage rate, w.



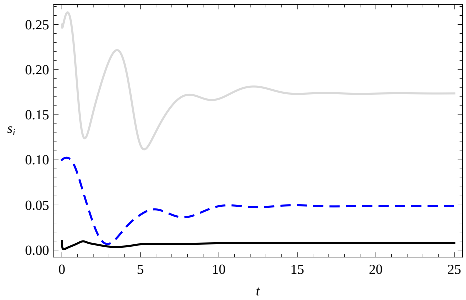
(c)Sectoral employment, $l_i q_i$.



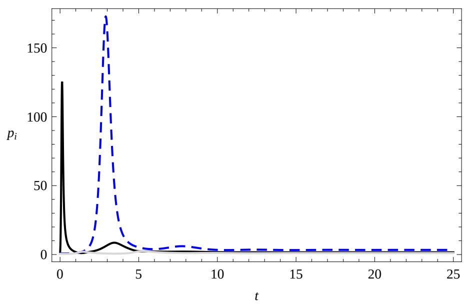
(d)Aggregate expenditure, $\alpha_w m_w + \alpha_c m_c$.



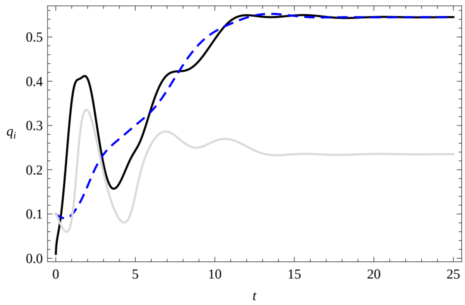
(e)Real demand, $\frac{\alpha m_w}{\mathbf{p}\mathbf{w}^{\mathrm{T}}} w_i + \frac{\alpha m_c}{\mathbf{p}\mathbf{c}^{\mathrm{T}}} c_i$.



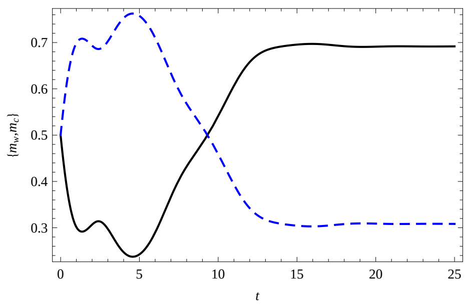
(f)Inventories, $\mathbf{s}$.



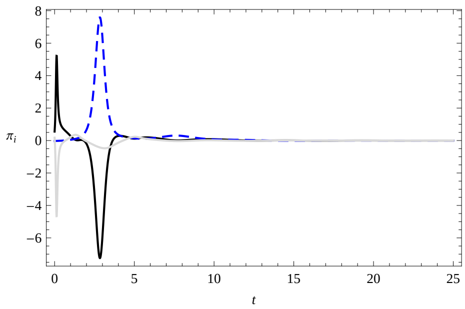
(g)Prices, $\mathbf{p}$.



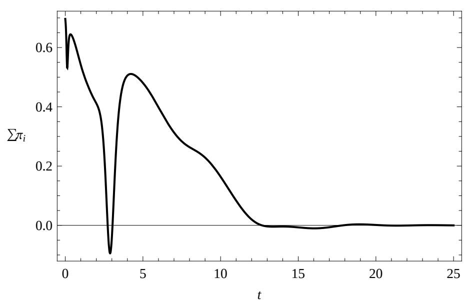
(h)Quantities $\mathbf{q}$.



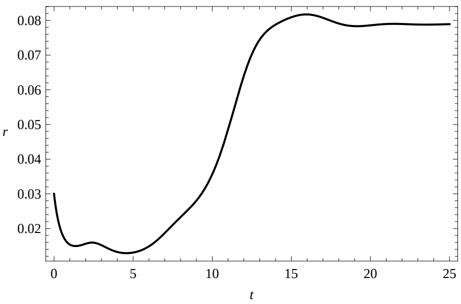
(i)Worker savings, m_w (black), and capitalist savings, m_c (dashed).



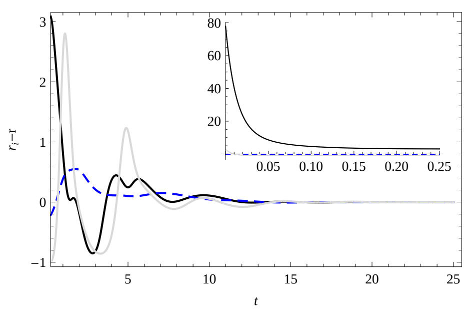
(j)Profit of enterprise, π_i.



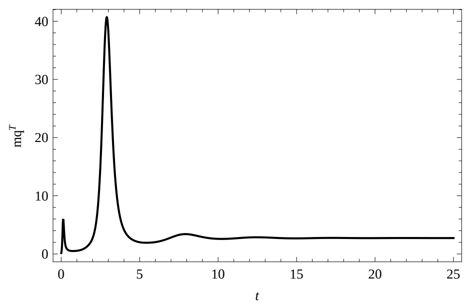
(k)Total profit of enterprise, $\sum_{i=1}^{3} \pi_i$.



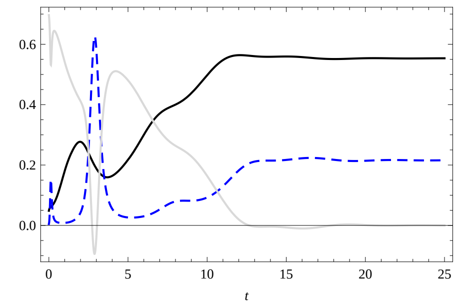
(l)Interest rate, r.



(m)Rate of return minus the interest rate, $r_i - r$. Main graph: $0.32 \leq t \leq 25$; inset: $0 \leq t \leq 0.32$ (the return in the corn sector is very high).



(n)Main graph: money-capital 'tied up' in production, or total outstanding debt, $\mathbf{m}\mathbf{q}^{\mathrm{T}}$.



(o)The distribution of income. Wages: black, interest: dashed, profit: gray.

FIG. 3. A numerical example of convergence to equilibrium.
Note: this economy follows a profit-led growth trajectory until it reaches a self-replacing equilibrium. Sectors: corn (black), iron (dashed), sugar (gray).

example the employment level rises (see Figure 3(a)) because in general real demand outstrips the capacity of the economy to supply commodities in the required amounts (e.g., Figure 3(f) graphs the inventory stock, which initially depletes to satisfy the excess demand). More workers are required to meet the demand, which causes a corresponding increase in the wage rate, shown in Figure 3(b). The division of labor adapts (see Figure 3(c)) until in equilibrium the scale and composition of the net product equals real demand, at which point inventory stocks stabilize.

Total household spending initially falls then at $t \approx 5$ steadily climbs to its maximum (see Figure 3(d)). Workers have a higher propensity to consume compared to capitalists. So the two demand regimes correspond to shifts in the distribution of money wealth between the two classes (see Figure 3(i)). Total spending always returns as income, either in the form of wages, interest or profit. So the stock of money is conserved and the trajectory of capitalist savings exactly mirrors worker savings.

Real demand hits two lows prior to $t \approx 5$ (see Figure 3(e)) caused by relatively low total household spending and two dramatic price spikes (see Figure 3(g)) that function to ration temporarily scarce commodities (corn and iron). But supply adjusts, consumer inflation dissipates, and after $t \approx 5$ real demand steadily rises.

Expansion of output is profit-led. Figure 3(k) plots the total profit of enterprise. In general, total profit is either positive during gravitation or close to zero near equilibrium. The exception is a short period at $t \approx 3$ where losses in the corn and sugar sectors outweigh profit in the iron sector (see Figure 3(j)).

Firms sell inventory to satisfy excess demand. Low inventory causes price spikes. Price spikes tend to raise sectoral profits (compare the price spikes, graphed in Figure 3(g), with profit of enterprise, graphed in Figure 3(j)). Industrial capitalists can therefore gain a higher return than the cost of borrowing and invest in production (see Figure 3(m), especially the initial high return in the corn sector). The new funds are used to increase the scale of production (see Figure 3(h), especially the initial high growth in the corn sector). Figure 3(j) plots total profits per sector, which initially exhibit wide fluctuations, indicating differential returns on money invested, until settling to a uniform zero profit rate at equilibrium, at which point activity levels are stable (Figure 3(h)).

Figure 3(l) graphs the interest rate, which fluctuates with the total stock of loanable funds (see inset of figure 3(n)). Figure 3(n) graphs total loans advanced, which is sensitive to the price structure and the scale of production. Total interest income is a function of the volume of lending and the interest rate. So, for instance, the high price of iron at $t \approx 3$ increases costs of production and therefore the volume of borrowing, which results in more interest income for rentiers (Figure 3(o)). In the same period profit of enterprise falls (Figure 3(o)). Why is this? Given a level of total household spending a dramatic spike in interest income means less income in the form of profit and wages. Industrial capitalists are subject to a cost-push 'profit squeeze', which at root derives from the relative scarcity of real-capital, specifically iron. The high costs of production throttle growth (see Figure 3(h) at $t \approx 3$). But this contraction is temporary. Labor and real-capital is reallocated to iron production, which increases supply, lowering its price and therefore costs of production in general. The volume of lending falls and profits of enterprise recover.

This single example is indicative but does not exhaust the range of dynamics the model generates.

V. STABILITY OF 'LONG-PERIOD' EQUILIBRIUM

Numerical simulations indicate that the equilibrium is locally asymptotically stable. So economies in the domain of attraction of the long-period position gravitate toward it. But a proof of local stability for this model is an outstanding problem. So this statement remains a conjecture for now. A special-case of this model, with a simpler quantity adjustment function and zero interest income, is provably locally stable. But the specific proof is not entirely satisfying since it utilizes vector Lyapunov functions, which do not yield a straightforward economic interpretation [17].

