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ABSTRACT

Recent research by Koopman, Wang and Wei (AER, February 2014) has provided a framework to decompose a country's total gross exports into four parts: (a) exports of domestic value-added that are eventually absorbed abroad, (b) exports of domestic value added that eventually return home, (c) foreign value added embedded in the country's exports, and (d) pure double counted items due to intermediate goods crossing borders multiple times. While the KWW framework already has useful applications, many other applications would need decompositions at the sector, bilateral, or bilateral sector level. Such generalizations are challenging. In this paper, we overcome these challenges and derive a decomposition methodology appropriate for these levels. We present a number of results based on applying our methodology to the World Input-Output Database (WIOD) for 40 countries and 35 industries from 1995 to 2011.

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1. Introduction

As it has been widely recognized, the increasing offshoring or segmentation of production across national borders has necessitated not only a framework to extract exports of value added from the official gross exports data but also a framework to quantify the structure of global production sharing. The official trade data do not reveal such information directly. An active recent literature has been devoted to measuring different aspects of international production sharing including Feenstra (1998), Feenstra and Hanson (1998), Feenstra and Jensen (2009), Hummels, Ishii, and Yi (2001), Yi (2003), Jorgenson and Vu (2011), Daudin et al (2011), Jorgenson and Timmer (2011), Johnson and Noguera (2012), Stehrer, Foster, and de Vries (2012), Antras (2013), Antras et al (2012), Baldwain and Lopez-Gonzalez (2013), and Timmer, et. al (2013), among others. The key concepts proposed in these papers include vertical specialization (VS for short) or its variations such as VS1 and VS1*, which are typically measured at the country level but not at the sector or bilateral level, and value added exports (VAX for short), which is typically measured at the country or bilateral level but not at the bilateral sector level. The exact relationships among these concepts are established in a recent contribution by Koopman, Wang, and Wei (forthcoming). A collection of papers in the volume edited by Mattoo, Wang, and Wei (2013) represents some of the latest thinking on the subject from both the academic community and international policy institutions such as the World Trade Organization, the OECD, the World Bank, and the IMF.

In a survey of work on quantifying global production sharing, Antras (2013, Chapter 1) calls the VAX ratio – the ratio of value added that is exported and ultimately absorbed abroad to gross exports, as proposed by Johnson and Noguera (2012) – the "state of art" and "an appealing inverse measure of the importance of vertical specialization in the world production." However, the VAX ratio concept needs to be improved in two important ways. First, as we will point out, the VAX ratio, as currently defined in the literature, is not well behaved at either sector, bilateral, or bilateral sector level[1]. The key to understanding this point is a distinction between a forward-linkage based measure of

[1] The VAX ratio at these levels is not upper-bounded by one. Indeed, it can take on the value of infinity when the gross exports are zero. An alternative measure that arises from our framework will be naturally bounded between zero and one at any level of disaggregation.

value added exports, which includes indirect exports of a sector's value added via gross exports from other sectors of the same exporting country, and a backward-linkage based measure of value added exports, which is value added from all sectors of a given exporting country embodied in a given sector's gross exports. For example, a forward-linkage based measure of exports of value added in US electronics sector includes that sector's value added embodied in US gross exports from automobile and chemical sectors, but excludes the value added contributions from these sectors embodied in the gross exports of US electronics. In comparison, a backward-linkage based measure of US value added embodied in US electronics exports includes value added contributions from other US sectors such as services and automobiles to the production of US electronics gross exports, but excludes the value added contributions from US electronics sector to the gross exports of other sectors such as US automobiles. Such a distinction is critical at the sector, bilateral, or bilateral sector level (and hence the VAX ratio, which is based on forward-linkages as defined in the literature, is not well behaved), but the distinction disappears at the country aggregate level (and hence the VAX ratio is only well behaved at this level). Since most international value chains are at the sector or country-sector level, one needs a well-behaved VAX ratio at these levels too.

Second, the VAX ratio, even after it is properly re-defined, still does not capture some of the important features of international production sharing. Let us consider a hypothetical example: both the US and Chinese electronics exports to the world can have an identical ratio of value added exports to gross exports (say, 50% for each) but for very different reasons. In the Chinese case, the VAX ratio is 50% because half of the Chinese gross exports reflect foreign value added (say value added from Japan, Korea, or even the United States). In contrast, for the US exports, half of the gross exports are US value added in intermediate goods that are used by other countries to produce goods that are exported back to the United States. So only half of the US value added that is initially exported is ultimately absorbed abroad; the US VAX ratio is 50% even if it does not use any foreign value added in the production of its electronics exports. In this example, China and the United States occupy very different positions on the global value chain but the two countries' VAX ratios would not reveal this important difference. To provide the additional information, our decomposition framework will go beyond just simply extracting value added exports in a country-sector's gross exports.

Koopman, Wang, and Wei (forthcoming, subsequently referred to as KWW) provide a unified mathematical framework to decompose a country's total gross exports into nine value-added and double counted components. Conceptually, the nine components can be grouped into four buckets: The first bucket gives a country's value added exports that is absorbed abroad, exactly as defined by Johnson and Noguera (2012). The second bucket gives the part of a country's domestic value added that is first exported but eventually returned home. While it is not a part of a country's exports of value added that stays abroad, it is a part of the exporting country's GDP. The third bucket is foreign value added used in the production of a country's exports and eventually absorbed by other countries. The forth bucket consists of so called "pure double counted terms" arising from intermediate goods that cross border multiple times. Some of the terms in the fourth bucket double count value added originated in the home country, while others double count value added originated in foreign countries. Other measures of international production sharing in the existing literature such as VS, VS1, VS1*, and VAX are shown to be some linear combinations of the terms in KWW's decomposition formula.

While the KWW framework already has many useful applications (as discussed in the KWW paper), an important limitation of the framework is that the decomposition is only done at a country's aggregate trade level, not at the sector, bilateral, or bilateral sector level[2]. Major challenges exist to generalize the framework in that direction. In producing exports in any given sector, not only value added from other sectors in the same country will be used, but also value added produced by potentially all sectors in other countries also need to be accounted for. Such an accounting framework has never been developed in a comprehensive way before. This is our goal.

Generalizing the KWW approach to the bilateral/sector level is not a trivial exercise; it cannot be achieved by simply applying the KWW gross exports decomposition formula to bilateral/sector level data. Conceptually, domestic value added can be decomposed

[2] The calculation of domestic value added that is ultimately absorbed abroad can be done at the bilateral and sector level. Indeed, some examples are given in KWW (forthcoming). However, the computations of the other three components that could sum to 100% bilateral/sector trade flows are not done in KWW..

from both the producer and the user’s perspective. On one hand, domestic value added created in a home sector can be exported indirectly through other sectors’ gross exports; On the other hand, domestic value added that is embedded in a sector's gross exports can include value added from other home sectors. These two concepts are different. Mathematically, additional adjustment terms have to be derived to properly account for other sectors/countries' value-added contributions to a given country-sector’s gross exports, in addition to properly measuring how that country-sector’s value-added is used in its own intermediate and final goods exports, so that all its value added and double counted components can sum to 100% of gross exports at the country-sector level. What makes the earlier work (KWW) at the country aggregate level relatively easier is that the difference between the decomposition from the producer and user’s perspectives disappears after aggregating to the economy-wide level. A useful decomposition formula also has to have the property that all the decomposition terms from the bilateral/sector level gross export must be internally consistent so that they can sum up to the decomposition equation given in KWW at the aggregate level.

This paper’s main contribution is to provide a new and comprehensive methodological framework that decomposes bilateral sector level gross exports into various value added and double counted terms. While it does not directly examine causes and consequences of changing structure of vertical specialization, reliable measurements made possible by such an accounting methodology are necessary for investigating these research questions.

The paper is organized as follows. Section 2 presents a derivation of our methodological framework, starting with some examples with two sectors and two or three countries. The most general case of G countries and N sectors is presented in Appendix H. Section 3 reports selected empirical decomposition results based on the World Input-Output Database (WIOD) and discusses how bilateral/sector level gross exports accounting results may help to measure international production sharing or a particular country/sector's position and participation in global production network. Section IV provides some concluding remarks.

2. Concepts and Methodology

2.1 Leontief insight and the decomposition of final goods and value-added production

All the decomposition methods in the recent vertical specialization and trade in value-added literatures are rooted in Leontief (1936). His work demonstrated that the amount and type of intermediate inputs needed in the production of one unit of output can be estimate based on the input-output (IO) structures across industries. Using the linkages across industries and countries, gross output in all stages of production that is needed to produce one unit of final goods can be traced. When the gross output flows associated with a particular level of final demand are known, value added production and trade can be simply derived by multiplying these flows with the value added to gross output ratio in each country/industry.

To better understand how such Leontief insight is used in the decomposition, let us assume a two-country (home and foreign) world, in which each country produces goods in N differentiated tradable industries. Goods in each sector can be consumed directly or used as intermediate inputs, and each country exports both intermediate and final goods to the other.

All gross output produced by country *S* must be used as either an intermediate good or a final good at home or abroad, or

$$X^s = A^{ss}X^s + Y^{ss} + A^{sr}X^r + Y^{sr} \qquad r,s = 1,2 \tag{1}$$

Where X^s is the N×1 gross output vector of country *S*, Y^{sr} is the N×1 final demand vector that gives demand in country *R* for final goods produced in *S*, and A^{sr} is the N×N IO coefficient matrix, giving intermediate use in *R* of goods produced in *S*. The two-country production and trade system can be written as an ICIO model in block matrix notation

$$\begin{bmatrix} X^s \\ X^r \end{bmatrix} = \begin{bmatrix} A^{ss} & A^{sr} \\ A^{rs} & A^{rr} \end{bmatrix} \begin{bmatrix} X^s \\ X^r \end{bmatrix} + \begin{bmatrix} Y^{ss} + Y^{sr} \\ Y^{rs} + Y^{rr} \end{bmatrix} \tag{2}$$

With rearrange, we have

$$\begin{bmatrix} X^s \\ X^r \end{bmatrix} = \begin{bmatrix} I - A^{ss} & -A^{sr} \\ -A^{rs} & I - A^{rr} \end{bmatrix}^{-1} \begin{bmatrix} Y^{ss} + Y^{sr} \\ Y^{rs} + Y^{rr} \end{bmatrix} = \begin{bmatrix} B^{ss} & B^{sr} \\ B^{rs} & B^{rr} \end{bmatrix} \begin{bmatrix} Y^s \\ Y^r \end{bmatrix} \tag{3}$$

where B^{sr} denotes the N×N block Leontief inverse matrix, which is the total requirement matrix that gives the amount of gross output in producing Country S required for a one-unit increase in final demand in country R. Y^s is an N×1 vector that gives global use of S' final goods., including domestic final goods sales Y^{ss} and final goods exports Y^{sr}. The intuition behind the Leontief inverse or the Leontief insight is as follows: when $1 export is produced, a first round of value-added is generated. This is the direct domestic value-added induced by the $1 export. To produce that export, intermediate inputs have to be used. The production of these intermediate inputs also generates value-added. This is the second round or indirect domestic value added induced by the $1 export. Such a process to generate indirect value-added can be traced to additional round of production throughout the economy, as intermediate inputs are used to produce other intermediate inputs. The total domestic value-added induced by the $1 export thus is equal to the sum of direct and all rounds of indirect domestic value-added generated from the $1 export production process. Expressing this process mathematically using the terms defined above, we have

$$\begin{aligned} DVS &= V + VA + VAA + VAAA + ... = V(I + A + A^2 + A^3 + ...) \\ &= V(I - A)^{-1} = VB \end{aligned} \tag{4}$$

It can be shown that the power series of matrix A is convergent and the inverse matrix $B = (I - A)^{-1}$ exists as long as A is in full rank (Miller and Jones, 2009).

Define V^s as a 1×N direct value-added coefficient vector. Each element of V^s gives the share of direct domestic value added in total output. This is equal to one minus the intermediate input share from all countries (including domestically produced intermediates):

$$V^s = u[I - A^{ss} - A^{rs}] \tag{5}$$

where u is a 1×N unity vector. When N=2, the corresponding inter-country input-output (ICIO) account can be described by Table 1 below.

Table 1: 2-country and 2-sector ICIO table

	Country	Intermediate Use				Final Demand		Total gross output
		S		R		S	R	
Country	Sector	1	2	1	2			
S	**1**	z_{11}^{ss}	z_{12}^{ss}	z_{11}^{sr}	z_{12}^{sr}	y_1^{ss}	y_1^{sr}	x_1^s
	2	z_{21}^{ss}	z_{22}^{ss}	z_{21}^{sr}	z_{22}^{sr}	y_2^{ss}	y_2^{sr}	x_2^s
R	**1**	z_{11}^{rs}	z_{12}^{rs}	z_{11}^{rr}	z_{12}^{rr}	y_1^{rs}	y_1^{rr}	x_1^r
	2	z_{21}^{rs}	z_{22}^{rs}	z_{21}^{rr}	z_{22}^{rr}	y_2^{rs}	y_2^{rr}	x_2^r
Value-added		va_1^s	va_2^s	va_1^r	va_2^r			
Total input		x_1^s	x_2^s	x_1^r	x_2^r			

Where x_1^s is gross output of the first sector in country S, va_1^s is direct value added of the first sector in country S, y_1^{sr} is final goods produced by the first sector in country S for consumption in ROW (Country R), and z_{11}^{sr} is intermediate goods produced in the first sector of Country S and used for the first sector production in Country R. Other variables can be interpreted similarly. Equations (2) and (3) can be re-written as follows:

$$\begin{bmatrix} x_1^s \\ x_2^s \\ x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} a_{11}^{ss} & a_{12}^{ss} & a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{ss} & a_{22}^{ss} & a_{21}^{sr} & a_{22}^{sr} \\ a_{11}^{rs} & a_{12}^{rs} & a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rs} & a_{22}^{rs} & a_{21}^{rr} & a_{22}^{rr} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \\ x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} y_1^{ss} + y_1^{sr} \\ y_2^{ss} + y_2^{sr} \\ y_1^{rs} + y_1^{rr} \\ y_2^{rs} + y_2^{rr} \end{bmatrix} \tag{2a}$$

$$\begin{aligned} \begin{bmatrix} x_1^s \\ x_2^s \\ x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1} \begin{bmatrix} y_1^{ss} + y_1^{sr} \\ y_2^{ss} + y_2^{sr} \\ y_1^{rs} + y_1^{rr} \\ y_2^{rs} + y_2^{rr} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{ss} + y_1^{sr} \\ y_2^{ss} + y_2^{sr} \\ y_1^{rs} + y_1^{rr} \\ y_2^{rs} + y_2^{rr} \end{bmatrix} \end{aligned} \tag{3a}$$

where a_{11}^{sr} is the direct IO coefficient that gives units of the intermediate goods produced in the first sector of Country S that are used in the production of one unit of gross output in the first sector of Country R, b_{11}^{ss} is the total IO coefficient that gives the total amount of the gross output of first sector in Country S needed to produce an extra unit of the first sector's final good in Country S (which is for consumption in both Countries S and R). Other coefficients have similar economic interpretations.