Some authors (e.g., [18]) observe that existing pure cross-dual models are unstable and lack convergence properties. But such conclusions are premature since the full range of pure cross-dual models has yet to be explored. For instance, I have simulated many qualitatively different but close variants of this model. The numerical simulations indicate that all these variants are locally stable. I expect, therefore, that a general proof of local stability, applicable to a wide class of cross-dual models, can be found.

The coordination of millions of independent production activities in a large-scale market economy is neither perfect or equitable but nonetheless 'one should be far more surprised by the existing degree of coordination than by the elements of disorder' [19]. The model developed in this paper, notwithstanding the theoretical simplifications, is a powerful example that the classical cross-dual theory of competition can be formalized to provide a coherent and successful explanation of the homeostatic kernel of generalized commodity production.

Next we examine some of the properties of the economic equilibrium.

VI. THE NATURAL PRICE EQUILIBRIUM

I restrict the analysis to $\mathcal{D} = \{[\mathbf{p}(t), \mathbf{q}(t), m_c(t)] \in \overline{\mathbb{R}}_+^{2n+1} : \mathbf{l}\mathbf{q}^{\mathrm{T}}(t) < L\}$, i.e. economically relevant equilibria.

VI.1. Zero profit of enterprise

Activity levels adjust according to profit rate differentials. By definition activity levels are constant in equilibrium. Hence in equilibrium profit of enterprise is uniformly zero and there is no incentive to reallocate capital.

Lemma 2. Profits are zero in equilibrium, $\pi_i = 0$ for all i.

Proof. Substitute $\frac{\mathrm{d}q_i}{\mathrm{d}t} = 0$ into quantity adjustment equation (8) to get $\pi_i = 0$ for all i. □

Profit of enterprise is a disequilibrium phenomenon deriving from imbalances between supply and demand. Profit represents an arbitrage opportunity that attracts capital. But the scramble for profit has the unintended consequence of reducing imbalances between supply and demand, which eliminates arbitrage opportunities and causes the rate of profit to fall. In equilibrium supply and demand are exactly equal and profit of enterprise is zero.

The zero profit condition is equivalent to the equality of the rate of return and the interest rate.

Lemma 3. The equilibrium rate of return in all sectors equals the equilibrium interest rate, $r_i^* = r^*$ for all i.

Proof. Substitute $\frac{\mathrm{d}q_i}{\mathrm{d}t} = 0$ into quantity adjustment equation (9) to get $r_i^* = r^*$ for all i. □

In equilibrium, the rate of return is uniform across sectors and equal to the interest rate. The equilibrium price structure does not provide an incentive for industrial capitalists to alter their production plans. In reality, of course, other sources of anticipated reward or loss, not included in this model, motivate capitalists to change the scale of production.

VI.2. Total income equals aggregate expenditure

Money conservation identity (13) implies that total household spending always returns either in the form of wage income, interest or profit. Since profit of enterprise is zero in equilibrium the total income must consist of wages and interest.

Lemma 4. Equilibrium total income is total wages and interest income, which equals the equilibrium aggregate expenditure:

$$\mathbf{l}\mathbf{q}^{*\mathrm{T}} w^* + \mathbf{m}^*\mathbf{q}^{*\mathrm{T}} r^* = \alpha_w m_w^* + \alpha_c m_c^*.$$

Proof. By Lemma 2, $\pi_i = 0$ for all i. Substitute the zero profit condition into equation (13). □

Out-of-equilibrium the aggregate demand may transfer from one class to another during its circulation. Equilibrium is simpler: no transfer occurs and both workers and capitalists earn what they spend.

Lemma 5. In equilibrium workers earn what they spend,

$$\mathbf{l}\mathbf{q}^{*\mathrm{T}} w^* = \alpha_w m_w^*$$

Proof. Set $\frac{\mathrm{d}m_w}{\mathrm{d}t} = 0$ in equation (1). □

Lemma 6. In equilibrium capitalists earn what they spend,

$$\mathbf{m}^*\mathbf{q}^{*\mathrm{T}} r^* = \alpha_c m_c^*$$

Proof. Set $\frac{\mathrm{d}m_c}{\mathrm{d}t} = 0$ in equation (4) and use the zero profit condition of Lemma 2 to yield the conclusion. □

These properties are an instance of Kalecki's aphorism [20, Ch. 3] that capitalists earn what they spend while workers spend what they earn [21, Ch. 3]. Kalecki's aphorism holds in equilibrium but not out-of-equilibrium when workers' saving may be non-zero.

VI.3. A positive rate of interest

Interest income, unlike profit of enterprise, is an *ex ante* cost of production not an *ex post* residual. There are no circumstances in which it can be eliminated given the institutional arrangements of capitalist production. In consequence, the interest rate is always positive in equilibrium.

Lemma 7. Positive equilibrium capitalist savings, $m_c^* > 0$, imply a positive equilibrium interest rate, $r^* > 0$.

Proof. Equation (12) and $m_c^* > 0$ implies $r^* > 0$. □

Profit rates are not the price of any particular thing. But in equilibrium the interest rate, or price of money-capital, is the price of a particular bundle of commodities.

Lemma 8. The equilibrium interest rate is the cost of capitalist consumption per unit of money-capital supplied to production,

$$r^* = \mathbf{p}^*\bar{\mathbf{c}}^{*\mathrm{T}}, \tag{15}$$

where

$$\bar{\mathbf{c}}^{*\mathrm{T}} = \frac{1}{\mathbf{m}^*\mathbf{q}^{*\mathrm{T}}} \frac{\alpha_c m_c^*}{\mathbf{p}^*\mathbf{c}^{\mathrm{T}}} \mathbf{c}^{\mathrm{T}}$$

is the vector of capitalist consumption divided by the total volume of borrowing.

Proof. Equilibrium capitalist consumption is $(\alpha_c m_c / \mathbf{p}^*\mathbf{c}^{\mathrm{T}})\mathbf{c}^{\mathrm{T}}$. Divide by the total borrowing, $\mathbf{m}^*\mathbf{q}^{*\mathrm{T}}$, to get the the rate of capitalist consumption per unit of money-capital, $\bar{\mathbf{c}}^{*\mathrm{T}}$. Then

$$\mathbf{p}^*\bar{\mathbf{c}}^{*\mathrm{T}} = \frac{\alpha_c m_c^*}{\mathbf{m}^*\mathbf{q}^{*\mathrm{T}}}. \tag{16}$$

Lemma 6 equates total interest income to capitalist expenditure, $\alpha_c m_c = \mathbf{m}^*\mathbf{q}^{*\mathrm{T}} r$. Substitute into (16) and the conclusion follows. □

In section I.10 we noted that Marx rejected the existence of a 'natural' rate of interest on the grounds that 'money-capital', unlike a reproducible commodity such as corn, lacks a cost of production; hence its price is simply a market price set by competition.

In equilibrium the interest rate does not have a natural rate defined in terms of a technical cost of production. But in equilibrium this 'price of money-capital' corresponds to the cost of production of a composite commodity, specifically capitalist consumption per unit of money-capital 'tied-up' in production. Although the trajectory of the interest rate is controlled by non-technical, 'conventional' factors, such as the interest rate elasticity and the stock of loanable funds, nonetheless it is both law-governed and causally connected to a real cost of production.

VI.4. Positive inventories

Stored inventories do not decay. Firms do not attempt to reduce their absolute level of inventory but only modify their prices in response to relative changes in inventory. In consequence, inventories are positive in equilibrium.

Lemma 9. Positive equilibrium prices, $\mathbf{p}^* > \mathbf{0}$, imply positive equilibrium stocks of inventory, $\mathbf{s}^* > \mathbf{0}$.

Proof. Equation (10) and $p_i^* > 0$ implies $s_i^* > 0$. $\square$

VI.5. Involuntary unemployment

Lemma 10. A positive equilibrium wage, $w^* > 0$, implies positive unemployment, $0 < \mathbf{l}\mathbf{q}^{*\mathrm{T}} < L$.

Proof. Equation (11) implies,

$$L - \mathbf{l}\mathbf{q}^{*\mathrm{T}} = k_w^{1/\eta_w} \left(\frac{1}{w^*}\right)^{1/\eta_w}.$$

If $w^* > 0$ then the RHS of this equation is positive; hence $L - \mathbf{l}\mathbf{q}^{*\mathrm{T}} > 0$ and the conclusion follows. $\square$

Market adjustment, including the pooling and lending of loanable funds does not, by itself, guarantee full employment. In equilibrium labor is efficiently allocated between the different sectors of production but the economy in general does not operate at full capacity.