The direct value added coefficient vector (equation 5) can be re-written as follows:

$$v_j^c \equiv va_j^c \big/ x_j^c = 1 - \sum_i^2 a_{ij}^{sc} - \sum_i^2 a_{ij}^{rc} \quad (c = s, r \quad j = 1,2)\,. \tag{5a}$$

Then we can define the total value added coefficient (*VB*) matrix, or the value added multiplier as named in the input-output literature:

$$VB = \begin{bmatrix} v_1^s & v_2^s & v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} + v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} + v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \\ v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} + v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} + v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix}^T \tag{6}$$

where *T* denotes matrix transpose operation. Each element of the last term in VB equals unity.

Condensing the final demand vector in (3a) as:

$$\begin{bmatrix} y_1^{ss} + y_1^{sr} & y_2^{ss} + y_2^{sr} & y_1^{rs} + y_1^{rr} & y_2^{rs} + y_2^{rr} \end{bmatrix}^T = \begin{bmatrix} y_1^s & y_2^s & y_1^r & y_2^r \end{bmatrix}^T,$$

the decomposition of the country/sector level value-added and final goods production as a direct application of the Leontief insight can be expressed as follows:

$$\begin{aligned} \hat{V}B\hat{Y} &= \begin{bmatrix} v_1^s & 0 & 0 & 0 \\ 0 & v_2^s & 0 & 0 \\ 0 & 0 & v_1^r & 0 \\ 0 & 0 & 0 & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^s & 0 & 0 & 0 \\ 0 & y_2^s & 0 & 0 \\ 0 & 0 & y_1^r & 0 \\ 0 & 0 & 0 & y_2^r \end{bmatrix} \\ &= \begin{bmatrix} v_1^s b_{11}^{ss} y_1^s & v_1^s b_{12}^{ss} y_2^s & v_1^s b_{11}^{sr} y_1^r & v_1^s b_{12}^{sr} y_2^r \\ v_2^s b_{21}^{ss} y_1^s & v_2^s b_{22}^{ss} y_2^s & v_2^s b_{21}^{sr} y_1^r & v_2^s b_{22}^{sr} y_2^r \\ v_1^r b_{11}^{rs} y_1^s & v_1^r b_{12}^{rs} y_2^s & v_1^r b_{11}^{rr} y_1^r & v_1^r b_{12}^{rr} y_2^r \\ v_2^r b_{21}^{rs} y_1^s & v_2^r b_{22}^{rs} y_2^s & v_2^r b_{21}^{rr} y_1^r & v_2^r b_{22}^{rr} y_2^r \end{bmatrix} \end{aligned} \tag{7}$$

This matrix gives the estimates of sector and country sources of value-added in each country's final goods production. Each element in the matrix represents the value added from a source sector of a source country directly or indirectly used in the production of final goods (absorbed in both the domestic and foreign market) in the source country. Looking at the matrix along the row, for example, the first element of the first row, $v_1^s b_{11}^{ss}(y_1^{ss}+y_1^{sr})$ is country S's first sector value added embodied in its final goods production for both the first sector's domestic sales and exports. The second element, $v_1^s b_{12}^{ss}(y_1^{ss}+y_1^{sr})$, is country S's first sector value added embodied in its second sector's final goods production. The third and fourth elements $v_1^s b_{11}^{sr}(y_2^{rs}+y_2^{rr})$ and $v_1^s b_{12}^{sr}(y_2^{rs}+y_2^{rr})$ are country S's first sector value added embodied in country R's final goods production in its first and second sector respectively. Therefore, summing up the first row of the matrix we have Country S's total value added created by production factors employed in its first sector. In other words, it equals GDP by industry of the first sector in country S. Express this mathematically:

$$\begin{aligned} va_1^s \text{ or } GDP_1^s &= \left[v_1^s b_{11}^{ss} y_1^{ss} + v_1^s b_{12}^{ss} y_2^{ss} + v_1^s b_{11}^{ss} y_1^{sr} + v_1^s b_{12}^{ss} y_2^{sr}\right] \\ &+ \left[v_1^s b_{11}^{sr} y_1^{rs} + v_1^s b_{12}^{sr} y_2^{rs} + v_1^s b_{11}^{sr} y_1^{rr} + v_1^s b_{12}^{sr} y_2^{rr}\right] = v_1^s (b_{11}^{ss} y_1^s + b_{11}^{sr} y_1^r + b_{12}^{ss} y_2^s + b_{12}^{sr} y_2^r) = v_1^s x_1^s \end{aligned} \tag{8}$$

Looking at the $\hat{V}B\hat{Y}$ matrix along the column, the second elements in the first column, $v_2^s b_{21}^{sr}(y_1^{ss}+y_1^{sr})$, for example, is country S's second sector value added embodied in country S's production of its first sector's final goods, and the third and fourth elements, $v_1^r b_{11}^{rs}(y_1^{ss}+y_1^{sr})$ and $v_1^r b_{21}^{rs}(y_1^{ss}+y_1^{sr})$ are country R's (foreign) value added embodied in country S's production of its first sector's final goods. Adding up all elements in the first column equals the value of final goods production by country S's first sector ,i.e:

$$(v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} + v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs}) y_1^s = y_1^s \tag{9}$$

In summary, sum of the $\hat{V}B\hat{Y}$ matrix across columns along the row accounts for how each country's domestic value-added originated in a particular sector is used by the sector and all its downstream countries/sectors. It measures the value-added contribution made by production factors employed at the producing sector in the source country to the destination country/sectors. While sum of the $\hat{V}B\hat{Y}$ matrix across the rows along the column, measures the country/sector sources of value-added in each country's final goods production, accounts for all upstream countries/sectors' value-added contribution to a specific country/sector's final goods output, and thus decomposes a particular sector's final goods and services into its various country and sector sources. Based on the identity given by equation (6), all these sources should sum to 100% of the value of the final products.

Therefore, the supply-side perspective (summing across columns along the row) decomposes how each country's GDP by industry is used, directly or indirectly to satisfy domestic or foreign final demand, while the user-side perspective (sum across rows along the column), decomposes a country/sector's final goods and services into its original country/sector sources. As an example, in the electronic sector, the supply-side perspective would include the value added created by production factors employed at the electronics sector and incorporated into gross exports of electronics itself (direct domestic value-added exports), as well as in exports of computers, consumer appliances, and automobiles (indirect domestic value-added exports). In other words, it decomposes GDP (domestic value-added) by industries according to where (i.e., which sector/country) it is used. Such a perspective is consistent with the literature on factor content of trade. On the other hand, decomposition from a user's perspective will include all upstream sectors/countries' contributions to value added in a specific sector/country's exports. In the electronics sector, it includes value added in the electronics sector itself as well as value added in inputs from all other upstream sectors/countries (such as glass from country A, rubber from country B, transportation and design from the home country) used to produce electronics for exports by the home country (direct/indirect domestic value added in exports and foreign value-added in exports). Such a perspective aligns well with case studies of supply chains of specific sectors and products, as the iPod or iPhone examples frequently cited in the literature.

These two different ways to decompose value-added and final goods production each have their own economic interpretation and thus different roles in economic analysis. However, they are equivalent in the aggregate because global value-added production equals global final demand.[3]

After understanding how value added (GDP) and final goods production at the

[3] See the proofs in Timmer et al. (2013) and Koopman, Wang and Wei (forthcoming).

sector level can be correctly decomposed based on the Leontief insight (equation (7) or the $\hat{V}B\hat{Y}$ matrix), we can better understand various decomposition methods proposed in the literature.

There are several attempts to estimate trade in value added and decompose value-added and final goods production based on the Leontief insight and inter-country input-output database in recent years. Timmer et al (2013) decompose final goods production based on backward looking linkage in the details that WIOD data allow. For example, their method can provide estimates on how much contribution an unskilled worker employed in the Chinese steel industry makes to a car produced in Germany, or how much contribution a skilled US worker in the electronic industry made to a computer consumed by a Chinese consumer. Johnson and Noguera (2012) estimate value-added content of trade based on forward-looking linkage. However, they only measure sector value added absorbed by foreign countries, and do not take GDP absorbed in the domestic market into account, thus cannot recognize the conceptual difference between value-added exports and domestic value-added in exports, which also include domestic value-added first exported but finally returns home. In this sense, their decomposition of value-added production is incomplete. Most importantly, their gross exports to value-added exports ratio (VAX) at the sector level cannot be used as share of gross exports, since they compute value-added exports by forward-looking linkages, which gives the amount of domestic value-added created in the source country’s production sector absorbed in the destination country; but in the source country's sector gross exports, there is also domestic value-added from other domestic sectors. Therefore, their VAX ratio tends to under-estimate the true domestic value-added exports to gross exports ratio for downstream sectors, while over-estimate the true VAX ratio for upstream sectors, and produce unreasonably large numbers for country/sector with tiny exports.

To compute value added by forward or backward linkages, such as what is pursued by Johnson and Noguera (2012 and Timmer et al (2013), one only needs to apply the insight of Lenotief directly (i.e., using Leontief inverse multiplied with the final demand), but does not need to decompose intermediate goods trade. However, in order to decompose the gross exports into various value added components and pure double counted terms, such as what is pursued by KWW (forthcoming), one would need to go beyond the original Leontief insight and find a way to decompose intermediate trade.

In Leontief's time of 1930s-1960s, intermediate goods trade is relatively unimportant. Today, it is about two thirds of the world gross trade. So being able to decompose intermediate goods trade has become more crucial, and KWW has made a useful step at performing such a decomposition at the level of a country's aggregate exports.

2.2 Decomposition for the 2-country 2 Sector Case

For ease of understanding, we continue our discussion with the two-country, two sector ICIO model specified in the previous section. We first lay out the basic gross output and exports accounting identities at the sector level and then propose a way to fully decompose a country's gross exports into the sum of components that include both the country's domestic value added in exports and various double-counted components. We then use a three country, two sector model to discuss what additional components will be involved once the third country effects are taken into account. Finally, we present the G-country M-sector model (in Appendix H) and highlight how our decomposition formula in this most general case is different from that in the three-country two-sector model. We also provide numerical examples following our analytical model to show intuitively how our accounting equation works.

Let us now consider the case of two countries and two sectors. First, the gross exports of country S can be decomposed into two parts: final goods exports and intermediate goods exports:

$$E^{sr} = \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} = \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \tag{10}$$

As we already show in previous section that the final goods exports can be easily decomposed into domestic and foreign value-added by directly applying the Leontief insight, we concentrate our discussion on the decomposition of intermediate goods exports. Based on the Leontief insight, gross output of the two countries can also be decomposed according to where they are ultimately absorbed to sustain the production of both countries' final demand:

$$BY = \begin{bmatrix} x_1^{ss} & x_1^{sr} \\ x_2^{ss} & x_2^{sr} \\ x_1^{rs} & x_1^{rr} \\ x_2^{rs} & x_2^{rr} \end{bmatrix} = \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{ss} & y_1^{sr} \\ y_2^{ss} & y_2^{sr} \\ y_1^{rs} & y_1^{rr} \\ y_2^{rs} & y_2^{rr} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}^{ss} y_1^{ss} + b_{12}^{ss} y_2^{ss} + b_{11}^{sr} y_1^{rs} + b_{12}^{sr} y_2^{rs} & b_{11}^{ss} y_1^{sr} + b_{12}^{ss} y_2^{sr} + b_{11}^{sr} y_1^{rr} + b_{12}^{sr} y_2^{rr} \\ b_{21}^{ss} y_1^{ss} + b_{22}^{ss} y_2^{ss} + b_{21}^{sr} y_1^{rs} + b_{22}^{sr} y_2^{rs} & b_{21}^{ss} y_1^{sr} + b_{22}^{ss} y_2^{sr} + b_{21}^{sr} y_1^{rr} + b_{22}^{sr} y_2^{rr} \\ b_{11}^{rs} y_1^{ss} + b_{12}^{rs} y_2^{ss} + b_{11}^{rr} y_1^{rs} + b_{12}^{rr} y_2^{rs} & b_{11}^{rs} y_1^{sr} + b_{12}^{rs} y_2^{sr} + b_{11}^{rr} y_1^{rr} + b_{12}^{rr} y_2^{rr} \\ b_{21}^{rs} y_1^{ss} + b_{22}^{rs} y_2^{ss} + b_{21}^{rr} y_1^{rs} + b_{22}^{rr} y_2^{rs} & b_{21}^{ss} y_1^{sr} + b_{22}^{ss} y_2^{sr} + b_{21}^{sr} y_1^{rr} + b_{22}^{sr} y_2^{rr} \end{bmatrix} \tag{11}$$

Following KWW (forthcoming), we label the 4 by 2 matrix on the left hand side of equation (11) as the “gross output decomposition matrix.” It fully decomposes each country’s gross outputs according to where it is absorbed. For example, x_1^{ss} is the first sector's gross output in country S that is eventually absorbed by country S’s final demand, x_1^{sr} is the first sector's gross output in country S that is eventually absorbed by country R’s final demand. They add up to the first sector's gross output in country S: $x_1^{s} = x_1^{ss} + x_1^{sr}$.