VI.6. Prices of production

Proposition 1. Equilibrium prices in terms of the equilibrium wage, w^*, and interest rate, r^*, are

$$\mathbf{p}^* = (\mathbf{p}^*\mathbf{A} + \mathbf{l}w^*)(1 + r^*), \tag{17}$$

where

$$\begin{aligned} w^* &= k_w \frac{1}{(L - \mathbf{l}\mathbf{q}^{*\mathrm{T}})^{\eta_w}} \\ r^* &= r_0 m_{c_0}^{\eta_c} \frac{1}{(M - m_w^*)^{\eta_c}}. \end{aligned} \tag{18}$$

Proof. The zero profit condition of Lemma 2 implies

$$\begin{aligned} p_i^*(\mathbf{A}_{(i)}\mathbf{q}^{*\mathrm{T}} + &\frac{\alpha_c m_c^*}{\mathbf{p}^*\mathbf{c}^\mathrm{T}} c_i + \frac{\alpha_w m_w^*}{\mathbf{p}^*\mathbf{w}^\mathrm{T}} w_i) = \\ &q_i^*(\mathbf{p}^*\mathbf{A}^{(i)} + l_i w^*)(1 + r^*). \end{aligned} \tag{19}$$

Set $\frac{\mathrm{d}s_i}{\mathrm{d}t} = 0$ in equation (6) to get

$$q_i^* = \mathbf{A}_{(i)}\mathbf{q}^{*\mathrm{T}} + \frac{\alpha_c m_c^*}{\mathbf{p}^*\mathbf{c}^\mathrm{T}} c_i + \frac{\alpha_w m_w^*}{\mathbf{p}^*\mathbf{w}^\mathrm{T}} w_i. \tag{20}$$

Substitute the RHS of (20) into (19) to get $p_i^* q_i^* = q_i^*(\mathbf{p}^*\mathbf{A}^{(i)} + l_i w^*)(1 + r^*)$. Simplify, write in vector form and use expressions (11) and (12) to yield the conclusion. $\square$

Equation (17) is structurally equivalent to the standard equation for profit-equalizing prices of production with wages paid *ex ante* (i.e., advanced by capitalists) studied extensively in static, linear production theory (e.g., see [22–24] etc.) This dynamic model therefore embeds the linear production price model as a special case at equilibrium.

Equation (17) has two unknowns, the equilibrium wage and profit rate. In linear production theory the two unknowns are fixed by specifying a normalization condition, an arbitrary *numéraire*, and a distributional variable, either the wage or profit rate. But the formal theory itself does not determine absolute prices and the distribution of income (e.g., Sraffa [25]). This dynamic model includes money and a theory of the distribution of income, specifically how wages and profits interact through time. In consequence when we fully characterize the equilibrium of the system below we also determine absolute prices and the distribution of income without reference to an arbitrarily chosen *numéraire*.

In equilibrium costs and revenues balance in all sectors and the same uniform rate of gross profit prevails. This profit however consists entirely of interest income. Profit of enterprise, in this deterministic model, is a disequilibrium phenomenon and therefore is almost always non-uniform. The uniform 'profit rate' in equilibrium interpretations of linear production systems should be interpreted as an *ex ante* uniform rate of interest that prevails in financial markets, and therefore a cost of production, and not an *ex post* profit or 'surplus' generated at the level of enterprises. This point-of-view is consistent with Farjoun and Machover's (1989) critique that the assumption of 'uniform profits' in linear production theory is empirically false.

VI.7. Equality of the net product and final demand

Proposition 2. Equilibrium quantities in terms of the equilibrium net product, $\mathbf{n}^* = \mathbf{w}^* + \mathbf{c}^*$, are

$$\mathbf{q}^* = \mathbf{n}^*(\mathbf{I} - \mathbf{A}^\mathrm{T})^{-1} \tag{21}$$

where

$$\mathbf{w}^* = \frac{\alpha_w m_w^*}{\mathbf{p}^*\mathbf{w}^\mathrm{T}}\mathbf{w}$$

is the equilibrium real wage and

$$\mathbf{c}^* = \frac{\alpha_c m_c^*}{\mathbf{p}^*\mathbf{c}^\mathrm{T}}\mathbf{c} = \frac{\alpha_c(M - m_w^*)}{\mathbf{p}^*\mathbf{c}^\mathrm{T}}\mathbf{c}$$

is the equilibrium real consumption of capitalists.

Proof. Set $\frac{\mathrm{d}s_i}{\mathrm{d}t} = 0$ in equation (6) to get

$$q_i^* = \mathbf{A}_{(i)}\mathbf{q}^{*\mathrm{T}} + \frac{\alpha_c m_c^*}{\mathbf{p}^*\mathbf{c}^\mathrm{T}} c_i + \frac{\alpha_w m_w^*}{\mathbf{p}^*\mathbf{w}^\mathrm{T}} w_i. \tag{22}$$

Use Lemma 1, write in vector form and rearrange to yield the conclusion. $\square$

Equation (21) can be written as $\mathbf{q}^* = \mathbf{q}^*\mathbf{A}^\mathrm{T} + \mathbf{n}^*$. Interpret this equation as stating that the equilibrium scale of production consists of the collection of commodities used-up as means of production, $\mathbf{q}^*\mathbf{A}^\mathrm{T}$ (the circulating real-capital), and the net product, $\mathbf{n}^*$, which is final consumption. The equilibrium activity levels in the economy are therefore determined by the current technique and the composition and scale of final consumption. Final consumption, however, is itself determined by aggregate monetary demand and the prevailing price structure. The real and the monetary aspects of the economy are intertwined.

Equation (21) is also identical to a standard linear production equation of circulating capital (e.g., see the discussion of the open Leontief system in Pasinetti [22, Ch. 4]). So this dynamic system embeds the complete linear production model at its equilibrium point.

VI.8. The long-period position: a monetary equilibrium

The long-period position – that is equilibrium prices, quantities and the distribution of savings – is defined by a system of nonlinear simultaneous equations. Aside from special cases the system requires numerical methods to solve. Nonetheless the qualitative structure of the equations has some salient features. I briefly sketch some of them and their implications.

Corollary 11. The $2n+1$ unknowns – equilibrium absolute prices, $\mathbf{p}^*$, quantities, $\mathbf{q}^*$, and the distribution of savings, represented by savings m_w^* – are jointly determined by the following $2n+1$ system of nonlinear simultaneous equations,

$$\mathbf{p}^* = \left(\mathbf{p}^*\mathbf{A} + (\frac{\alpha_w m_w^*}{\mathbf{l}\mathbf{q}^{*\mathrm{T}}})\mathbf{l}\right)(1+r^*) \tag{23}$$

$$\mathbf{q}^* = \mathbf{q}^*\mathbf{A}^{\mathrm{T}} + \frac{\alpha_w m_w^*}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}}\mathbf{w} + \frac{\alpha_c(M - m_w^*)}{\mathbf{p}^*\mathbf{c}^{\mathrm{T}}}\mathbf{c} \tag{24}$$

$$m_w^* = \frac{k_w}{\alpha_w}\frac{\mathbf{l}\mathbf{q}^{*\mathrm{T}}}{(L - \mathbf{l}\mathbf{q}^{*\mathrm{T}})^{\eta_w}}, \tag{25}$$

where constant $k_w = w_0(L - \mathbf{l}\mathbf{q}_0^{\mathrm{T}})^{\eta_w}$ and r^* is a function of m_w^* as specified by equation (18).

Proof. Equilibrium prices are given by Proposition 1. By Proposition 5 the equilibrium wage rate, w^*, can be replaced by $\alpha_w m_w/\mathbf{l}\mathbf{q}^{*\mathrm{T}}$. The equilibrium interest rate, r^*, can be replaced by a function of equilibrium worker savings (equation (18)). These replacements yield equation (23). Equation (24) is given directly by Proposition 2. Lemma 5 gives equilibrium worker savings as $m_w^* = \frac{1}{\alpha_w}\mathbf{l}\mathbf{q}^{*\mathrm{T}}w^*$. Use equation (11) to replace the equilibrium wage rate by a function of the equilibrium level of employment to yield equation (25). □

Equation system (23,24,25) implicitly defines the long-period position. Prices, activity levels and the distribution of income are all interrelated.

The long-period position is differentially sensitive to the initial conditions in the economy. For example, the equilibrium is entirely independent of initial prices, $\mathbf{p}_0$, and initial inventory levels, $\mathbf{s}_0$. As might be expected, out-of-equilibrium scarcity prices of reproducible commodities turn out to be irrelevant not only to the determination of equilibrium prices but any aspect of the long-period position. So stable economies differentiated only by their market prices and initial stocks of inventory all converge to the same economic state. Market prices are therefore transient phenomena that affect the path the economy takes to equilibrium but not the equilibrium itself.

Corollary 12. Equilibrium workers' savings, m_w^*, are implicitly defined by

$$m_w^* = \frac{1}{\alpha_w}\frac{1}{1+r^*}\mathbf{l}(\mathbf{I}-\mathbf{A})^{-1}\left(\frac{\alpha_w m_w^*}{\mathbf{d}\mathbf{w}^{\mathrm{T}}}\mathbf{w}^{\mathrm{T}} + \frac{\alpha_c m_c^*}{\mathbf{d}\mathbf{c}^{\mathrm{T}}}\mathbf{c}^{\mathrm{T}}\right),$$

where $\mathbf{d} = \mathbf{l}[\mathbf{I} - \mathbf{A}(1+r^*)]^{-1}$. m_w^* is therefore a function of constants $\mathbf{A}$, $\mathbf{l}$, $\mathbf{w}$, $\mathbf{c}$, η_c, α_w, α_c and a subset of the initial conditions, m_{w_0}, m_{c_0} and r_0.