From equation (11), the first sector's gross output in country S that is ultimately absorbed by country S’s final demand can be further decomposed into four parts: $x_1^{ss} = b_{11}^{ss} y_1^{ss} + b_{12}^{ss} y_2^{ss} + b_{11}^{sr} y_1^{rs} + b_{12}^{sr} y_2^{rs}$. The first part, $b_{11}^{ss} y_1^{ss}$ is the first sector's gross output in country S ultimately absorbed by domestic final demand of the first sector in country S. The second part, $b_{12}^{ss} y_2^{ss}$ is the gross output of the first sector in country S ultimately absorbed by the second sector's domestic final demand in country S. The third part, $b_{11}^{sr} y_1^{rs}$ is the gross output of the first sector in country S used to produce first sector's final goods exports in country R that are ultimately consumed by county S. The last part, $b_{12}^{sr} y_2^{rs}$ is the gross output of the first sector in country S used to produce second sector's final goods exports in country R that are ultimately absorbed by county S. The first two terms collectively are the first sector's gross output of country S directly absorbed at home, and the last two terms collectively are the first sector's gross output of country S that is initially exported as intermediate goods to country R which use them to produce final goods that are shipped back to country S as final goods and ultimately absorbed there. The other seven terms in the gross output decomposition matrix (11) can be interpreted in a similar way. Based on equation (11), the gross output of country R can be decomposed into the following four components according to where they are finally absorbed:

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} x_1^{rs} + x_1^{rr} \\ x_2^{rs} + x_2^{rr} \end{bmatrix} = \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \quad (12)$$

Insert equation (12) into the last term of equation (10), we can decompose Country S's gross intermediate goods exports according to where they are absorbed:

$$\begin{aligned} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\ &+ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \end{aligned} \quad (13)$$

The first term in equation (13) is the part of country S's intermediate goods exports used by Country R to produce final goods and consumed by country R; the second term is the part of country S's intermediate goods exports used by country R to produce final goods exports that are shipped back to country S; the third term is the part of country S's intermediate goods exports that are used by country R to produce intermediate exports, shipped back to country S and used by country S to produce its domestic consumed final goods; the last term is the part of country S's intermediate goods exports used by country R to produce intermediate goods exports that are shipped back to country S to produce final goods exports to country R and are consumed there. These four terms completely decompose country S's intermediate exports according to where they are finally absorbed.

From equation (2), the gross output production and use balance conditions, we know

$$\begin{aligned} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} + \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \end{aligned} \quad (14)$$

Re-arranging:

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \quad (15)$$

Define:

$L = \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1}$ as local Leontief inverse, then equation (15) can be re-written as

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \qquad (16)$$

Therefore, the intermediate goods exports by country S can also be decomposed into two components according to where it is used similar to a single country IO model:

$$\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \qquad (17)$$

Equations (13) and (17), the two important intermediate goods exports decomposition equations (in level), together with the global value-added multiplier adding-up condition defined in equation (6) and the local value-added multipliers defined below , are the key equations in deriving our gross exports decomposition formula.

From equation (6), we can obtain country S's domestic and foreign value-added multiplier as follows:

$$V^s B^{ss} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} & v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \qquad (18)$$

$$V^r B^{rs} = \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} = \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} & v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \qquad (19)$$

Also from equation (6) we know that the sum of equations (18) and (19) equals unity. In a single country IO model, domestic value-added multiplier can be calculated as

$$V^s (I - A^{ss})^{-1} = V^s L^{ss} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} & v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \qquad (20)$$

Using equations (18)-(20), and define "#" as element-wise matrix multiplication operation[4], the value of country S' gross intermediate exports can be decomposed as

$$
\begin{aligned}
\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \left\{\begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} + \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix}\right\} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix}\right\} \\
&= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix}\right\} + \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix}\right\} \\
&= \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix}\right\} \\
&+ \left\{\begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix}\right\} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix}\right\}
\end{aligned} \tag{21}
$$

Inserting equations (13) and (17) into equation (21), we can obtain the full decomposition of country S's intermediate goods exports:

$$
\begin{aligned}
\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}\right\} \\
&+ \left\{\begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix}\right\} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}\right\}
\end{aligned} \tag{22}
$$

[5]

[4] For example, when a matrix is multiplied by n $\times$ 1column vector, each row of the matrix is multiplied by the corresponding row of the vector.

Finally, based on the Leontief insight, country S's final goods exports can be decomposed into domestic and foreign value-added as follows:

$$\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \tag{23}$$

Combining equations (22) and (23), we obtain country S's gross exports decomposition equation:

$$\begin{aligned}
E^{sr} &= \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} = \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \\
&= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \quad (1) \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \quad (2) \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \right\} \quad (3) \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \quad (4) \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \right\} \quad (5) \\
&+ \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \quad (6) \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \quad (7) \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \quad (8) \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \right\} \quad (9)
\end{aligned} \tag{24}$$

Similarly, we can derive the decomposition of country R's gross exports in a similar way. To save space, we list the equation in appendix B.

Equation (24) indicates that the gross exports of a country can be completely

[5] For a detailed derivation, please see Appendix B.

decomposed into the sum of nine terms. It is an extension of equation (13) in KWW; with the domestic pure double counting term being further split by production related to final and intermediate goods exports respectively. To better understand each term in this accounting equation, we provide the following detailed economic interpretations:

The first term is domestic value added embodied in the final exports of the first and second sectors in country S. Each of them has two parts: domestic value added created by the sector itself and domestic value added created by the other sector embodied in the sector's final exports.

The second term is domestic value added embodied in country S's first and second sector's intermediate exports which are used by country R to produce final goods, y_1^{rr} and y_2^{rr}, and are consumed in R.

These two terms are domestic value added embodied in country S's gross exports which are ultimately absorbed by country R. They are value added exports of country S.

The third term is domestic value added embodied in country S's first and second sector's intermediate exports used to produce country R's final exports, i.e. country S's imports of final goods from R.

The fourth term is domestic value added embodied in country S's first and second sector's intermediate exports that are used by country R to produce intermediate exports and return to country S via its intermediate imports to produce its domestic final goods. These two terms are domestic value added embodied in the first and second sector's gross exports which returned home and are finally consumed in country S.

The first four terms are the domestic value added (GDP) embodied in the first and second sectors' gross exports of country S, which include value-added created from all sectors in country S.

The fifth term is domestic value added of country S's first and second sector's intermediate exports which return home as its first and/or second sector's intermediate imports and are used for production of country S's both sector's final exports and are finally consumed in country R. They are part of the value-added in country S's final exports and already counted once by the first term of equation (24), therefore it is a domestic double counted portion caused by the back and forth intermediate goods trade in order to produce final goods exports in country S.

The sixth term is domestic value added of country S's first and second sector's intermediate exports that return home as intermediate imports and are used for production of country S's intermediate exports to country R. It is also a domestic double counted portion caused by the back and forth intermediate goods trade in order to produce intermediate goods exports in country S.

Since the second to the sixth terms come from the first 5 terms of equation (22), their sum equals domestic content of the first and second sector's intermediate exports of country S. Therefore, sum of the first to the sixth term is domestic content of the first and second sector's gross exports, $\sum_{i}^{2} v_i^s b_{i1}^{ss} e_1^{sr}$ and $\sum_{i}^{2} v_i^s b_{i2}^{ss} e_2^{sr}$. A detailed mathematical proof is given in Appendix B.

The seventh term is foreign value added used in country S's first and second sector's final goods exports. Each of them also has two parts: foreign value-added from the sector itself and the other sector used to produce final exports from country S. Adding up the first and the seventh terms accounts 100% of the value of the final exports in country S by sector.

The eighth term is foreign value added used to produce the first and second sector intermediate exports of country S, which are then used by country R to produce its domestic final goods. Summing the seventh and eighth term, the two elements in the resulted vector are total foreign value added embodied in the first and second sectors' exports of country S, respectively.

The ninth term is foreign value added embodied in the first and second sector's intermediate exports used by country R to produce its final and intermediate exports, which is the foreign double counted term of country S's exports. Adding up the eighth and ninth term yields the foreign content of the first and second sector's intermediate exports.

Therefore, the seventh to the ninth terms are the foreign content of the first and second sector's gross exports of country S, $\sum_{i}^{2} v_i^r b_{i1}^{rs} e_1^{sr}$ and $\sum_{i}^{2} v_i^r b_{i2}^{rs} e_2^{sr}$.

It is easy to show that the aggregation of the two sectors in equation (24) results in equation (13) in KWW. A detailed proof is given in Appendix D.

2.2.2 Two perspectives on domestic value added exports

As discussed in section 2.1, domestic value added exports at the sector level can be measured from two perspectives. They can be estimated by aggregating the $\hat{V}BY$ matrix along different directions. The first measure is based on the forward linkages in the IO literature by summing up the off-diagonal elements across the columns along the rows in the $\hat{V}BY$ matrix. It measures how a country's GDP by industry is used to produce exports that are absorbed by the destination countries. It is consistent with the factor content method in the international trade literature and is the same as what is defined in Johnson and Norgera (2012) and equation (9) (for the two country case) or (30) (for the general case of G countries and N sectors) in KWW (2014).

The second measure is based on backward linkages in the IO literature. It decomposes a particular sector’s final products according to its value-added sources. It measures each source country's value-added embodied in particular sector’s gross export flows, regardless of value-added creating sectors in the source country, that is absorbed by each destination country. This measure is consistent with the GVC case studies in the literature. At the country aggregate bilateral level, these two value-added trade flow measures are exactly the same, but at the sector level they are quite different.

Without loss of generality, let us use vt_1^{sr} to denote value-added exports by the first sector of country S (producer’s perspective, forward-looking linkage) and $vt(e_1^{sr})$ as country S's domestic value-added in the first sector's gross exports that is absorbed in country R (user perspective, backward looking linkage). Then vt_1^{sr} and $vt(e_1^{sr})$ can be fully decomposed as follows:

$$vt_1^{sr} = \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \tag{25}$$
$$= v_1^s b_{11}^{ss} y_1^{sr} + v_1^s b_{12}^{ss} y_2^{sr} + v_1^s b_{11}^{sr} y_1^{rr} + v_1^s b_{12}^{sr} y_2^{rr}$$

which is an extension of equation (9) of KWW in the two sector case.

$$vt\left(e_1^{sr}\right) = \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix}^T \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} + \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix}^T \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \quad (26)$$
$$= v_1^s b_{11}^{ss} y_1^{sr} + v_2^s b_{21}^{ss} y_1^{sr} + v_1^s l_{11}^{ss} \sum_i^2 \sum_j^2 a_{1i}^{sr} b_{ij}^{rr} y_j^{rr} + v_2^s l_{21}^{ss} \sum_i^2 \sum_j^2 a_{1i}^{sr} b_{ij}^{rr} y_j^{rr}$$

which is part of the first two terms in equation (24).

Based on the properties of Leontief Inverse, we know

$$\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (27)$$

Therefore,

$$\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} + \begin{bmatrix} -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{sr} & -a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} = 0 \quad (28)$$

Rearrange:

$$\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \quad (29)$$

Inserting equation (29) into equation (25) and re-arrange, we have

$$vt_1^{sr} = \begin{bmatrix} v_1^s & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & 0 \end{bmatrix} \left\{ \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \quad (30)$$
$$= v_1^s b_{11}^{ss} y_1^{sr} + v_1^s b_{12}^{ss} y_2^{sr} + v_1^s l_{11}^{ss} \sum_i^2 \sum_j^2 a_{1i}^{sr} b_{ij}^{rr} y_j^{rr} + v_1^s l_{12}^{ss} \sum_i^2 \sum_j^2 a_{2i}^{sr} b_{ij}^{rr} y_j^{rr}$$

Comparing equations (26) and (30), the first and third term of the measures of value-added exports are the same. They are value-added created from the first sector of country S embodied in country S's first sector's gross exports, or the direct value-added exports of the first sector. However, the second and the last term of the two measures are different. Therefore, the difference between vt_1^{sr} and $vt(e_1^{sr})$ equals

$$vt_1^{sr} - vt\left(e_1^{sr}\right) = \left[v_1^s b_{12}^{ss} y_2^{sr} + v_1^s l_{12}^{ss} \sum_j^2 \sum_k^2 a_{2j}^{sr} b_{jk}^{rr} y_k^{rr} \right] - \left[v_2^s b_{21}^{ss} y_1^{sr} + v_2^s l_{21}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rr} y_k^{rr} \right] \quad (31)$$

The two terms in the first square brackets of equation (31) are the second and last term from equation (30), representing value added created by the first sector of country S but embodied in gross exports of the second sector in country S and are finally consumed in country R (indirect value-added exports of the second sector embodied in the first sector's gross exports) and hence has no relation with the first sector's gross exports. The two terms in the second square brackets of equation (31) are the second and last term from equation (26), representing the second sector's value added that is embodied in the gross exports of the first sector produced by country S and is finally consumed in country R (indirect value-added exports of the second sector embodied in the first sector's gross exports). Unless these indirect value-added exports terms equal to each other, country S's value-added exports from its first sector cannot be equal to its domestic value-added embodied in the first sector's gross exports absorbed in country R.