Proof. Proposition 1 implies

$$\mathbf{p}^* = \mathbf{l}(\mathbf{I} - \mathbf{A}(1+r^*))^{-1}w^*(1+r^*) = \mathbf{d}w^*(1+r^*). \tag{26}$$

Proposition 2 implies

$$\mathbf{q}^* = \left(\frac{\alpha_w m_w^*}{\mathbf{p}^*\mathbf{w}^{\mathrm{T}}}\mathbf{w} + \frac{\alpha_c m_c^*}{\mathbf{p}^*\mathbf{c}^{\mathrm{T}}}\mathbf{c}\right)(\mathbf{I} - \mathbf{A}^{\mathrm{T}})^{-1}. \tag{27}$$

Substitute (26) into (27) and pre-multiply both sides by direct labor coefficients $\mathbf{l}$ to yield the scalar equation,

$$\mathbf{l}\mathbf{q}^{*\mathrm{T}}w^* = \frac{1}{1+r^*}\mathbf{l}(\mathbf{I}-\mathbf{A})^{-1}\left(\frac{\alpha_w m_w^*}{\mathbf{d}\mathbf{w}^{\mathrm{T}}}\mathbf{w}^{\mathrm{T}} + \frac{\alpha_c m_c^*}{\mathbf{d}\mathbf{c}^{\mathrm{T}}}\mathbf{c}^{\mathrm{T}}\right).$$

From Proposition 5, $\mathbf{l}\mathbf{q}^{*\mathrm{T}}w^* = \alpha_w m_w$. □

Corollary 12 implies that – for a given technique, $[\mathbf{A}, \mathbf{l}]$, and composition of demand, $\mathbf{w}$ and $\mathbf{c}$ – the equilibrium class distribution of savings, represented by m_w^*, and therefore the equilibrium income distribution and aggregate expenditure, are entirely independent of market prices, the scale of production and the dynamics of the labor market. Income shares are instead determined by a set of nominal factors, which we shall call 'monetary factors', specifically propensities to consume, the initial distribution of savings, and the interest rate 'policy', represented by r_0 and elasticity η_c. So convergent economies with the same monetary factors all converge to the same nominal income distribution. Income shares are therefore insensitive to a wide range of economic disturbances, such as market prices, activity levels and labor market conditions. This result is suggestive since the relative stability of income shares is a notable feature of actual capitalist economies [27].

The total wage bill, $\mathbf{l}\mathbf{q}^{*\mathrm{T}}w^*$, or equivalently the aggregate expenditure of workers, is fixed by monetary factors. The conditions in the labor market – for instance wage elasticity η_w – then determine the wage rate and level of employment consistent with this level of expenditure.

The long-period position has a Sraffian structure. Equilibrium prices depend on income shares (and not the other way around). Income shares are fixed primarily by the dynamics of the interest rate. Sraffa [25] suggested that price equation (17) might be closed by 'the level of the money rates of interest'. The supply of homogeneous money-capital is distinguished from the supply of heterogeneous physical capital; hence the model avoids the conceptual errors exposed by the Cambridge capital critique [28].

This model shares commonalities with Post Keynesian approaches [29, Ch. 7]; for example, money is not neutral but has real effects; the interest rate is a conventional or exogenous variable that lacks a 'natural' rate; the supply of credit is endogenous and not constrained by the 'money supply'; and the long-period equilibrium is determined by the principle of 'effective demand', such that there is a 'limit to the profitable expansion of output' [30, pg. 71] before full employment is reached. Many authors have noted that Keynes' vision of a capitalist economy as a 'monetary production economy' is along many dimensions consistent with Marx's analysis (e.g., [29, 31–33]).

VII. NATURAL PRICES AND OBJECTIVE COSTS

We will now use the model to address a foundational question in the theory of economic value.

As production is reorganized to meet final demand the link between market prices and scarcity is dissipated. In 'long-period' equilibrium relative natural prices are independent of initial endowments and the composition of demand and are instead determined by the technique and the distribution of income (see Proposition 1). Scarcity does not explain equilibrium prices. What cost principle, then, are market prices attracted toward?

The answer is simple and both dissonant and consonant with the classical labor theory of value: prices of production are proportional to total labor costs. But to be able to give this answer first requires clarity and precision about the meaning of 'total labor costs'.

VII.1. Labor-values in a 'worker-only' economy

This model includes a dynamic 'worker-only' economy as a special-case by setting $\alpha_c = 0$ and $r_0 = 0$, which removes capitalist demand and interest income, and setting $\frac{\mathrm{d}m_w}{\mathrm{d}t} = \mathbf{l}\mathbf{q}^{\mathrm{T}} w - \alpha_w m_w + \frac{\mathrm{d}m_c}{\mathrm{d}t} = \mathbf{l}\mathbf{q}^{\mathrm{T}} w - \alpha_w m_w + \sum \pi_i$, which distributes profit of enterprise to worker households (which we can interpret as local profit sharing in worker-owned firms or as a global social dividend).

Figure 4 depicts an example social accounting matrix for a worker-only economy, which produces corn, iron and sugar. The real wage coefficients, $\bar{w}_i(t)$, which are commodity inputs to labor households per unit of labor supplied, are then

$$\bar{\mathbf{w}}(t) = \frac{1}{\mathbf{l}\mathbf{q}^{\mathrm{T}}} \frac{\alpha_w m_w}{\mathbf{p}\mathbf{w}^{\mathrm{T}}} \mathbf{w}^{\mathrm{T}} = [\bar{w}_i(t)],$$

which vary in time. Coefficients $\bar{w}_i(t)$ synchronize the consumption of the real wage with the supply of labor; they do not imply that the consumption of the real wage is necessarily funded by the wages of that labor (e.g., workers may be dissaving).

The social accounting matrix immediately tells us that l_i units of labor are directly required to produce commodity i. But we can also calculate the total labor required to reproduce commodity i; that is, the total labor, operating not just in one sector but in parallel in the economy as a whole, supplied simultaneously to output 1 unit of commodity i and replace all the indirect commodity inputs used-up during its production. Marx, following the Ricardian socialist, Thomas Hodgskin [34, 35], on occasion referred to this concept as 'coexisting labor': '[Raw] cotton, yarn, fabric, are not only produced one after the other and from one another, but they are produced and reproduced *simultaneously*, alongside one another. What appears as the effect of antecedent labor, if one considers the production process of the individual commodity, presents itself at the same time as the effect of coexisting labor, if one considers the *reproduction process* of the commodity, that is, if one considers this production process in its continuous motion and in the entirety of its conditions, and not merely an isolated action or a limited part of it. There exists not only a cycle comprising various phases, but all the phases of the commodity are simultaneously produced in the various spheres and branches of production. If the same peasant just plants flax, then spins it, then weaves it, these operations are performed in succession, but not simultaneously as the mode of production based on the division of labor within society presupposes.' [36] A labor-value is simply the total coexisting labor required to reproduce a commodity.

To calculate labor-value we proceed as follows: beginning at sector i, where we imagine 1 output is produced, we recursively trace all input paths backwards in the social accounting graph, counting labor inputs along the way, a procedure known as 'vertical integration' [37].

For example, production of unit i requires direct labor l_i plus a bundle of input commodities $\mathbf{A}^{(i)}$. During the production of unit i the bundle of inputs are simultaneously replaced by an expenditure of direct labor $\mathbf{l}\mathbf{A}^{(i)}$ operating in parallel in other sectors. But this production itself requires as input another bundle of commodities $\mathbf{A}\mathbf{A}^{(i)}$, which are also simultaneously replaced with the expenditure of an additional amount of direct labor $\mathbf{l}\mathbf{A}\mathbf{A}^{(i)}$ operating in parallel. To count all the coexisting labor, λ_i, working in parallel we must continue the sum; that is,

$$\begin{aligned}\lambda_i &= l_i + \mathbf{l}\mathbf{A}^{(i)} + \mathbf{l}\mathbf{A}\mathbf{A}^{(i)} + \mathbf{l}\mathbf{A}^2\mathbf{A}^{(i)} + \dots \\ &= l_i + \mathbf{l}(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \dots)\mathbf{A}^{(i)} \\ &= l_i + \mathbf{l}(\sum_{n=0}^{\infty} \mathbf{A}^n)\mathbf{A}^{(i)}. \qquad (28)\end{aligned}$$

The total coexisting labor is a reduction of commodity i to the simultaneous expenditure of work occurring in different sectors of the economy that all contribute to its reproduction. The vector $\boldsymbol{\lambda}$ of coexisting labor required to reproduce the unit bundle $\mathbf{u} = [1]$ is, from equation (28),

$$\begin{aligned}\boldsymbol{\lambda} &= \mathbf{l} + \mathbf{l}(\sum_{n=0}^{\infty} \mathbf{A}^n)\mathbf{A} \\ &= \mathbf{l}\sum_{n=0}^{\infty} \mathbf{A}^n.\end{aligned}$$

Since the technique is productive the infinite series converges to a finite value. The Leontief inverse $(\mathbf{I} - \mathbf{A})^{-1}$ is an alternative representation of the infinite series; hence,

$$\boldsymbol{\lambda} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}$$

and the vector of coexisting labor required to reproduce unit commodities is identical to the standard, and well-known, formula for labor-values, $\mathbf{v} = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1}$; or

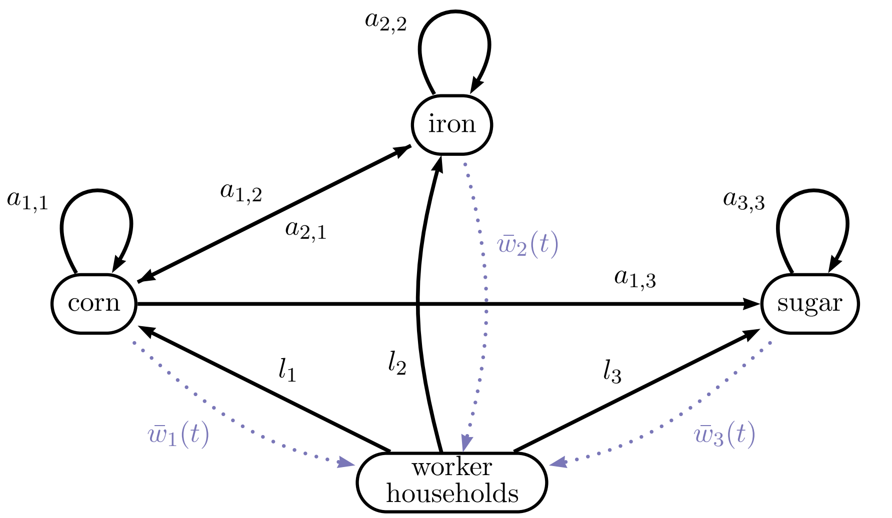
$$\mathbf{v} = \mathbf{v}\mathbf{A} + \mathbf{l}. \qquad (29)$$

This equation was probably first written down by Dmitriev (1868 – 1913) who translated the classical concept of 'labor embodied' into a mathematical formula [38, 39]. Dmitriev's formula is now standard (e.g., [22, 25, 40, 41]).