Similarly, the difference between $vt(e_2^{sr})$ and vt_2^{sr} equals

$$vt\left(e_2^{sr}\right) - vt_2^{sr} = \left[v_1^s b_{12}^{ss} y_2^{sr} + v_1^s l_{12}^{ss} \sum_i^2 \sum_j^2 a_{2i}^{sr} b_{ij}^{rr} y_j^{rr}\right] - \left[v_2^s b_{21}^{ss} y_1^{sr} + v_2^s l_{21}^{ss} \sum_i^2 \sum_j^2 a_{1i}^{sr} b_{ij}^{rr} y_j^{rr}\right] \quad (32)$$

It is easy to show that the sum of equations (31) and (32) equals 0. This means that when aggregating the two sectors together, the difference between country S's value added exports and country S's domestic value-added in gross exports absorbed in country R at the sector level cancels out. Therefore, at country aggregate, the two value-added exports measures should equal each other.

Extending the equation (31) and (32) to the n-sector case, the value-added exports to country R produced by sector k of country S vt_k^{sr} and the country S's domestic value-added in sector k's gross exports absorbed in country R can be expressed as

$$vt_k^{sr} = \sum_i^n v_k^s b_{ki}^{ss} y_i^{sr} + \sum_i^n v_k^s b_{ki}^{sr} y_i^{rr} = \sum_i^n v_k^s b_{ki}^{ss} y_i^{sr} + \sum_i^n \sum_j^n \sum_u^n v_k^s l_{ki}^{ss} a_{ij}^{sr} b_{ju}^{rr} y_u^{rr} \quad (33)$$

$$vt(e_k^{sr}) = \sum_i^n v_i^s b_{ik}^{ss} y_k^{sr} + \sum_i^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rr} y_u^{rr} \quad (34)$$

It is easy to show:

$$vt(e_k^{sr}) - vt_k^{sr} = \left[\sum_{i\neq k}^n v_i^s b_{ik}^{ss} y_k^{sr} + \sum_{i\neq k}^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rr} y_u^{rr}\right] - \left[\sum_{i\neq k}^n v_k^s b_{ki}^{ss} y_i^{sr} + \sum_{i\neq k}^n \sum_j^n \sum_u^n v_k^s l_{ki}^{ss} a_{ij}^{sr} b_{ju}^{rr} y_u^{rr}\right] \quad (35)$$

The two terms in the first square brackets of equation (35) are other sectors' value added embodied in sector k's gross exports produced by country S and finally consumed in country R in final and intermediate goods respectively. They increase the domestic value-added in sector k's gross exports. Similarly, the two terms in the second square brackets of equation (35) are the value added created by sector k but embodied in other sectors' intermediate goods respectively. Thus they reduce the value-added created in sector k that can be embodied in sector k's gross exports. Therefore, the two measures of value-added exports at the sector level are not equal in general. Understanding this fact is important for us to define the value-added to gross exports ratio at the country/sector level properly. We will discuss this in more details when the third country effect can be explicitly accounted in the three country model. Here will give the following proposition and sketch a mathematical proof.

Proposition A: In a two country world (home country S and rest of the world R), $vt(e_k^{sr})$, the sector level value-added exports measure based on backward linkage, is always less than or equal to sector level gross exports. Therefore value-added exports to gross exports ratio is upper-bounded at 1.

Proof:

From equation (6), the sum of total value added coefficient equals unity, i.e

$$\sum_i^n v_i^s b_{ik}^{ss} + \sum_i^n v_i^r b_{ik}^{rs} = \sum_i^n v_i^r b_{ik}^{rr} + \sum_i^n v_i^s b_{ik}^{sr} = 1 \qquad (k = 1 \quad \cdots \quad n)$$

$$\sum_i^n v_i^s l_{ik}^{ss} + \sum_i^n \sum_j^n a_{ji}^{rs} l_{ik}^{ss} = \sum_i^n v_i^r l_{ik}^{rr} + \sum_i^n \sum_j^n a_{ji}^{sr} l_{ik}^{rr} = 1 \qquad (k = 1 \quad \cdots \quad n)$$

Therefore,

$$\sum_i^n v_i^s b_{ik}^{ss} \leq 1, \quad \sum_i^n v_i^r b_{ik}^{rr} \leq 1, \quad \sum_i^n v_i^s l_{ik}^{ss} \leq 1, \quad \sum_i^n v_i^r l_{ik}^{rr} \leq 1$$

Insert these inequalities into equation (34):

$$vt(e_k^{sr}) = \sum_i^n v_i^s b_{ik}^{ss} y_k^{sr} + \sum_i^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rr} y_u^{rr} \leq y_k^{sr} + a_{kj}^{sr} b_{ju}^{rr} y_u^{rr} \leq e_k^{sr}$$

Rearranging the above inequality, we have

$$\frac{vt\left(e_k^{sr}\right)}{e_k^{sr}} \leq 1$$

Proposition B: In a two country world (home country S and rest of the world R), vt_k^{sr}, the sector level value-added exports measure based on forward linkage is always less than or equal to sector level value-added production. Therefore, the value-added export to GDP by industry ratio is upper-bounded at 1.

Proof:

$$vt_k^{sr} = \sum_i^n v_k^s b_{ki}^{ss} y_i^{sr} + \sum_i^n v_k^s b_{ki}^{sr} y_i^{rr} = v_k^s \left[\sum_i^n b_{ki}^{ss} y_i^{sr} + \sum_i^n b_{ki}^{sr} y_i^{rr} \right] \leq v_k^s x_k^s$$
$$= v_k^s \left[\sum_i^n b_{ki}^{ss} y_i^{ss} + \sum_i^n b_{ki}^{ss} y_i^{sr} + \sum_i^n b_{ki}^{sr} y_i^{rs} + \sum_i^n b_{ki}^{sr} y_i^{rr} \right]$$

The intuition behind these two propositions is simple. As shown earlier in the two sector case， the sector level direct value-added exports are the same for both value-added trade measures ($v_k^s b_{kk}^{ss} y_k^{sr} + v_k^s l_{kk}^{ss} a_{kj}^{sr} b_{ju}^{rr} y_u^{rr}$), but the sector level indirect value-added exports can be very different. The indirect value-added exports in the forward looking linkage based value-added exports measure ($\sum_{i \neq k}^n v_k^s b_{ki}^{ss} y_i^{sr} + \sum_{i \neq k}^n \sum_j^n \sum_u^n v_k^s l_{ki}^{ss} a_{ij}^{sr} b_{ju}^{rr} y_u^{rr}$) are the sector's value-added embodied in other sector's gross exports, which have no relation with the sectors gross exports. Therefore, the value-added exports to gross exports ratio defined by Johnson and Noguera at the sector level is not proper since its denominator (sector gross exports) does not include the indirect value-added exports from other sectors in its denominator. It is common that a sector has no gross exports, but its products are used by other domestic industries as intermediate inputs and thus it has indirect value-added exports through other sectors. In such case, the VAX ratio will become infinitive. However, because such indirect value-added exports are part of the total value-added created by the same sector, a value-added exports to GDP by industry ratio can be properly defined based on forward looking linkage.

2.2.3 Domestic value-added in gross exports at the sector level

Following KWW (2014), we define a country's value-added exports differently from "domestic value added in a country's gross exports". The latter is the sum of a country's value-added exports and its domestic value-added that was first exported but eventually returns and is consumed at home. The second concept only considers where the value added is originated regardless where it is ultimately absorbed. In comparison, a country's "value added exports" refers to a subset of "domestic value added in a country's gross exports" that is ultimately absorbed abroad. Such a conceptual difference naturally extends to the sector level measures and can be computed from two different directions as the sector level measure of value-added exports.

Based on the Leontief insight and equation (7), and using equation (29) and (A4) country S's GDP by sector can be decomposed as

$$
\begin{aligned}
GDP^s &= \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \right\} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \right\} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \right\} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&= \begin{bmatrix} \sum_i^2 v_1^s b_{1i}^{ss} y_i^{sr} \\ \sum_i^2 v_2^s b_{2i}^{ss} y_i^{sr} \end{bmatrix} + \begin{bmatrix} \sum_i^2 v_1^s l_{1i}^{ss} \sum_j^2 \sum_k^2 a_{ij}^{sr} b_{jk}^{rr} y_k^{rr} \\ \sum_i^2 v_2^s l_{2i}^{ss} \sum_j^2 \sum_k^2 a_{ij}^{sr} b_{jk}^{rr} y_k^{rr} \end{bmatrix} + \begin{bmatrix} \sum_i^2 v_1^s l_{1i}^{ss} \sum_j^2 \sum_k^2 a_{ij}^{sr} b_{jk}^{rr} y_k^{rs} \\ \sum_i^2 v_2^s l_{2i}^{ss} \sum_j^2 \sum_k^2 a_{ij}^{sr} b_{jk}^{rr} y_k^{rs} \end{bmatrix} \\
&+ \begin{bmatrix} \sum_i^2 v_1^s l_{1i}^{ss} \sum_j^2 \sum_k^2 a_{ij}^{sr} b_{jk}^{rs} y_k^{ss} \\ \sum_i^2 v_2^s l_{2i}^{ss} \sum_j^2 \sum_k^2 a_{ij}^{sr} b_{jk}^{rs} y_k^{ss} \end{bmatrix} + \begin{bmatrix} \sum_i^2 v_1^s l_{1i}^{ss} y_i^{ss} \\ \sum_i^2 v_2^s l_{2i}^{ss} y_i^{ss} \end{bmatrix} \qquad (36)
\end{aligned}
$$

Country R's GDP can be expressed in a similar way:

$$
\begin{aligned}
GDP^r &= \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \left\{ \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \right\} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix} \# \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&= \begin{bmatrix} \sum_i^2 v_1^r b_{1i}^{rr} y_i^{rs} \\ \sum_i^2 v_2^r b_{2i}^{rr} y_i^{rs} \end{bmatrix} + \begin{bmatrix} \sum_i^2 v_1^r l_{1i}^{rr} \sum_j^2 \sum_k^2 a_{ij}^{rs} b_{jk}^{ss} y_k^{ss} \\ \sum_i^2 v_2^r l_{2i}^{rr} \sum_j^2 \sum_k^2 a_{ij}^{rs} b_{jk}^{ss} y_k^{ss} \end{bmatrix} + \begin{bmatrix} \sum_i^2 v_1^r l_{1i}^{rr} \sum_j^2 \sum_k^2 a_{ij}^{rs} b_{jk}^{ss} y_k^{sr} \\ \sum_i^2 v_2^r l_{2i}^{rr} \sum_j^2 \sum_k^2 a_{ij}^{rs} b_{jk}^{ss} y_k^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} \sum_i^2 v_1^r l_{1i}^{rr} \sum_j^2 \sum_k^2 a_{ij}^{rs} b_{jk}^{sr} y_k^{rr} \\ \sum_i^2 v_2^r l_{2i}^{rr} \sum_j^2 \sum_k^2 a_{ij}^{rs} b_{jk}^{sr} y_k^{rr} \end{bmatrix} + \begin{bmatrix} \sum_i^2 v_1^r l_{1i}^{rr} y_i^{rr} \\ \sum_i^2 v_2^r l_{2i}^{rr} y_i^{rr} \end{bmatrix}
\end{aligned} \tag{37}
$$

Subtracting global GDP from global gross exportsyields residuals as follows:

$$
\begin{aligned}
&\mu E^{sr} + \mu E^{rs} - \mu GDP^s - \mu GDP^r = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}^{-1} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \left\{ \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ 2 \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} + 2 \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} - \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \right\} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \\
&- \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} - \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}
\end{aligned} \tag{38}
$$

(38)[6]

Equation (38) shows clearly that besides the value added produced and consumed at home (the last two terms), which is not a part of either country's gross exports, the seventh and eighth term in equation (24) (the second term in (38)), and the seventh and eighth terms in the gross exports decomposition equations of country R (the first term in (38)) given in Appendix B, are double counted only once as foreign value-added in the other country's gross exports. Because the third and fourth terms in (24) reflect part of the countries' GDP, they are not double counted from the global GDP point of view. In

[6] A step by step derivation is given in Appendix E.

comparison, the third and fourth term in equation (38) and the same with the fifth, sixth and ninth term in equations (24) are counted twice relative to the global GDP since they are not a part of either country's GDP. This explains the reason why we would like to label the fifth, sixth and ninth term in equations (24) as the "pure" double counted terms to differentiate them from those double counted domestic and foreign value-added in gross export statistics (the third, fourth, seventh and eighth term in the gross exports decomposition equations). The pure double counted terms are greater than zero only when there is two-way intermediate goods trade as pointed by KWW (forthcoming).

Just as the sector level measures of value-added exports can be defined from either the supply-side or user's perspective (i.e., based forward and backward looking linkage), the sector level domestic value-added in gross exports can also be defined in these two different directions.

The user's perspective measure for the sector level domestic value-added in sector gross exports for country S can be defined by directly taking the first four terms from equation (24) as follows:

$$\begin{aligned} dv(E^{sr}) = &\begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\ &+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \right\} \\ &+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \end{aligned} \tag{39}$$

Obviously, it is the domestic value-added portion of country S's gross exports.

To define measure of sector level domestic value-added in country S's gross exports from the producer's perspective, we start from the first four terms of equation (36)

$$\begin{aligned} dv^{sr} = &\begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix} \# \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \end{aligned} \tag{40}$$

In fact, they are equivalent to summing up the $\hat{V}BY$ matrix in equation (16) across columns along the rows and then subtracting the part of domestic value added that is directly consumed at home, as the last item is not a part of the either country's exports as shown in equation (38). Equation (40) is a generalization of equation (22) in KWW for the two-country and two-sector setting. Detailed derivations can be found in Appendix E**.**

2.2.4 A numerical example for the 2-country 2-sector case

We now provide a numerical example to illustrate various concepts discussed above. Suppose a simple 2-country, 2-sector ICIO table as summarized in table 2 below:

Table 2

Country / Sector		S		R		YS	YR	Gross output
		S1	S2	R1	R2			
S	S1	1	1	0	0	1	0	3
	S2	0	1	0	1	0	1	3
R	R1	1	0	1	0	0	1	3
	R2	0	0	1	1	1	0	3
Value-added		1	1	1	1			
Total input		3	3	3	3			

Gross intermediate and final good exports matrix is

$$E = EI + EF = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

The direct input coefficient matrix A, Global Leontief inverse Matrix B, Local Leontief inverse matrix L and direct value-added coefficient vector V can be easily computed as

$$A = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 \end{bmatrix} \quad B = \begin{bmatrix} 8/5 & 4/5 & 1/5 & 2/5 \\ 1/5 & 8/5 & 2/5 & 4/5 \\ 4/5 & 2/5 & 8/5 & 1/5 \\ 2/5 & 1/5 & 4/5 & 8/5 \end{bmatrix} \quad L = \begin{bmatrix} 3/2 & 3/4 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 3/2 & 0 \\ 0 & 0 & 3/4 & 3/2 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Using decomposition equation (24), we can fully decompose country S and R’s gross exports into the nine value-added and double counted components as reported in table 3, detailed computation is listed in Appendix F.

Table 3: Gross exports decomposition results: 2-country, 2-sector numerical example

E	T1	T2	T3	T4	T5	T6	T7	T8	T9	VAX_F J&N	VAX_B WWZ	VAX_B Ratio	VAX_F Ratio
S1	0	0	0	0	0	0	0	0	0	1/3	0	0%	∞
S2	4/5	1/5	2/5	1/10	1/20	1/20	1/5	1/20	3/20	2/3	1	50%	33%
R1	0	2/5	1/5	1/20	1/10	1/20	0	1/10	1/10	1/3	2/5	40%	33%
R2	3/5	0	0	0	0	0	2/5	0	0	2/3	3/5	60%	67%
ST	4/5	1/5	2/5	1/10	1/20	1/20	1/5	1/20	3/20	1	1	50%	50%
RT	3/5	2/5	1/5	1/20	1/10	1/20	2/5	1/10	1/10	1	1	50%	50%

This example shows that the two measures of value-added exports only equal to each other at the aggregate level and are different at the sector level. It also shows that when a sector does not have gross exports (S1), but its output is used as intermediate inputs for the other domestic sector that exports (S2), we will have an infinitive VAX ratio. Only the new backward linkage based VAX ratio defined in this paper has the desired property and economic interpretations at the country-sector level as we demonstrate analytically in the previous sub-section.

	Terms in Table 3 and equation (24)
T1	Domestic value added in final exports
T2	Domestic value added in intermediate exports used to produce importer’s domestic final goods
T3	Domestic value added in intermediate exports used to produce importer’s final exports
T4	Domestic value added in intermediate exports used to produce importer’s intermediate exports and return home country to produce its domestic final goods
T5	Domestic value added in intermediate exports first used to produce importer’s intermediate exports and eventually return to home country to produce its final exports, Domestic double counted term i due to the production of final exports
T6	Domestic value added in intermediate exports used to produce importer’s intermediate exports and return home country to produce its intermediate exports, Domestic double counted term due to the production of intermediate exports
T7	Foreign value added in final exports
T8	Foreign value added in intermediate exports used to produce importer’s domestic final goods
T9	Foreign value added in intermediate exports used to produce importer’s gross exports, Foreign double counted term due to the production of intermediate and final exports.

2.3 The Case of Three Countries and Two Sectors

Examining a three-country case in detail is useful for two reasons: (i) it exhibits nearly all the richness of the fully general multi-country analysis, and (ii) analytical solutions remain tractable and continue to have intuitive explanations.

We use a superscript s, to represent the source country, r to represent the destination country, and t to represent the third country and define the country set G = {s, r, t}. Based on the Leontief insight, from a three-country two-sector ICIO model we can obtain country R's gross output in terms of final demand as follows[7]:

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} x_1^{rs} + x_1^{rr} + x_1^{rt} \\ x_2^{rs} + x_2^{rr} + x_2^{rt} \end{bmatrix} = \begin{bmatrix} \sum_l^G \sum_g^G \sum_j^2 b_{1j}^{rg} y_j^{lg} \\ \sum_l^G \sum_g^G \sum_j^2 b_{2j}^{rg} y_j^{lg} \end{bmatrix} \qquad l, g \in G \tag{41}$$

Therefore, the gross output of country R can be decomposed into the following nine components according to where they are finally absorbed:

$$\begin{aligned} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{st} \\ y_2^{st} \end{bmatrix} \\ &+ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \\ &+ \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{ts} \\ y_2^{ts} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} + \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \end{aligned} \tag{42}$$

Insert equation (42) into the last term of equation (10), we can decompose country S's gross intermediate goods exports according to where and how they are absorbed as follows:

$$\begin{aligned} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} \\ &+ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} \\ &+ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{ts} \\ y_2^{ts} \end{bmatrix} \\ &+ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \left\{ \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} y_1^{st} \\ y_2^{st} \end{bmatrix} \right\} \end{aligned} \tag{43}$$

Comparing equation (43) with equation (13), the intermediate goods exports decomposition equation in 2-country, 2-sector case, the first, fifth, and seventh term in equation (43) are exactly the same as the first three terms in equation (13) and have the similar economic interpretations. They are part of country S's intermediate goods exports used by partner country R to produce final goods and are consumed there, to produce

[7] Detailed gross output decomposition matrix is given in Appendix A

final goods exports that are shipped back to country S, and to produce intermediate exports and are shipped back to country S and are used by country S to produce domestically consumed final goods. The last term in the two equations also has similar economic interpretation. Both are part of country S's intermediate goods exports used by importing country R to produce intermediate goods exports that are shipped back to country S to produce its final goods exports that are consumed abroad. However, in equation (13), the term only has final goods exports to country R, while in equation (43), the term also includes final goods exports to the third country, country T.

Four additional terms appear in equation (43), the sencond, third, fourth and sixth term, all of which are related to the third country, T. The second term is country S's intermediate exports used by the direct importer, country R, to produce intermediate good that is exported to the third country T for production of finally goods consumed in country T The third term is country S's intermediate exports used by the direct importer, country R, to produce final exports which are ultimately absorbed by the third country T. The fourth term is country S's intermediate exports used by the direct importer country R to produce intermediate exports to the third country T for production of final exports that return back to the direct importer (R). The sixth term is country S's intermediate exports used by the direct importer(R) to produce intermediate exports to the third country T for its production of final exports that return back to country S. These additional terms are measures of different pattern of production sharing that involve more than two countries (direct trade partners) and only can be observed in a three or more country models.

The eight terms in equation (43) completely decompose country S's intermediate exports according to where and how they are finally absorbed.

In the three-country ICIO model, the gross output production and use balance, or the row balance condition becomes:

$$\begin{aligned}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} + \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} a_{11}^{rt} & a_{12}^{rt} \\ a_{21}^{rt} & a_{22}^{rt} \end{bmatrix}\begin{bmatrix} x_1^t \\ x_2^t \end{bmatrix} + \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} + \begin{bmatrix} e_1^{rt} \\ e_2^{rt} \end{bmatrix} = \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix}\end{aligned} \tag{44}$$

Re-arrange:

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}^{-1} \begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix}$$
$$= \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix} \tag{45}$$

Therefore, the intermediate goods exports by country S can also be decomposed into two components according to where it is used similar to a single-country IO model:

$$\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix} \tag{46}$$

Equation (46) is almost the same as equation (17), except its last term on the RHS also includes country S's exports to country T, i.e country S's total gross exports to the world.

It is important to note that the value-added multipliers of country S and R are exactly the same in the 3-country, 2-sector model as in the 2-country, 2-sector case. As specified in equations (18) and (19), the value-added multiplier of country T can be defined in a similar way:

$$V^t B^{ts} = \begin{bmatrix} v_1^t & v_2^t \end{bmatrix} \begin{bmatrix} b_{11}^{ts} & b_{12}^{ts} \\ b_{21}^{ts} & b_{22}^{ts} \end{bmatrix} = \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} & v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \tag{47}$$

and the sum of equations (18),(19) and (47) equals unity, i.e:

$$V^s B^{ss} + V^r B^{rs} + V^t B^{ts} = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{48}$$

Using equations (18), (19), (47) and (20), the value of country S' gross intermediate exports in the 3-country, 2-sector model can be decomposed in a similar way as the 2-country 2-sector case as follows:

$$\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} = \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\}$$
$$+ \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\} \tag{49}$$
$$+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\} + \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\}$$

Insert equations (43) and (46) into equation (49), we can obtain the full decomposition of country S's intermediate goods exports in the 3-country, 2-sector model as:

$$
\begin{aligned}
\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} &= \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix}\begin{bmatrix} y_1^{ts} \\ y_2^{ts} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\left\{\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} y_1^{st} \\ y_2^{st} \end{bmatrix}\right\} \\
&+ \left\{\begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix}\right\} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix}\right\} \\
&+ \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix}\right\}
\end{aligned} \tag{50}
$$

Finally, based on the Leontief insight, country S's final goods exports can be decomposed into country S, R, and T 's value-added as follows:

$$
\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \tag{51}
$$

Combine equations (50) and (51), we obtain the gross exports decomposition equation in the 3-country, 2-sector model:

$$
\begin{aligned}
\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} &= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{ts} \\ y_2^{ts} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \left\{ \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} y_1^{st} \\ y_2^{st} \end{bmatrix} \right\} \\
&+ \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^r b_{11}^{rs} + v_2^r b_{21}^{rs} \\ v_1^r b_{12}^{rs} + v_2^r b_{22}^{rs} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^t b_{11}^{ts} + v_2^t b_{21}^{ts} \\ v_1^t b_{12}^{ts} + v_2^t b_{22}^{ts} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{r*} \\ e_2^{r*} \end{bmatrix} \right\}
\end{aligned}
\tag{52}
$$

Equation (52) is similar to equation (24) but with seven additional terms, all of which involve the third country T.

Four of them are domestic value-added components. The third term is country S’s domestic value-added in its intermediate exports used by the direct importer (country R) to produce intermediate exports to the third country T for production of county T's domestic final goods; the fourth term is domestic value-added in country S’s intermediate exports used by the direct importer (country R) producing final goods exports to the third country T; the fifth term is domestic value-added in country S’s intermediate exports used by the direct importer (R) to produce intermediate exports to the third country T for its production of final goods exports that are shipped back to the direct importer country R;

and the seventh term is domestic value-added in country S's intermediate exports used by the direct importer (R) to produce intermediate exports to the third country T for its production of final goods exports that are shipped back to the source country S.

Two of the seven additional terms are foreign value added components. The thirteenth term is foreign value added from the third country T used by country S's first and second sectors to produce final exports from country S. Adding up the first (domestic value-added from source country S), eleventh (foreign value-added from country R) and thirteenth term (foreign value-added from country T) accounts for 100% of the value of the final exports in country S by sector.

The fourteenth term is foreign value added from the third country T used to produce the first and second sectors' intermediate exports of country S, which are then used by country R to produce its domestic final goods. Summing the eleventh though the fourteenth term, the two elements in the resulted vector are total foreign value added embodied in the first and second sectors' gross exports of country S respectively.

Similar to the ninth term in equation (24), the last two terms in equation (52) are foreign value added (value-added from country R and T) embodied in the first and second sectors' intermediate exports used by country R to produce its final and intermediate goods exports to the world (sum of exports to country S and T), which are foreign double counted terms in country S's gross exports.

Coming to the rest of the six terms in equation (52), the first, second, sixth, eighth, ninth and tenth terms have similar economic interpretations as the first six terms in equation (24), so we do not repeat them here to save space. The sixteen terms completely decompose bilateral gross exports from country S to country R into different value-added and double counted components, and their sum equals 100% of bilateral trade flows at the sector level. The complete gross exports accounting at the bilateral/sector level made by equations (52) is also diagrammed in Figure 1.

With equation (52), our bilateral gross exports decomposition equation in hand, we are ready to discuss how to properly define the value-added exports to gross exports ratio (VAX ratio) and double counting measure at the bilateral and the sector level. It is easy to show that the VAX ratio based on forward-looking linkage as defined by Johnson and

Noguera (2012) is not well behaved at the bilateral/sector level and cannot be used as a summary measure of value-added content of gross exports and double counting except at the country aggregate level. The proper measure can be defined at the bilateral/sector level is domestic value-added in exports that are absorbed in foreign countries. The reason is very intuitive. Since bilateral value-added trade between country S and R is value-added produced in country S but absorbed in country R by definition, due to indirect value-added trade via third countries, two countries can have a large volume of value added trade (measured by forward linkages) even when they have no gross trade. Therefore, it is not possible to define a well-behaved ratio of value-added exports to gross exports that is upper bounded at 1, which can be used as a summary measure of value-added content of gross trade and double counting at the bilateral level. The correct measure is the share of domestic value-added that is absorbed abroad in gross exports. No destination or partner country constraints should be imposed as long as the value-added finally stays and is absorbed abroad. Decomposition equation (52) shows clearly that bilateral value-added trade flows between two countries could deviate from gross bilateral trade due to the existence of terms 3, 4, 5. While summing up 100% gross bilateral exports flows, these value-added terms have to be counted. However, when the bilateral/sector decomposition is aggregated to the country/sector level, the VAX ratio computed based on backward linkage equals domestic value-added in exports that stay abroad. At the country/sector level, because there is no need to impose any particular destination country (as long as no value added returns home), the definition of value-added exports can be consistently maintained. In such a case, only the backward linkage based VAX ratio defined in this paper is upper bounded at 1, while the forward linkage based VAX ratio is not.

Proposition C: In a three or more country world, $dv\left(e_k^{sr}\right)$, domestic value-added in gross bilateral exports of sector k of country S, is always less than or equal to e_k^{sr}, the sector level bilateral gross exports. Therefore domestic value-added in exports to gross exports ratio is upper-bounded at 1.