We will now examine two related properties of labor-values, in the context of a worker-only economy, which are subtle and normally overlooked.

VII.1.1. The independence of labor-values from the real wage

The recursive procedure of vertical integration ignores some input paths in the social accounting matrix. The real



$$\begin{bmatrix} \mathbf{A} & \bar{\mathbf{w}}^{\mathrm{T}}(t) \\ \mathbf{l} & 0 \end{bmatrix} = \begin{bmatrix} a_{1,1} & 0 & a_{1,3} & \bar{w}_1(t) \\ a_{2,1} & a_{2,2} & 0 & \bar{w}_2(t) \\ 0 & 0 & a_{3,3} & \bar{w}_3(t) \\ l_1 & l_2 & l_3 & 0 \end{bmatrix}$$

FIG. 4. Social accounting graph for an example 3-sector worker-only economy.

wage inputs to worker households, depicted as dashed arcs in figure 4, are not traced backwards. So the direct labor used-up to produce the real wage, which maintains and reproduces the working class, is excluded as a component of the cost of reproduction of commodity i. Why is this coexisting labor ignored?

A labor-value is the answer to the question, 'What is the total labor time required to reproduce 1 unit of a commodity?' To answer the question we assume a counter-factual: that the real cost measure, in this case quantities of labor, represents a resource that is used-up and therefore *not reproduced* during the reproduction of the commodity. A labor-value does not measure the labor cost *both* to reproduce 1 unit of the commodity *and* to reproduce the labor that was used-up. The fact that labor is required to reproduce the real wage (and therefore the workers) during the production of the commodity is irrelevant: the meaning of a labor-value requires that this additional cost of reproducing labor be excluded. It would make no sense to measure the cost of reproducing the very resource that is serving as the measure of cost.

We can look at this another way. Any system of measurement defines a standard unit (e.g., the 'meter'). We do not ask, 'How many meters are in one meter?' since the measure of the standard unit is by definition a unit of the standard. In a labor theory of value the question, 'What is the labor-value of one unit of direct labor?' is similarly ill-formed: the real cost of 1 hour of labor, *measured by labor time*, is 1 hour. No further reduction is possible or required. The self-identity of the standard of measure is a conceptual necessity in any system of measurement. So whether workers consume one bushel or a thousand bushels of corn to supply a unit of direct labor makes no difference to the labor-value of that unit of direct labor: an hour of labor-time is an hour of labor-time, period. The procedure of vertical integration over the social accounting matrix therefore always terminates at labor inputs and does not further reduce labor inputs to the real wage.

For example, Marx writes that the expression 'labor-value of labor-power', where labor-power is the capacity to supply labor, denotes the 'difficulty of production' of the real wage, which is the conventional level of consumption that reproduces the working class. In contrast, the expression 'labor-value of labor' embodies a confusion: 'the value of labor is only an irrational expression for the value of labor-power'. The expression, taken literally, is analogous to querying the color of a logarithm [4] or the time on the sun [42]. 'Labor is the substance, and the immanent measure of value, but *has itself no value*.' [43, pg. 503].

Labor-values, as a conceptual necessity, are independent of the real wage.

VII.1.2. Labor-values as total labor costs

Lemma 13. In a worker-only economy the supply of labor from worker households equals the labor-value of the real wage plus the change in the labor-value of inventories,

$$\mathbf{l}\mathbf{q}^{\mathrm{T}} = \frac{\alpha_w m_w}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}\mathbf{v}\mathbf{w}^{\mathrm{T}} + \sum_{i=1}^{n} v_i \frac{\mathrm{d}s_i}{\mathrm{d}t}, \tag{30}$$

where $(\alpha_w m_w / \mathbf{p}\mathbf{w}^{\mathrm{T}})\mathbf{v}\mathbf{w}^{\mathrm{T}}$ is the labor-value of the real wage.

Proof. Sum equation (6), with $\alpha_c = 0$, for all i and replace $\mathbf{v}(\mathbf{I} - \mathbf{A})\mathbf{q}^{\mathrm{T}}$ by $\mathbf{l}\mathbf{q}^{\mathrm{T}}$ using equation (29) and the conclusion follows. $\square$

Equation (30) is an aggregate labor-flow equation that reconceptualizes the circulation of multiple commodity types as the circulation of a single labor 'substance'. Interpret the equation as follows: $\mathbf{lq}^{\mathrm{T}}$ labor is input to the system of production. Since labor-value is conserved in exchange this labor reappears as output either in the form of the real wage or a change in the labor embodied in the inventory stock. The real wage represents the aggregate demand for labor time. Out-of-equilibrium the aggregate demand mismatches the supply of labor; in consequence, the inventory stock flexes to accommodate, either increasing the store of unsold goods, or depleting the store to meet excess demand.

In equilibrium aggregate labor-flow equation (30) collapses to the equality

$$\mathbf{vw}^{*\mathrm{T}} = \mathbf{lq}^{*\mathrm{T}}, \tag{31}$$

where $\mathbf{w}^* = (\alpha_w m_w^* / \mathbf{p}^* \mathbf{w}^{\mathrm{T}}) \mathbf{vw}^{\mathrm{T}}$ is the equilibrium real wage. What is the meaning of this surprising equality?

In linear production theory a 'vertically integrated' sector or subsystem [37] is a conceptual reclassification of an economy that cuts across the 'horizontal' boundaries of work location and firm ownership and consists of a self-replacing slice of economic activity dedicated to the reproduction of a specific commodity type. As we compute the labor-value of a commodity, by recursively vertically integrating over the social accounting matrix, we are also, as a side-effect, constructing the vertically integrated sector that reproduces the commodity. The labor-value of a commodity is also the total direct labor supplied to its vertically integrated sector.

Denote l as total labor supplied and split equation (31) into two:

$$l = \mathbf{lq}^{*\mathrm{T}},$$
$$l = \mathbf{vw}^{*\mathrm{T}}.$$

Pasinetti [37] interprets equation (31) as expressing two different ways of classifying, or disaggregating, the total labor l. The expression $l = \mathbf{lq}^{*\mathrm{T}}$ classifies the total labor 'according to the criterion of the industry in which [it is] required'. The expression $l = \mathbf{vw}^{*\mathrm{T}}$ classifies the total labor 'according to the criterion of the vertically integrated sector for which [it is] directly and indirectly required'. The situation is as follows: in equilibrium workers consume real wage $\mathbf{w}^*$ while supplying $\mathbf{lq}^{*\mathrm{T}}$ units of labor (when we count in terms of the direct labor supplied to each sector); and, by the definition of labor-value, $\mathbf{vw}^{*\mathrm{T}}$ units of labor are supplied during the reproduction of the real wage $\mathbf{w}^*$ (when we count in terms of the coexisting labor performed in each vertically integrated sector). Workers consume labor (in the form of the real wage) equal to the quantity they supply (in the form of direct labor activity); that is, in equilibrium the supply and demand for labor are equal.

Labor-values, in the context of a worker-only economy, measure *total labor costs* because they reduce *all* real costs to labor costs, except the cost of reproducing labor itself, i.e. the real wage. Hence in equilibrium the total labor cost of the real wage completely exhausts the supply of labor.

VII.1.3. 'That early and rude state'

The classical proposition that 'long-period' equilibrium prices are proportional to labor-values in a worker-only economy, where capitalist income is absent, is not controversial; indeed, in the context of static, equilibrium models, even modern critics of the labor theory of value accept this (e.g., [40, 41, 44]).

Theorem 14. In a worker-only economy equilibrium prices are proportional to labor-values,

$$\mathbf{p}^* = \mathbf{v}w^*,$$

where the constant of proportionality is the equilibrium wage.

Proof. By definition of a worker-only economy $r^* = 0$ and therefore equilibrium prices, by Proposition 1, are $\mathbf{p}^* = \mathbf{p}^*\mathbf{A} + \mathbf{l}w$; and hence $\mathbf{p}^* = \mathbf{l}(\mathbf{I} - \mathbf{A})^{-1} w^* = \mathbf{v}w^*$. □

A commodity's natural price is the wage bill of the total coexisting labor required to reproduce it. So commodities that require more labor-time to produce sell at higher prices in equilibrium.

Monetary costs and labor costs are proportionate because in both accounting systems all commodities are reduced to a single measure, either total monetary costs or total labor costs. The dual accounting systems are then related by the price of labor.

Smith famously argued that prices reflect labor costs in 'that early and rude state of society' [3, pg. 54] but after the 'accumulation of stock' (i.e., the existence of profit income) prices no longer bear such a proportionate relationship to labor costs.

VII.2. Labor-values in the presence of 'profits on stock'

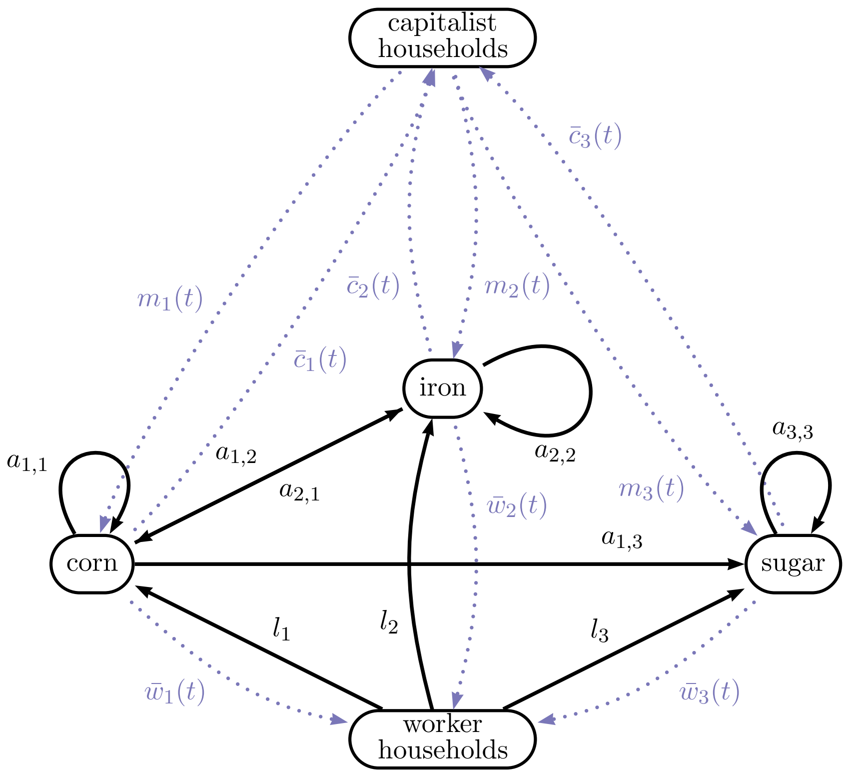
Figure 5 is the same 3-sector economy, producing corn, iron and sugar, but with additional material relations: the money-capital supplied to production, $\mathbf{m}(t)$, and the real consumption of capitalists, $\bar{\mathbf{c}}(t)$. As before, 1 unit of corn directly requires $a_{1,1}$ units of corn, $a_{2,1}$ units of iron, and l_1 units of labor but additionally a loan of $m_1(t)$ money-capital is now required to finance the production of unit output. In this model, where firms are not self-financing, the money-capital coefficients are the costs of production,

$$\mathbf{m}(t) = \mathbf{pA} + \mathbf{l}w = [m_i(t)].$$

A capitalist consumption coefficient is the quantity of commodity i consumed per unit of money-capital supplied,

$$\bar{\mathbf{c}}(t) = \frac{1}{\mathbf{mq}^{\mathrm{T}}} \frac{\alpha_c m_c}{\mathbf{pc}^{\mathrm{T}}} \mathbf{c} = [\bar{c}_i(t)],$$

Coefficients $\bar{c}_i(t)$ synchronize capitalist consumption with the supply of money-capital and vary with the price structure and capitalist savings. The coefficients do not imply that capitalist consumption is necessarily funded by interest on money-capital (e.g., capitalists may be dissaving and out-of-equilibrium also receive profit of enterprise).



$$\begin{bmatrix} \mathbf{A} & \bar{\mathbf{w}}^{\mathrm{T}}(t) & \bar{\mathbf{c}}^{\mathrm{T}}(t) \\ \mathbf{l} & 0 & 0 \\ \mathbf{m}(t) & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{1,1} & 0 & a_{1,3} & \bar{w}_1(t) & \bar{c}_1(t) \\ a_{2,1} & a_{2,2} & 0 & \bar{w}_2(t) & \bar{c}_2(t) \\ 0 & 0 & a_{3,3} & \bar{w}_3(t) & \bar{c}_3(t) \\ l_1 & l_2 & l_3 & 0 & 0 \\ m_1(t) & m_2(t) & m_3(t) & 0 & 0 \end{bmatrix} .$$

FIG. 5. Social accounting graph for an example 3-sector capitalist economy. Capitalist production introduces new material relations in the economy: money-capital requirements per unit output, $m_i(t)$, and capitalist consumption per unit of money-capital supplied, $\bar{c}_i(t)$.

VII.2.1. The divergence of technical and total labor costs

To calculate a labor-value we again recursively follow input paths in the social accounting graph adding labor costs along the way. The production of a unit of commodity i requires direct labor l_i and a bundle of input commodities $\mathbf{A}^{(i)}$. The input commodities $\mathbf{A}^{(i)}$ used-up are simultaneously replaced by the application of additional direct labor $\mathbf{l}\mathbf{A}^{(i)}$. But the presence of 'profits on stock' introduces something new. In this case production, unlike the worker-only economy, is controlled by the social relation of capital. The production of unit i additionally requires the loan of money-capital $m_i(t)$ (see the dashed input edges from capitalist households to the system of production in Figure 5).

Money-capital is not produced and therefore does not directly incur real costs. Nobody 'makes' money-capital, even in circumstances where money is a commodity. The inclusion of an explicit financial sector, including the labor of capital management, would not alter the essential fact that lending is a mere transfer of existing means of payment. Hence Marx's rejection of a 'natural' rate of interest.

But although there are no direct labor costs there are indirect labor costs associated with the supply of money-capital. Capitalists do not supply money-capital for free, either nominally or in real terms. In parallel with the production of unit i and the supply of money-capital $m_i(t)$, the capitalist class consumes commodity bundle $m_i(t)\bar{\mathbf{c}}^{\mathrm{T}}(t)$. So a quantity of coexisting labor, $\mathbf{l}m_i(t)\bar{\mathbf{c}}^{\mathrm{T}}(t)$, is indeed used-up during the supply of money-capital, specifically the coexisting labor employed to reproduce the goods that capitalists consume during the reproduction of unit i.

The standard formula for labor-value does not vertically integrate the input paths corresponding to the supply of money-capital. Money-capital is treated as an irreducible terminus, on the same footing as the supply of labor (e.g., all the dashed input edges in Figure 5 are ignored). In consequence, standard labor-values do not count the coexisting labor employed to reproduce capitalist consumption goods as a real cost of production. Should this labor be counted as a cost?

Quite simply, the answer depends on what we want to measure. Standard labor-values provide a purely technical measure of labor costs. For example, the reciprocal of a standard labor-value is a productivity index that measures the amount of the commodity produced by a unit of coexisting labor. Standard labor-values facilitate productivity comparisons across time independent of the distribution of income (e.g., see Flaschel [45]). But if we want to measure total labor costs then, in the context of capitalist production, we cannot use standard labor-values. By definition total labor costs reduce *all* real costs to labor costs, except the cost of reproducing labor itself. But standard labor-values do not include the labor cost of reproducing the capitalist class; hence, they do not measure total labor costs. This is not a matter of interpretation – it is a simple yet overlooked fact.

Note that the labor required to produce capitalist consumption goods is not part of the cost of reproducing labor and therefore necessarily excluded, as a conceptual necessity, from any definition of labor-value.

In a monetary production economy, like capitalism, money-capital although *sui generis* is nonetheless a necessary material prerequisite to production. In consequence, the total coexisting labor required to reproduce a commodity varies with the additional labor cost of reproducing the commodities that capitalists consume during the supply of money-capital necessary to finance production. A commodity cannot be produced without workers simultaneously performing 'tributary' labor for a capitalist class.

The presence of 'profits on stock', i.e. production under capitalist rules of distribution, causes technical and total labor costs to diverge. If we aim to calculate the *total* coexisting labor required to reproduce a commodity then we must treat money-capital as a *bona fide* commodity and reduce it to its (indirect) labor cost.

VII.2.2. The cause of the 'transformation problem'

Smith and Ricardo both understood that price-value proportionality breaks down in the presence of 'profits on stock' (e.g., Theorem 14 does not apply in capitalist conditions). Clearly this is a problem for a theory of value that purports to explain the structure of natural prices.

Marx proposed therefore that prices of production are *transformed* labor-values that function to redistribute the monetary representation of the labor-value of commodities such that capitalists share the available pool of surplus labor-value in proportion to the size of the money-capitals they advance. In Marx's view the divergence of prices of production from labor-values is an apparent contradiction or distortion that is necessarily generated by the specific distributional rules of capitalism.

The transformation is a nominal redistribution that neither creates or destroys labor-value. So Marx postulated three aggregate equalities between prices and labor-values: (i) the rate of profit is equal to the ratio of total surplus labor-value to the total labor-value of capital advanced (e.g., $r^* \equiv \frac{\mathbf{v}\mathbf{c}^{*\mathrm{T}}}{\mathbf{v}\mathbf{A}\mathbf{q}^{*\mathrm{T}}+\mathbf{v}\mathbf{w}^{*\mathrm{T}}}$); (ii) 'the sum of the profits in all spheres of production must equal the sum of the surplus-values' ([4], p. 173) (e.g., $\mathbf{p}^*\mathbf{c}^{*\mathrm{T}} \equiv \mathbf{v}\mathbf{c}^{*\mathrm{T}}w^*$); and (iii) 'the sum of the prices of production of the total social product equal the sum of its [labor-]value' ([4], p. 173) (e.g., $\mathbf{p}^*\mathbf{q}^{*\mathrm{T}} \equiv \mathbf{v}\mathbf{q}^{*\mathrm{T}}w^*$). These conservation rules maintain the quantitative link between labor costs and monetary costs. On this basis Marx could argue that labor-value remained the ultimate referent and regulator of natural price even in conditions of 'profits on stock'.

But, as is well known, when Marx's proposal is translated into linear production theory, his conservation claims in general cannot hold (for more detail see [46]). This is an incontrovertible mathematical fact. Since 'what does not hold in the special case cannot claim general validity' [47] price cannot measure labor-value and 'there is no rigorous quantitative connection between the labor time accounts arising from embodied labor coefficients and the phenomenal world of money price accounts' [48]. Prices of production are not conservative transforms of labor-values and Marx's solution appears to fail.