Proof: From equation (52), domestic value-added in gross exports equals the sum of the first eight terms:

$$
\begin{aligned}
\begin{bmatrix} e_1^{sr} \\ 0 \end{bmatrix} &= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tt} \\ y_2^{tt} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rt} \\ y_2^{rt} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{tr} \\ y_2^{tr} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rt} & b_{12}^{rt} \\ b_{21}^{rt} & b_{22}^{rt} \end{bmatrix} \begin{bmatrix} y_1^{ts} \\ y_2^{ts} \end{bmatrix} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}
\end{aligned}
$$

Re-write into scalar notation:

$$
\begin{aligned}
dv\left(e_1^{sr}\right) &= \sum_i^2 v_i^s b_{i1}^{ss} y_1^{sr} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_u^2 a_{1j}^{sr} b_{ju}^{rr} y_u^{rr} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_u^2 a_{1j}^{sr} b_{ju}^{rt} y_u^{tt} \\
&+ \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_u^2 a_{1j}^{sr} b_{ju}^{rr} y_u^{rt} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_u^2 a_{1j}^{sr} b_{ju}^{rt} y_u^{tr}
\end{aligned} \tag{53}
$$

Extend to N sectors:

$$
\begin{aligned}
dv\left(e_k^{sr}\right) &= \sum_i^n v_i^s b_{ik}^{ss} y_k^{sr} + \sum_i^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rr} y_u^{rr} + \sum_i^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rt} y_u^{tt} \\
&+ \sum_i^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rr} y_u^{rt} + \sum_i^n \sum_j^n \sum_u^n v_i^s l_{ik}^{ss} a_{kj}^{sr} b_{ju}^{rt} y_u^{tr}
\end{aligned} \tag{54}
$$

Since both $\sum_i^n v_i^s b_{ik}^{ss}$ and $\sum_i^n v_i^s l_{ik}^{ss}$ are less than 1,

$$
\begin{aligned}
dv\left(e_k^{sr}\right) &\le y_k^{sr} + a_{kj}^{sr} b_{ju}^{rr} y_u^{rr} + a_{kj}^{sr} b_{ju}^{rt} y_u^{tt} + a_{kj}^{sr} b_{ju}^{rr} y_u^{rt} a_{kj}^{sr} b_{ju}^{rt} y_u^{tr} \\
\le e_k^{sr} &= y_k^{sr} + a_{kj}^{sr} b_{ju}^{rr} y_u^{rr} + a_{kj}^{sr} b_{ju}^{rt} y_u^{tt} + a_{kj}^{sr} b_{ju}^{rr} y_u^{rt} a_{kj}^{sr} b_{ju}^{rt} y_u^{tr} + a_{kj}^{sr} b_{ju}^{rr} y_u^{rs} \\
&+ a_{kj}^{sr} b_{ju}^{rt} y_u^{ts} + a_{kj}^{sr} b_{ju}^{rs} y_u^{ss} + a_{kj}^{sr} b_{ju}^{rs} y_u^{sr} + a_{kj}^{sr} b_{ju}^{rs} y_u^{st}
\end{aligned} \tag{55}
$$

$e_k^{sr} \ge dv\left(e_k^{sr}\right)$, i.e., $\dfrac{dv\left(e_k^{sr}\right)}{e_k^{sr}} \le 1$.

2.3.2 A numerical example

Relative to the example in Section 2.2.4, an additional country T is added. The corresponding ICIO table is listed in following tables:

Table 4

Country / Sector		S1	S2	R1	R2	T1	T2	YS	YR	YT	Gross output
		S		R		T					
S	S1	1	1	0	0	0	0	9/10	1/10	0	3
	S2	0	1	0	1	0	0	1	0	0	3
R	R1	0	0	1	1	0	0	0	1	0	3
	R2	0	0	1	1	0	0	0	1	1	4
T	T1	1	0	0	0	1	0	1	0	0	3
	T2	0	0	0	0	1	1	0	0	1	3
Value-added		1	1	1	1	1	2				
Total input		3	3	3	4	3	3				

Gross intermediate and final good exports matrix is

$$E=EI+EF=\begin{bmatrix}0 & 1/10 & 0\\0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 1/10 & 0\\0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 1\\2 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

The direct input coefficient matrix A, Global Leontief inverse Matrix B, Local Leontief inverse matrix L and direct value-added coefficient vector V can be easily computed as

$$A=\begin{bmatrix}1/3 & 1/3 & 0 & 0 & 0 & 0\\0 & 1/3 & 0 & 1/4 & 0 & 0\\0 & 0 & 1/3 & 1/4 & 0 & 0\\0 & 0 & 1/3 & 1/4 & 0 & 0\\1/3 & 0 & 0 & 0 & 1/3 & 0\\0 & 0 & 0 & 0 & 1/3 & 1/3\end{bmatrix} B=\begin{bmatrix}3/2 & 3/4 & 3/20 & 3/10 & 0 & 0\\0 & 3/2 & 3/10 & 3/5 & 0 & 0\\0 & 0 & 9/5 & 3/5 & 0 & 0\\0 & 0 & 4/5 & 8/5 & 0 & 0\\3/4 & 3/8 & 3/40 & 3/20 & 3/2 & 0\\3/8 & 3/16 & 3/80 & 3/40 & 3/4 & 3/2\end{bmatrix}$$

$$L=\begin{bmatrix}3/2 & 3/4 & 0 & 0 & 0 & 0\\0 & 3/2 & 0 & 0 & 0 & 0\\0 & 0 & 9/5 & 3/5 & 0 & 0\\0 & 0 & 4/5 & 8/5 & 0 & 0\\0 & 0 & 0 & 0 & 3/2 & 0\\0 & 0 & 0 & 0 & 3/4 & 3/2\end{bmatrix}$$

$$V=\begin{bmatrix}1/3 & 1/3 & 1/3 & 1/4 & 1/3 & 1/3\end{bmatrix}$$

Applying decomposition equation (52), we can fully decompose each of the three countries' gross bilateral exports into the sixteen value-added and double counted components as reported in table 5. Detailed computation is listed in Appendix G

This example shows that VAX ratio based on both forward-looking and backward-looking linkage cannot be meaningfully defined for bilateral gross exports at both the sector and aggregate levels due to the presence of indirect value-added trade via third countries. There are 6 out of the 12 bilateral VAX ratios at the sector level defined by Johnson and Noguera (2012) that go to infinity (Column 21 of table 5). For bilateral VAX ratio at the sector level based on backward linkage, there are four cases with an undefined VAX ratio (column 23). The aggregate bilateral VAX ratio is also undefined in 2 out of the 6 routes in this simple example for both the forward-and backward-looking linkage based VAX ratios (row ST and TR). The only meaningful measure for bilateral flows is the share of domestic value-added that stays abroad with no restriction on particular foreign country as its final destination (column 20). It is also the proper measure of value-added content of trade for bilateral flows; one minus this quantity can be used as an index of double counting in official bilateral gross trade statistics.

Summing all bilateral partners over sectors, we obtain the country/sector level VAX ratio based on backward-looking linkage that is consistent with the definition of value-added exports, while VAX ratio defined as Johnson and Noguera remains undefined. When either summing over all bilateral partners or summing over all sectors, the two measures of value-added exports equal each other, but are still different from domestic value-added that stays abroad due to indirect value-added trade via third countries. Only at the country aggregate level, the three measures are exactly the same. This clearly demonstrates that Johnson and Noguera's VAX ratio can only be used as the measure of factor content of trade and (1 minus VAX ratio) double counting at the country aggregate level. All the economic reasoning and econometric work based on their bilateral and sector level VAX ratios are very likely to be incorrect.

To save space and focus attention on the economic intuition behind decomposition equations rather than complicated technical details, the general case with arbitrary numbers of countries and sectors is discussed in Appendix H for interested readers

3. Decomposition Results for 40 Economies during 1995-2011

In this section, we apply our gross trade accounting method to the World Input-output Database (WIOD), which was developed by a consortium of eleven European research institutions with funding from the European Commission. The database is a time series of inter-country input-output (ICIO) tables from 1995 to 2011, covering 40 economies including most major industrialized countries and major emerging market trading nations. Timmer et al. (2012) provide a detailed description of this database.

Our entire decomposition results consist of many GN (1435) by G (41) and GN by GN matrices each year, collectively taking up more than 25 gigabytes of storage space. We will provide these results on our website. To illustrate the estimation outcomes in a manageable manner for its usefulness, we now provide a series of examples, which are selected and processed from subsets of the detailed results. To facilitate digestion, we organize them into a few categories:

(A) Decomposing a country's sector level gross exports into four buckets: domestic value added that is ultimately absorbed abroad (DVA for short), foreign value added used in the production for the exports (FVA for short), returned value added or the portion of domestic value added that is initially exported but ultimately returned home by being embedded in the imports from other countries and consumed at home (RDV for short), and pure double counted terms (PDC) due to the back and forth intermediate goods trade.

(B) Decomposing the FVA by source in a country's sector-level exports;

(C) Distinguishing and comparing a backward-looking measure of domestic value added embedded in a country-sector's gross exports and a forward-looking measure of value added that is originated from a country-sector but is embedded in the gross exports from all sectors of that country;

(D) A new measure of revealed comparative advantage that takes into account both domestic and international production sharing.

(E) Decomposing bilateral-sector trade;

(F) Characterizing types of international production sharing at the sector level.

(G) Tracing structures of Vertical Specialization (VS) across countries and over time.

3A. Decomposing gross exports at the country-sector level

We first look at the decomposition for the gross exports of US transport equipment sector (WIOD sector 15). The transport equipment sector, including automobiles and airplanes, is one of the most important export sectors for the United States. The decomposition for 1995-2011 is presented in Table 6a. Column 2 records the gross exports in millions of dollars (current price). In the next four columns, we report the four major components, expressed in percentage of gross exports. Figure 2a provides a graphic presentation of the decomposition results.

On average, domestic value added that is ultimately absorbed abroad (DVA) is about 70% of gross exports, while foreign value added (FVA) that is embedded in US transport equipment exports is somewhere between 12-22% of the gross exports.

The remaining parts consist of returned domestic value added (RDV) and pure double counting (PDC). When we compare with the next example (Mexico's electronics exports), we will see that the RDV share for the US transport equipment exports is relatively high. This suggests that a fraction of the US exports are parts and components that are used as intermediate inputs in the production of transport equipment or refined components that are re-imported back to the United States.

The four components exhibit different trends. Clearly, the FVA share has increased over time from 12.1% in 1995 to 21.9% in 2011. In comparison, the RDV share has declined from 12.6% to 5.0% during the same period.

We note that our definition of DVA at the sector level differs from that of VAX defined by Johnson and Noguera (2012). The VAX measure describes the amount of domestic value added that originates in this sector (transport equipment) that is exported and absorbed abroad. It excludes domestic value added originated in other sectors (e.g., services) that is exported via the transport equipment sector. The ratio of VAX in gross exports for the US transport equipment sector is 33.3% in 1995 and 24.8% in 2011, which are less than half of the DVA shares in gross exports in the corresponding years.

We now look at the decomposition of Mexico electronics (WIOD sector 14) exports. The results are presented in Table 6b and Figure 2b, respectively. An important feature of this example is the relatively high share of foreign value added in Mexico's exports in this sector. Indeed, FVA is often higher than DVA, driven in a significant part by imported components in the Marquiladora factories. Note that RDV is too small to be visible in Figure 2b.

3B. Decomposing foreign value added (FVA) by source countries

We go back to Example 3.A1, the US transport equipment exports. Our decomposition formula allows us to trace the FVA in that sector to the original countries. The evolution of the top five foreign suppliers of value added in the US transport equipment sector is presented in Figure 3a. In the 1990s and early 2000s, Japan and Canada are the top two suppliers of foreign value added. However, in more recent years, China has become the top supplier, followed by Canada and Mexico.

The example of the FVA in German exports in transport equipment is presented in Figure 3b. Unsurprising, France, Italy, United States, and the United Kingdom are always among the top suppliers of foreign value added in German transport equipment production throughout the sample. In terms of relative growth, China stands out in its own league, from starting as a relatively minor supplier in the 1990s to becoming the number one supplier by 2011. Poland is also a fast grower as a supplier to Germany, though its growth rate pales in comparison to China.

The third example is FVA in Mexico’s electronics exports, presented in Figure 3c. The United States is the leading supplier of foreign value added to Mexico throughout the sample. However, in terms of relative changes, the most striking feature of Figure 4c is the rapid rise in China, and a corresponding decline in the United States. The graph suggests that, in a few years, China may very well overtake the United States as the leading foreign supplier of value added to Mexico’s electronics sector.

3C. Two Concepts of Exports of Domestic Value Added at the Country-sector Level

The distinction of the two concepts can be seen via an example of German business services exports. The first measure of value added is from a recipient or importing country’s perspective (user perspective), and the domestic value added embedded in the German business service exports includes German domestic value added from other German sectors used as inputs in the production of German business service exports. This notion of domestic value added exports is called a “backward-looking” measure. Columns 2-5 of table 8 provide a “backward-looking” decomposition, similar to example A1 and A2. In particular, DVA is the domestic value added from all sectors in Germany that is embedded in its business service sector exports that are ultimately absorbed abroad. Unsurprisingly, all the other terms, RDV, FVA and PDC are relatively small. The DVA is about 93% of the gross exports for that sector.

A second measure, or a "forward-looking" notion of value added exports takes into account all value added that is originated in German business service exports but is either directly exported by the service sector or indirectly exported by other German sectors (producer perspective). For example, if German automobile exports uses German business services, that constitutes indirect exports of value added from the German business services, This particular indirect value added exports are a part of the forward-linkage based exports of value added from the German service sector (although they are also part of a backward-linkage exports of German value added that is embedded in German automobile gross exports).