Ricardo [49] had identified a key problem: prices of production depend on the distribution of income (i.e., the interest rate – see equation (17)) but labor-values do not (see equation (29)); therefore natural prices have an additional degree-of-freedom and vary independently of labor-values. A conservative relation cannot hold between prices and labor-values if the price structure is not determined solely by objective labor costs but also by 'another though much less powerful cause' ([49], p. 404-405)

The transformation problem is the primary ostensible reason for the modern rejection of the logical possibility of a labor theory of value. The problem has generated a large literature spanning well over one hundred years. Definitive modern critiques of Marx's theory are Samuelson [40] and Steedman [41]. Samuelson, for example, points out that, given the technique and real wage, one can determine (a) profits and prices and (b) labor-values. But due to the transformation problem there is in general 'no way' of relating (a) and (b); hence, labor-values are 'redundant'. A theory of economic value based exclusively on labor-cost simply cannot account for price phenomena.

But, at root, why does this problem exist at all?

Money-capital has a price, the interest rate, and an associated real cost, which, in this model, is the labor-cost of reproducing capitalist consumption goods. Prices of production count the price of money-capital as a monetary cost of production. But standard labor-values do not count the labor-cost of the supply of money-capital as a real cost of production. The price of money-capital refers to labor that is not counted; hence there cannot be a conservative transform between standard labor-values and prices. The asymmetrical treatment of the commodity money-capital – present as a monetary cost in the price system but absent as a real cost in the labor-value system – is the fundamental cause of the transformation problem. An accounting mismatch must necessarily arise if total monetary costs are compared to partial labor costs.

The philosopher Ryle [50] introduced the term 'category error' to refer to the conceptual error of expecting some concept or thing to possess properties that it cannot have.[5] Category errors generate theoretical difficulties that appear insoluble because they are ill-posed at their hidden conceptual foundations. Only conceptual analysis, i.e. the identi-

[5] The famous example is the visitor to a University who, after being shown around campus, asks, 'But where is the University?'.

fication and removal of the underlying category error, can resolve, or more properly 'dissolve', the problems.

The transformation problem is the theoretical manifestation of a category error that conflates technical and total labor costs.[6] Technical labor costs cannot have properties that belong to total labor costs (and vice versa). This is the deep conceptual error at the heart of the classical labor theory of value. This error has been, and continues to be, the major obstacle toward a deeper understanding of the relationship between social labor and monetary phenomena. For example, the error has obscured the existence of a conservative quantitative relation between prices of production and labor-time, as we will now show.

VII.2.3. Total labor costs: nonstandard labor values

'Nonstandard' labor-values [46, 51] are the labor-values that result when we include the real cost of capitalist consumption in the process of vertical integration. For example, and returning to the example in figure 5, in addition to $\mathbf{l}\mathbf{A}^{(i)}$ labor used-up to replace input commodities $\mathbf{A}^{(i)}$ we now also count the $\mathbf{l}m_i(t)\bar{\mathbf{c}}^{\mathrm{T}}(t)$ labor simultaneously used-up to replace consumption goods $m_i(t)\bar{\mathbf{c}}(t)$.

The matrix of capitalist consumption coefficients is

$$\mathbf{C}(t) = \bar{\mathbf{c}}^{\mathrm{T}}(t)\mathbf{m}(t) = [b_{i,j}(t)],$$

where each $b_{i,j}(t)$ is the quantity of commodity i capitalists consume per unit output of commodity j. Matrix $\mathbf{C}$ encapsulates the current real costs of supplying money-capital to fund production in the different sectors of the economy. The nonstandard approach reduces these commodity inputs to their labor costs.

Define the technique augmented by capitalist consumption as

$$\tilde{\mathbf{A}}(t) = \mathbf{A} + \mathbf{C}(t) = [\tilde{a}_{i,j}(t)],$$

where each $\tilde{a}_{i,j}(t)$ is the quantity of commodity i, including capitalist consumption, directly used-up per unit output of j. Now the production of commodity bundle $\mathbf{A}^{(i)} + m_i(t)\bar{\mathbf{c}}^{\mathrm{T}}(t) = \mathbf{A}^{(i)} + \mathbf{C}^{(i)}(t)$ itself uses-up the bundle of input commodities $\tilde{\mathbf{A}}(t)(\mathbf{A}^{(i)} + \mathbf{C}^{(i)}(t))$, which are simultaneously replaced with the expenditure of direct labor $\mathbf{l}\tilde{\mathbf{A}}(t)(\mathbf{A}^{(i)} + \mathbf{C}^{(i)}(t))$ operating in parallel. To count all the coexisting labor we continue the sum; that is,

$$\begin{aligned}
\tilde{\lambda}_i(t) &= l_i + \mathbf{l}(\mathbf{A}^{(i)} + \mathbf{C}^{(i)}(t)) + \mathbf{l}\tilde{\mathbf{A}}(t)(\mathbf{A}^{(i)} + \mathbf{C}^{(i)}(t)) + \\
&\qquad \mathbf{l}\tilde{\mathbf{A}}^2(t)(\mathbf{A}^{(i)} + \mathbf{C}^{(i)}(t)) + \dots \\
&= l_i + \mathbf{l}(\mathbf{I} + \tilde{\mathbf{A}}(t) + \tilde{\mathbf{A}}^2(t) + \dots)(\mathbf{A}^{(i)} + \mathbf{C}^{(i)}(t)) \\
&= l_i + \mathbf{l}(\sum_{n=0}^{\infty} \tilde{\mathbf{A}}^n(t))(\mathbf{A}^{(i)} + \mathbf{C}^{(i)}(t)).
\end{aligned}$$

So the vector $\tilde{\boldsymbol{\lambda}}$ of coexisting labor required to reproduce a unit bundle $\mathbf{u} = [1]$ of commodities is

$$\begin{aligned}
\tilde{\boldsymbol{\lambda}}(t) &= \mathbf{l} + \mathbf{l}(\sum_{n=0}^{\infty} \tilde{\mathbf{A}}^n(t))(\mathbf{A} + \mathbf{C}(t)) \\
&= \mathbf{l}\sum_{n=0}^{\infty} \tilde{\mathbf{A}}^n(t). \qquad (32)
\end{aligned}$$

We can write infinite series (32) as

$$\tilde{\boldsymbol{\lambda}}(t) = \mathbf{l}(\mathbf{I} - \tilde{\mathbf{A}}(t))^{-1} = \tilde{\mathbf{v}}(t); \qquad (33)$$

which gives nonstandard labor-values as

$$\tilde{\mathbf{v}}(t) = \tilde{\mathbf{v}}\tilde{\mathbf{A}}(t) + \mathbf{l}. \qquad (34)$$

This equation has a finite solution if the augmented matrix $\tilde{\mathbf{A}}$ has a dominant eigenvalue less than 1, which is the nonstandard analog of the requirement of a productive technique in the standard formula for labor-values [51]. A non-viable augmented matrix $\tilde{\mathbf{A}}$ indicates that more than 1 unit of commodity i is required to reproduce 1 unit of commodity i given the current rate of capitalist consumption. In such circumstances a self-reproducing equilibrium cannot be obtained by any possible combination of activity levels.

The standard formula for labor-values, $\mathbf{v} = \mathbf{v}\mathbf{A} + \mathbf{l}$, is a property of the technique. Labor costs are the sum of labor 'embodied' in means of production, $\mathbf{v}\mathbf{A}$, plus direct labor supplied, $\mathbf{l}$. In contrast, the nonstandard formula for labor-values, $\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\tilde{\mathbf{A}} + \mathbf{l}$, is a property of the social accounting matrix. Labor costs are the sum of labor 'embodied' in means of production, including the commodity money-capital, $\tilde{\mathbf{v}}\tilde{\mathbf{A}}$, plus direct labor supplied, $\mathbf{l}$.[7] Standard labor-values view all household consumption (whether workers and capitalists) as net output and not a cost of production; nonstandard labor-values, in contrast, view worker consumption as net output and capitalist consumption as a cost of production. In general, $\tilde{\mathbf{v}} > \mathbf{v}$. In the special case of a worker-only economy standard and nonstandard labor-values are identical.

The standard approach measures technical labor costs and the nonstandard approach measures total labor costs. Both are entirely consistent labor-cost accounting schemes.

Now that we have defined total labor costs we can construct an aggregate labor-flow equation appropriate for production in conditions of 'profits on stock'.

Lemma 15. In a capitalist economy the supply of labor from worker households equals the (nonstandard) labor-value of the real wage plus the change in the (nonstandard) labor-value of inventories,

$$\mathbf{l}\mathbf{q}^{\mathrm{T}} = \frac{\alpha_w m_w}{\mathbf{p}\mathbf{w}^{\mathrm{T}}}\tilde{\mathbf{v}}\mathbf{w}^{\mathrm{T}} + \sum_{i=1}^{n} \tilde{v}_i \frac{\mathrm{d}s_i}{\mathrm{d}t}. \qquad (35)$$

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[6] In previous publications [46, 51] I used the term 'labor-cost accounting error'. However, both the standard and nonstandard accounting schemes are entirely consistent and error-free accounting schemes. An error arises in how these schemes are applied.

[7] I put 'embodied' in scare quotes as a reminder of its non-physical meaning as discussed in [17].