If a sector does a lot of indirect exports of its sector-originated value added via other sectors, the forward-looking measure value added exports can in principle exceed that sector's direct gross exports because indirect exports of that sector's value added are not part of that sector's gross exports. This is indeed the case for German business services sector. German exports in other sectors commonly embed value added that is originally from the German business service sector. As we see in Column 4 of Table 8, as a result, the forward-looking measure of value added exports out of German business services sector is often 280% to 377% of the sector's gross exports. (In contrast, the backward-looking measure of total domestic value added in a sector's gross exports is bounded between 0 and 100%.)

These two measures of value added exports at the sector level are useful for different purposes. If one wishes to understand the fraction of a country-sector's gross exports that reflects a country's domestic value added, one should look at the backward-linkage based value added for that sector, which by our decomposition formula is DVA = gross exports – FVA-RDV-PDC. If one wishes to understand the contribution of all value added from a given sector to the country's aggregate exports, one should look at the forward- linkage based measure of value added exports.

We note briefly that our framework allows one to further decompose the backward-looking measure of DVA embedded in a country-sector's gross exports into finer components:

DVA = domestic value added embedded in that sector's gross exports in final goods (DVA_Fin) + domestic value added embedded in that sector's gross exports in intermediate goods that is absorbed in the direct importing country (DVA_INT) + domestic value added embedded in that sector's gross exports in intermediate goods but re-exported and ultimately absorbed outside the direct importing country (DVA_INTrex).

The decomposition results for the DVA in US transport equipment's gross exports are presented in Table 9. While column 2 reports the value of gross exports, Column 3 is the share of DVA in gross exports. Columns 4-6 reports the three components of the DVA: DVA_Fin, DVA_INT, and DVA_INTrex. As we can see, domestic value added in intermediate goods exports (DVA_INT+ DVA_INTrex) collectively is more important than DVA in final goods exports .

In principle, we can decompose each of the items into further details. For example, we can decompose DVA_INTrex into three components for this US industry.

DVA_INTrex = DVA in intermediate exports used by the direct importer to produce intermediate exports for production in third countries for their domestically consumed final goods (**DVA_INTrexI1)** **+** DVA in intermediate exports used by the direct importer producing final exports to third countries (**DVA_INTrexF)** **+** DVA in intermediate exports used by the direct importer producing intermediate exports to third countries to produce exports (**DVA_INTrexI2)**.

The numerical results for the decomposition of DVA in US transport equipment exports are presented in Columns 7-9 of Table 9, all expressed in percentages of DVA_INTrex.

We also note briefly that our framework allows distinguishing forward-linkage based value-added exports measure (VAX_F) from GDP by industry in a sectors' gross exports, which also includes forward-linkage based measure of domestic value-added in a given sector but finally returns home (RDV_F) in addition to VAX_F. Such difference is particular important for countries located on the top of a global value Chain. To save space, we report some selected industries examples from our decomposition results in Appendix I.

3D. A New Measure of Revealed Comparative Advantage

The previous discussion of the forward-looking measure of value added in a sector's exports naturally leads to a revised notion of a country-sector's revealed comparative advantage. The traditional definition of a country-sector's revealed comparative advantage (traditional RCA, for short) is the share of that country-sector's gross exports in the country's total gross exports relative to that sector's gross exports from all countries as a share of the world total gross exports.

Formally, denoting e_i^{r*} to be the export of good i of country R, and assuming that there are N commodities and G countries engaged in trade, then the traditional RCA is defined as:

$$TRCA_i^r = \frac{e_i^{r*}}{\sum_{i=1}^{n} e_i^{r*}} \Bigg/ \frac{\sum_{r}^{G} e_i^{r*}}{\sum_{i}^{n}\sum_{r}^{G} e_i^{r*}} \tag{56}$$

When the RCA exceeds one, the country is said to have a revealed comparative advantage in that sector; when the RCA is below one, the country is said to have a revealed comparative disadvantage in that sector.

The traditional RCA ignores both domestic production sharing and international production sharing. To be more specific, first, it ignores the fact that a country-sector's value added may be exported indirectly via the country's exports in other sectors. Indirect exports of a sector's value added should be included in a conceptually correct measure of a country's sector's comparative advantage. Second, it also ignores the fact that a country-sector's gross exports partly reflect foreign contents (which show up in both FVA and a portion of PDC). A conceptually correct measure of comparative advantage needs to exclude foreign-originated value added and pure double counted terms in gross exports but include indirect exports of a sector's value added through other sectors of the exporting country.

Taking into account both domestic and international production sharing, we propose to define a new measure of a country sector's revealed comparative advantage (new RCA for short) as the share of a country-sector's forward-looking measure of domestic value added in exports in the country's total domestic value added in exports relative to that sector's total forward-looking domestic value added in exports from all countries as a share of global value added in exports[8]. The new RCA measure, or NRCA for short, is:

$$NRCA_i^r = \frac{vax\_f_i^r + rva\_f_i^r}{\sum_{i=1}^{n}(vax\_f_i^r + rva\_f_i^r)} \Bigg/ \frac{\sum_r^G (vax\_f_i^r + rva\_f_i^r)}{\sum_i^n \sum_r^G (vax\_f_i^r + rva\_f_i^r)} \quad (57)$$

We now report two pairs of examples. First, we compute and plot the RCA, both the traditional and the new definitions, for China and the United States, respectively, in the sector of electric and optimal equipment. The time series profiles of the RCA for China, computed by both methods are presented in the left graph of Figure 5a. If one looks at the traditional measure of RCA, this is a strong comparative advantage sector for China, with the RCA exceeding 2.5 since 2007. In contrast, when our new measure of RCA is used, the RCA takes on a much lower value, about 1.8 in recent years.

The RCA for the US in this sector is plotted in the right graph of Figure 5a. We see an even bigger divergence between the new and traditional measures of RCA. By the traditional measure, electric and optical equipment has become a comparative disadvantage sector for the United States since 2003. However, by the new measure, not only this sector remains to be a comparative advantage sector for the United States, the strength of the RCA has in fact increased in recent years. The divergent trends in the new and traditional measures of the RCA for the United States illustrate the potential misleading nature of the traditional measure. While the traditional measure based on the gross trade data tells a seemingly sobering story of a decline in the US competitiveness in the manufacture of electrical and optical equipment, our new measure reveals the continued robustness of the US comparative advantage in the industry.

For the second pair of examples, we look at the RCA for India and Germany, respectively, in the business services sector. The traditional and new measures of the RCA for India are plotted in the left graph of Figure 5b, whereas the two measures for Germany are plotted in the right graph. India's business services exports are legendary

[8] Note that the measures of domestic value-added in exports used to compute revealed comparative advantage is based on forward looking linkage. That is, it includes the indirect exports a sector's value added through the gross exports of all downstream sectors. This is consistent with the factor content of trade in the trade literature.

due to media reports about Infosis, Wipro, and call centers. Interestingly, the strength of the RCA for Indian business services is weaker under the new measure than under the traditional measure. In contrast, German business services exports attract less media attention than its manufacturing sector export successes. However, while the business services appear to be a comparative disadvantage sector for Germany based on gross exports (with traditional RCA < 1 throughout 1995-2011), it is a comparative advantage sector by our new measure that takes into account domestic and international production sharing. For India, the domestic business services sector contributes relatively little to the production and exports of other sectors. For Germany, the opposite is the case; the domestic business services sector is a significant contributor to the production and exports of automobiles, machineries, and other products. Once indirect exports of domestic business services are taken into account, Indian's business service exports become much less impressive relative to Germany and many other developed countries.

3E. Decomposing Bilateral-Sector Level Exports

We first consider the US – China bilateral trade in electronics and optimal equipment. Among all bilateral sector level trade flows in recent years, this is the single biggest item in recent years measured by the volume of gross trade, with the sum of the two-way flows reaching 212 billion dollars in 2011. By the gross statistics, presented in Column 1 of Table 10, the trade is highly imbalanced, with the Chinese exports ($176.9 billion in 2011) being five times that of the US exports to China ($35.1 billion in 2011). If we separate exports of final goods versus that of intermediate goods (reported in Columns 2a and 2b of Table 10), we see that most of the Chinese exports consist of final goods, whereas most of the US exports consist of intermediary goods.

We provide a decomposition of the trade flows for selected years (1995, 2005, and 2011) in Columns (3)-(7) of Table 10. More precisely,

Column (1) = (3)+(4)+(5)+(6)+(7), where Column (3), DVA, represents the exporter's domestic value added that is ultimately absorbed by other countries, including both the direct importing country and other foreign countries; Column (4), RDV, is the part of domestic value added initially exported but ultimately returned home and is absorbed at home; Column (5), MVA, is the part of the FVA that comes from the direct importing country; Column (6), OVA, is the part of the FVA that comes from third countries; and finally, Column (7) is the pure double counted items.

Column (3) = (3a) + (3b) + (3c), that is, the DVA part is further decomposed into DVA in final goods, DVA in intermediate goods absorbed by the direct importer, and DVA in intermediate goods re-exported and ultimately absorbed in third countries.

The decomposition results show that the US and Chinese exports have very different structures. First, the DVA as a share of the gross exports is much higher for the US exports (81% in 2011) than for the Chinese exports (about 70% in 2011)[9]. Second, correspondingly, the FVA share (MVA+OVA) is higher for the Chinese exports than for the US exports. This is especially true for the OVA share in China. In other words, the US exports rely overwhelmingly on its own value added (only 2.1% from China and 7.6% from other countries in 2011), whereas the Chinese exports use more foreign value added, especially value added from third countries (with 3.1% from the United States and 23.1% from Japan, Korea, and all other countries). Third, whereas the RDV share is trivial for China, it is non-negligible for the United States (7.0% in 2011). This again reflects the different positions the two countries occupy on the global production chain. As the United States produces and exports parts and components, it is on the upstream of the chain; part of its value added in its exports returned home as embedded in imports from other countries. In comparison, China is in the downstream of the chain; very few of its value added come home as intermediary goods in other countries' exports. This also evidenced by China having a much higher proportion of FVA used in producing its final goods exports to the US, while the US has a higher share of FVA in producing its intermediate goods export to China.

The decomposition of DVA into (3a), (3b) and (3c) also reveals differences between the two exporters. In particular, the DVA in the Chinese exports to the United States is dominated by DVA in the final goods, where the DVA in the US exports is dominated by DVA in intermediate goods that is absorbed by China.

As a consequence of these differences in the structure of value added composition, the China – US trade balance in this sector looks much smaller when computed in terms

[9] Because WIOD data do not distinguish processing and normal trade, the domestic value added share for China is likely to be overestimated (Koopman, Wang and Wei, 2012).

of domestic value added than in terms of gross exports. In Column (8), we report forward-linkage based value added exports, or VAX_F. Because this concept captures value added originated in that sector in all downstream sectors of exports from the exporting country but excludes contributions of value added from other (upstream) domestic sectors to the electric and optical equipment sector, it is generally not the same as DVA at the bilateral sector level, and in our application, VAX_F is smaller than DVA (This is generally true for downstream sectors).

In Column (9) of Table 10, we report backward-linkage based value added exports, VAX_B, reflecting all exporting country's value added (from all upstream sectors) that is exported via this sector and absorbed by the direct importing country, including value-added embodied in the source country's gross exports to third countries, but finally absorbed by the partner country. Because exporter's domestic value added that is exported to and absorbed by a particular partner country indirectly via third countries can be either larger or smaller than exporter's domestic value-added embodied in its intermediate goods re-exported by the partner country to and absorbed by third countries, VAX_B (e.g., 76% and 85.2% for Chinese and US exports in 2011 respectively) at the bilateral sector level is generally different from DVA (69.6% and 80.8% of Chinese and US gross exports in 2011 respectively).

We report the US-China bilateral balance of trade in electric and optical equipment sector by gross and the two value-added trade measures in Figure 6a. It is important to understand that at the bilateral/sector level, DVA, different from both VAX_F and VAX_B (both of them deviate from gross trade flows), is only part of gross trade flows (so it is the only value-added measure that is consistent with bilateral gross trade flows), but not a measure of bilateral value-added trade flows, because it includes a portion of value-added that is absorbed by other countries (while both VAX_F and VAX_B are absorbed by the importing countries).

Our second example is the China-Japan trade in rubber and plastics. We perform a similar decomposition as above (details omitted to save space). We then plot the three different types of trade balance measures: in gross exports, VAX_F, and VAX_B, respectively in Figure 6b. As we can see, due to the vast differences in the structure of value added in exports by the two countries, the trade balance looks different, often with a sign switch, as we move from one measure to the other.

3F. Patterns of Production Sharing by Sector and Country

As our decomposition formula allows us to not only capture the vertical specialization (VS) share but also the source countries of FVA, we can use the information to characterize the type of production sharing arrangements by country and sector.

In Table 11, we report the average values of VS shares across all countries in 1995 and 2011, for each of the 35 sectors, in Columns 2 and 3, respectively. We sort the sectors in descending order of the value of the average VS share in 2011. The sectors with the highest VS shares in 2011 are electric (and optical) equipment, transport equipment, basic medals, machinery, and rubber and plastics. The sectors with the lowest VS shares are private household (services), education, real estate, public administration, and retail trade. These numbers and the sector order are hardly surprising.

For a given sector, we summarize the distribution of countries in each type of production sharing in Columns 4-6. If the VS share is less than 10%, we label that country-sector as following a national production arrangement. If the VS share exceeds 10%, we label the country-sector as following a cross-country production sharing. We further divide the latter into two categories: if intra-regional sourcing accounts for 60% or more of the VS, we label it as using regional production sharing; otherwise, we label it as using global production sharing. As an example, for the electric (and optical) equipment sector, only one country follows a national production arrangement, whereas 39 countries have significant cross-country production sharing. Within the latter group, 15 countries follow a regional sharing arrangement, and 24 countries follow a global sharing arrangement. By this set of definitions, we find that it is common to see a global production sharing arrangement in the electric equipment, machinery, rubber and plastics, air transport, water transport, textile and leather and footwear industries.