Proof. From equation (6) we get

$$\sum_{i=1}^{n} \tilde{v}_i \frac{\mathrm{d}s_i}{\mathrm{d}t} = \sum_{i=1}^{n} \tilde{v}_i \left(q_i - \mathbf{A}_{(i)}\mathbf{q}^{\mathrm{T}} - \frac{\alpha_c m_c}{\mathbf{p}\mathbf{c}^{\mathrm{T}}} c_i - \frac{\alpha_w m_w}{\mathbf{p}\mathbf{w}^{\mathrm{T}}} w_i \right)$$
$$= \tilde{\mathbf{v}}\mathbf{q}^{\mathrm{T}} - \tilde{\mathbf{v}}\mathbf{A}\mathbf{q}^{\mathrm{T}} - \frac{\alpha_c m_c}{\mathbf{p}\mathbf{c}^{\mathrm{T}}} \tilde{\mathbf{v}}\mathbf{c}^{\mathrm{T}} - \frac{\alpha_w m_w}{\mathbf{p}\mathbf{w}^{\mathrm{T}}} \mathbf{v}\mathbf{w}^{\mathrm{T}}. \tag{36}$$

Multiply nonstandard labor-value equation (34) by the activity levels to give $\tilde{\mathbf{v}}\mathbf{q}^{\mathrm{T}} = \tilde{\mathbf{v}}\tilde{\mathbf{A}}\mathbf{q}^{\mathrm{T}} + \mathbf{l}\mathbf{q}^{\mathrm{T}}$. Rearrange to get

$$\begin{aligned} \mathbf{l}\mathbf{q}^{\mathrm{T}} &= \tilde{\mathbf{v}}\mathbf{q}^{\mathrm{T}} - \tilde{\mathbf{v}}(\mathbf{A} + \mathbf{C})\mathbf{q}^{\mathrm{T}} \\ &= \tilde{\mathbf{v}}\mathbf{q}^{\mathrm{T}} - \tilde{\mathbf{v}}(\mathbf{A} + \bar{\mathbf{c}}^{\mathrm{T}}\mathbf{m})\mathbf{q}^{\mathrm{T}} \\ &= \tilde{\mathbf{v}}\mathbf{q}^{\mathrm{T}} - \tilde{\mathbf{v}}\mathbf{A}\mathbf{q}^{\mathrm{T}} - (\alpha_c m_c / \mathbf{p}\mathbf{c}^{\mathrm{T}})\tilde{\mathbf{v}}\mathbf{c}^{\mathrm{T}}. \end{aligned}$$

Substitute into equation (36) and the conclusion follows. □

As in the case of a worker-only economy the aggregate labor-flow equation relates labor inputs to labor outputs. Any out-of-equilibrium mismatches between the supply and demand for labor are accommodated by changes to the stock of inventory. In equilibrium the equation collapses to the equality

$$\tilde{\mathbf{v}}^* \mathbf{w}^{*\mathrm{T}} = \mathbf{l}\mathbf{q}^{*\mathrm{T}},$$

at which point the supply and demand for labor are equal. The interpretation of this equality is identical to the case of the worker-only economy analyzed in section VII.1.2 with one important difference: the concept of 'vertically integrated sector' has altered. The vertically integrated sectors that correspond to nonstandard labor-values now include the reproduction of capitalist consumption goods.

Pasinetti [52] first proposed to obtain more general measures of labor-value by extending vertical integration. In the context of a 'natural' (i.e., normative) economic system, with wages as the only source of income and nonuniform growth in real demand, Pasinetti calculates 'hyper-integrated labor coefficients' by extending vertical integration to additionally include the labor required to produce the commodities 'strictly necessary to expand such a circular process at a rate of growth'. Total labor costs, in this context, additionally include the coexisting labor used-up to expand production and depend on non-technical properties such as growth coefficients. Natural prices in Pasinetti's economy are proportional to hyper-integrated labor coefficients. Pasinetti's analysis is an example of how purely technical measures of labor cost can diverge from total labor costs even in the absence of 'profits on stock'. In fact, these kinds of divergences are ubiquitous and to be expected.

Nonstandard labor-values are a kind of hyper-integrated labor coefficient. For example, in the context of a capitalist economy in steady-state proportionate growth and zero capitalist consumption, nonstandard labor-values are formally identical to a special case of Pasinetti's hyper-integrated labor coefficients (see the appendix of [46].)

VII.2.4. This late and civilized state

We can now answer our original question: in equilibrium prices of production are proportional to nonstandard labor-values, i.e. total labor costs.

Theorem 16. Prices of production are proportional to equilibrium total labor costs,

$$\mathbf{p}^* = \tilde{\mathbf{v}}^* w.$$

Proof. Substitute $r^* = \mathbf{p}^*\bar{\mathbf{c}}^{*\mathrm{T}}$ from Lemma 8 into equilibrium price equation (17) to get

$$\mathbf{p}^* = (\mathbf{p}^*\mathbf{A} + \mathbf{l}w^*) + (\mathbf{p}^*\mathbf{A} + \mathbf{l}w^*)\mathbf{p}^*\bar{\mathbf{c}}^{*\mathrm{T}}. \tag{37}$$

Substitute $\mathbf{m}^* = \mathbf{p}^*\mathbf{A} + \mathbf{l}w^*$ into (37) and rearrange,

$$\begin{aligned} \mathbf{p}^* &= (\mathbf{p}^*\mathbf{A} + \mathbf{l}w^*) + \mathbf{m}^*\mathbf{p}^*\bar{\mathbf{c}}^{*\mathrm{T}} \\ &= \mathbf{p}^*\mathbf{A} + \mathbf{p}^*\bar{\mathbf{c}}^{*\mathrm{T}}\mathbf{m}^* + \mathbf{l}w^* \\ &= \mathbf{p}^*(\mathbf{A} + \bar{\mathbf{c}}^{*\mathrm{T}}\mathbf{m}^*) + \mathbf{l}w^* \\ &= \mathbf{p}^*\tilde{\mathbf{A}}^* + \mathbf{l}w^*. \end{aligned}$$

Hence $\mathbf{p}^* = \mathbf{l}(\mathbf{I} - \tilde{\mathbf{A}}^*)^{-1} w^* = \tilde{\mathbf{v}}^* w$, by equation (33). □

The price of production of a commodity is the wage bill of the total coexisting labor required to reproduce it. Commodities that cost more labor time to produce sell at proportionally higher prices in long-period equilibrium. The objective cost principle that regulates market prices, even in our late and civilized state, is total labor cost.

The classical authors believed that natural prices diverged from labor costs due to 'profits on stock'. This premise has been universally accepted. But it is false. In general, the Marxist tradition has accepted divergence and defended conservation of labor-value in price, whereas critics have also accepted divergence but denied conservation of labor-value in price. But both sides of the argument are mistaken: there is no divergence and there is aggregate conservation. The whole problematic is generated by a category error.

The labor costs that prices of production represent are nonstandard not standard labor-values. For example, in the nonstandard value system, it is easy to show that in equilibrium all Marx's expectations regarding the conservation of labor-time in price are met: (i) the interest rate equals the labor-value rate of profit,

$$r^* \equiv \frac{\mathbf{p}^*\mathbf{c}^{*\mathrm{T}}}{\mathbf{p}^*\mathbf{A}\mathbf{q}^{*\mathrm{T}} + \mathbf{p}^*\mathbf{w}^{*\mathrm{T}}} \equiv \frac{\tilde{\mathbf{v}}\mathbf{c}^{*\mathrm{T}}}{\tilde{\mathbf{v}}\mathbf{A}\mathbf{q}^{*\mathrm{T}} + \tilde{\mathbf{v}}\mathbf{w}^{*\mathrm{T}}},$$

(ii) profit is proportional to surplus labor-value,

$$\mathbf{p}^*\mathbf{c}^{*\mathrm{T}} \equiv \tilde{\mathbf{v}}\mathbf{c}^{*\mathrm{T}} w^*,$$

and (iii) total price is proportional to total value,

$$\mathbf{p}^*\mathbf{q}^{*\mathrm{T}} \equiv \tilde{\mathbf{v}}\mathbf{q}^{*\mathrm{T}} w^*.$$

In consequence, the standard criticisms of the labor theory of value do not apply: nonstandard labor-values are not redundant, since natural prices may be derived from the value system by scaling by the money wage w. In consequence, a theory of value based exclusively on labor-cost can account for price phenomena: labor-values and prices are 'two sides of the same coin'.

This conclusion, it should be emphasized, destroys the basis of any claim that a labor theory of value is logically incoherent because prices and labor-values are quantitatively incommensurable (e.g., [40, 41, 53]).

VIII. CONCLUSION

A major theoretical claim of the classical labor theory of value, at least Marx's version, is that monetary magnitudes, such as commodity prices, represent or refer to quantities of labor time in virtue of the causal relations instantiated by a market economy, independent of the consciousness of the economic actors that constitute it. Marx's labor theory of value claims that the unit of account represents labor time and nothing else (see, in particular, the opening three chapters of Capital, Vol. 1 and Rubin [54]). Smith, Ricardo and Marx all recognized that a substantive theoretical problem is raised for a labor theory of value due to the divergence of natural prices from labor-values. If labor-values cannot in theory fully account for the variation in long-period natural prices, which by construction are independent of the vagaries of supply and demand and the essential arbitrariness of market prices, then another 'less powerful cause' [49, pg. 404–405] must exist, other than labor costs, for their variation, in which case natural prices cannot refer exclusively to quantities of labor time and the labor theory of value is an incomplete, and inadequate, explanation of economic value. Hence Marx and Engels' explicit emphasis of the importance of the theoretical problem [55] and Marx's proposal of the aggregate conservation of labor-value in price [4].

Pilling [56] emphasizes that Marx's 'critique of political economy was *not* one which involved him finding a "constant" in terms of which everything could be quantified but of establishing the laws of mediation through which the "essence" of phenomena manifested itself as "appearance"'. The labor theory of value proposes to understand the economic laws that cause money to represent labor time, just as theories of physical law explain how and why a thermometer represents its local temperature. This paper makes three main contributions to this unfinished project: (i) in circumstances of a fixed technique and composition of demand the macrodynamics of classical competition cause market prices, which indicate out-of-equilibrium scarcity, to converge to profit-equalizing prices of production, (ii) the transformation problem is the theoretical manifestation of a category error that conflates technical and total labor costs, and (iii) equilibrium prices of production refer to objective labor costs even in circumstances of 'profits on stock' (Theorem 16).

Of course, from the perspective of the theory of economic value, these results raise as many questions as answers. But they are new questions.

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