We can also get more details about any particular sector. As an illustration, in Table 12, we zoom in on the transport equipment sector. We list the major developed and emerging market economies in the first column, ordered by the volume of gross exports in that sector in 2011 (recorded in the second column). The largest exporters of transport equipment are Germany, United States, Japan, France, Korea, and China.

For each country, we report the top markets for their transport equipment exports in Column 3. For example, for Germany, the largest markets are France, the United States and China. For the United States, the largest markets are Canada, Mexico, and China. In Column 4, we report vertical specialization as a share of the gross exports. All countries on this list have a relatively high VS share, often in excess of 30%. This confirms that transport equipment sector is highly integrated both regionally and globally; production in most countries relies on parts and components made in some foreign countries. In Columns 5 and 6, we report the share of VS coming from within the same region of the country and from outside the region, respectively. Generally speaking, European countries source heavily from other European countries, though they also import value added from outside the region. Most countries outside Europe tend to source globally, with value added from countries outside the region accounting for more than half of the overall VS.

If one wishes to test theories about determinants of offshoring and outsourcing, such information can be very useful.

3G. Tracing Structures of Vertical Specialization across Countries and over Time

Vertical specialization or VS, defined as foreign contents in a country's gross export is a summary statistic to measure international production sharing widely used in the literature (e.g., HIY, 2003, and Antras, 2013). However, as showed by our gross exports decomposition formula, there are different components within VS and each of these components have different economic meanings and represent different types of cross-country production sharing. For example, large share of foreign value-added in a country's final goods exports (FVA_FIN for short) may indicate that the country mainly engages in final assembling activities based on imported components and just participates in cross-country production sharing on the low end of a global value chain, while an increasing foreign value added share in a country's intermediate exports (FVA_INT for short) may imply the country is upgrading its industry to start producing intermediate goods for other countries, especially when more and more of these goods are exported to third countries for final goods production. The latter is a sign that the country is no longer at the bottom of the GVCs.

Pure double counting of foreign value-added in a country's exports (FDC for short) can only occur when there is back and forth trade of intermediate goods. An increasing weight of FDC share in VS indicates the deepening of cross-country production sharing. Intermediate goods have to cross national boarders multiple times before they are used into final goods production. Therefore, knowing the relative importance of these components and their trend of change over time in a country's total VS can help us to gauge the depth and pattern of cross-country production sharing and discover the major driver of the general increase of VS in a country's gross exports during the last two decades.

As shown in Table 13, across all countries and all sectors, the total foreign content (VS) sourced from manufacturing and services sectors used in world manufacturing goods production has increased by 8.3 percentage points (from 22.5% in 1995 to 30.8% in 2011, column 3). Interestingly, the VS structure information reported in the last three columns indicates that this net increase is mainly driven by an increase of FDC. This suggests that the international production chain is getting longer; over time, a rising portion of trade reflects intermediate goods made and exported by one country, used in the production of the next-stage intermediate goods and exported by another country to be used by the next country to produce yet another intermediate good. This progressively more trade of intermediate goods that cross national boarders multiple times is what gives the rising share of PDC.

Because the share of foreign value-added in final goods exports in total VS has declined by about 5 percentage points during the same period (from 44.5% in 1995 to 40.6 in 2011), and because the share of foreign value-added in intermediate goods exports) in total VS stayed almost constant, the increase of VS share in world manufacturing exports is driven mainly by an increase in FDC share (from 19.5% in 1995 to 25% in 2011). If this trend continues, the FDC share may reach the level of the FVA share and become an important feature of cross-country production sharing. If we add the shares of

FVA_INT and FDC, these two components involving intermediate goods trade have already accounted for about 60% of the total manufacture VS in 2011.

Of course, there is heterogeneity in the VS structure both across countries and across sectors, especially between industrialized and developing economies. Table 14 reports total VS and its structure in electric and optical equipment exports for six Asian countries: Japan, Korea, Taiwan, China, India and Indonesia. The three industrialized Asian economies are reported in the right panel. Despite their difference in the level of total VS shares, their VS structure is very similar: lower and declining in FVA_FIN, relatively stable in FVA_INT and rapid expanding in FCD. Taiwan's VS structure is an informative example (presented in right bottom 5 rows in Table 14). As Taiwan is an important supplier of parts and components, and crucially, as Taiwan often occupies several different positions on the global production chain (since it produces both inputs into chip making, memory chips themselves, and components that embed the chips), the collective shares of FDC and FVA_INT already exceed 80% of its total VS (or 40% of its gross exports) since 2005. In comparison, for other developing Asian countries such as China, India and Indonesia (presented in the left panel of table 14), the share of FVA_FIN still accounts for about 50% its total VS until 2011. However, there are also interesting differences among the three emerging Asian giants: the VS structure change during the 17 years for China was mainly driven by the decline of FVA_FIN and increase of FDC, while FVA_INT stayed relatively stable; for Indonesia, it was driven by the rapid expanding of both FVA_INT and FDC. Both of them increased more than 10 percentage points during this period, indicating that there was rapid upgrading of Indonesia's electric and optical equipment industries. While for India, the later-comer in Asian and global production network of electric and optical equipment, its share of FVA-FIN rises (from 38.2% in 1995 to 52.8% in 2011) and FVA_INT share continued to decline (from 40.2% in 1995 to 25.3% in 2011), while FDC share stayed relatively stable in the last 17 years. This might result from a strategic shift from import substitution to export oriented development; it is also consistent with a move from the upper stream portion of the production chain to a more downstream position as China and Indonesia did decades ago. These empirical evidences indicate that the structure of VS in addition to its total sums offer additional information about each country's respective positions in the global value chain.

We want to end this section with a note of caution in using sector-level decompositions. As discussed in KWW, the lack of information in current global ICIO database on how imported inputs are distributed among sector users within each country may introduce unknown noises into both sources of value added in gross exports and value added trade estimates at the sector level. If we focus on country/sector rankings rather than the exact numerical values, the impact of noises is likely to be smaller.

4. Concluding Remarks

This paper aims to deliver two outputs. The first is an accounting framework that allows one to decompose gross exports at the sector, bilateral, or bilateral sector level into four major parts: (a) domestic value added that is absorbed abroad, (b) domestic value added that is initially exported but eventually returned home, (c) foreign value added, and (d) pure double counting terms. Our framework in fact allows one to further decompose each of the four major parts above into finer components with economic interpretations. For example, we can decompose FVA in a country-sector's exports into different source countries; we can also trace exports of value added by channels, whether they are embedded in final goods exports, intermediate goods exports that are absorbed in the direct importing countries, or intermediate goods exports that are re-exported and absorbed outside the direct importing countries.

The second output is a decomposition of bilateral sector exports from 40 trading nations in 35 sectors from 1995 to 2011 based on the WIOD database. Because the full decomposition output takes up 25 gigabytes of space, we illustrate potential usefulness of the resulting data by a series of examples that are subsets of the overall decomposition output. For example, we show how we may meaningfully distinguish a forward-linkage based measure of domestic value added from a sector that indirectly exports of its value added through other sectors' gross exports from a backward-linkage based measure of value added that includes the value added contributions from other domestic sectors. Based on the decomposition results, we can also correct some shortcomings of a popular measure of revealed comparative advantage and derive a new measure that takes into account both domestic and international production sharing.

In principle, when new countries or years are added to the WIOD database, or an alternative inter-country input-output table becomes available, our accounting framework can be applied as well. So the accounting framework developed in this paper is not inherently tied to the WIOD database and can be a stand-alone tool to help us extract useful information from official trade statistics.

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Table 5: Gross exports decomposition results: 3-country, 2-sector numerical example

	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12	T13	T14	T15	T16	Gross exports	DVA	% of DVA	VAX_F J&N	VAX_F Ratio	VAX_B WWZ	VAX_B Ratio
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)
SR1	1/20	0	0	0	0	0	0	0	0	0	0	0	0	1/20	0	0	1/10	1/20	50%	1/5	200%	1/5	200%
SR2	0	9/20	0	3/10	0	0	0	0	0	0	0	0	0	0	3/20	1/10	1	3/4	75%	3/10	30%	3/10	30%
ST1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/10	∞	0	0%
ST2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/5	∞	3/10	∞
RT1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/5	∞	0	0%
RT2	3/5	0	0	0	0	0	0	0	0	0	1/10	0	0	3/10	0	0	1	3/5	60%	2/5	40%	3/5	60%
RS1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	0	0%	0	0%
RS2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	0	0%	0	0%
TS1	1	7/10	3/20	1/20	0	0	1/10	0	0	0	0	0	0	0	0	0	2	19/10	95%	17/20	43%	19/20	73%
TS2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	17/20	∞	1/4	∞
TR1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/10	∞	1/10	∞
TR2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/10	∞	1/10	∞
SR	1/20	9/20	0	3/10	0	0	0	0	0	0	0	0	0	1/20	3/20	1/10	11/10	4/5	73%	1/2	45%	1/2	45%
ST	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	3/10	∞	3/10	∞
RT	3/5	0	0	0	0	0	0	0	0	0	1/10	0	0	3/10	0	0	1	3/5	60%	3/5	60%	3/5	60%
RS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	0	0%	0	0%
TS	1	7/10	3/20	1/20	0	0	1/10	0	0	0	0	0	0	0	0	0	2	19/10	95%	17/10	85%	17/10	85%
TR	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0%	1/5	∞	1/5	∞
S	1/20	9/20	0	3/10	0	0	0	0	0	0	0	0	0	1/20	3/20	1/10	11/10	**4/5**	**73%**	**4/5**	**73%**	**4/5**	**73%**
R	3/5	0	0	0	0	0	0	0	0	0	1/10	0	0	3/10	0	0	1	**3/5**	**60%**	**3/5**	**60%**	**3/5**	**60%**
T	1	7/10	3/20	1/20	0	0	1/10	0	0	0	0	0	0	0	0	0	2	**19/10**	**95%**	**19/10**	**95%**	**19/10**	**95%**

Note: Terms in Table 5 and Equation 52

	Description
T1	DVA exports in final goods exports
T2	DVA in intermediate exports to the direct importer and is absorbed there
T3	DVA in intermediate exports used by the direct importer to produce intermediate exports for production of third countries' domestic used final goods
T4	DVA in Intermediate exports used by the direct importer producing final exports to third countries
T5	DVA in Intermediate exports used by the direct importer producing intermediate exports to third countries
T6	Returned DVA in final goods imports -from the direct importer
T7	Returned DVA in final goods imports -via third countries
T8	Returned DVA in intermediate imports
T9	Double counted DVA used to produce final goods exports
T10	Double counted DVA used to produce intermediate exports
T11	Direct importer's VA in source country’s final goods exports
T12	Direct importer's VA in source country’s intermediate goods exports
T13	Third countries' VA in final goods exports
T14	Third countries’ countries' VA in intermediate goods exports
T15	Direct importer’s VA double counted in exports production
T16	Third countries’ VA double counted in exports production

Table 6a: Decomposition of US Transport Equipment Exports(WIOD sector 15)

Year (1)	Gross Exports (2)	DVA Share (% of (2)) (3)	FVA Share (% of (2)) (4)	RDV Share (% of (2)) (5)	PDC Share (% of (2)) (6)
1995	90737	69.9	12.1	12.6	5.4
2000	124345	66.7	12.0	14.5	6.8
2005	150442	65.5	15.1	11.8	7.5
2007	194374	67.2	16.6	8.9	7.3
2009	158999	74.1	15.8	5.1	4.9
2010	179540	67.4	20.7	5.2	6.6
2011	198891	66.2	21.9	5.0	6.8

Notes

Term	Description
DVA	Domestic Value Added in exports absorbed by other countries
FVA	Foreign Value Added in exports that is absorbed in foreign countries
RDV	Domestic Value Added that is first exported but finally return home
PDC	Pure double counting due to two-way intermediate trade

Table 6b: Decomposition of Mexico Electrical and Optical Equipment Exports (WIOD sector 14)

Year (1)	Gross exports (2)	DVA Share (% of (2)) (3)	FVA Share (% of (2)) (4)	RDV Share (% of (2)) (5)	PDC Share (% of (2)) (6)
1995	17394	45.4	49.5	0.2	5.0
2000	46483	44.5	49.1	0.3	6.1
2005	54983	40.9	48.8	0.4	10.0
2007	69083	41.8	47.8	0.3	10.0
2009	56401	42.9	48.7	0.3	8.1
2010	67893	40.0	50.6	0.3	9.1
2011	71397	38.4	52.1	0.3	9.2

Table 7a: Main Source Countries for Foreign Value Added in US transport equipment exports (WIOD sector 15) (Unit: % of gross exports)

Year	1995	2000	2005	2010	2011
FVA as a share of gross exports	16.04	16.91	20.87	25.61	26.98
China	0.44	0.73	1.93	4.14	4.52
Canada	2.51	2.87	3.21	3.18	3.26
Mexico	0.94	1.25	1.80	2.69	2.90
Japan	3.65	2.84	2.55	2.75	2.70
Germany	1.24	1.24	1.69	1.76	1.85
Korea	0.62	0.48	0.69	0.95	1.05
United Kingdom	0.88	1.11	0.92	0.92	0.92

Table 7b: Main Source Countries for Foreign Value Added in German transport equipment exports (WIOD sector 15) (Unit: % of gross exports)

Year	1995	2000	2005	2010	2011
FVA as % of gross exports	20.98	27.53	29.82	33.09	34.43
China	0.27	0.57	1.07	2.66	2.86
France	2.65	2.98	2.93	2.70	2.79
Italy	1.97	2.19	2.34	2.14	2.45
USA	1.89	3.02	2.24	2.30	2.27
United Kingdom	1.65	2.36	2.25	1.85	1.96
Poland	0.48	1.04	1.36	1.58	1.67
Austria	1.47	1.41	1.68	1.67	1.65

Table 7c: Main Source Countries for Foreign Value Added in Mexico's electronics exports (WIOD sector 14)(Unit %)

Year	1995	2000	2005	2010	2011
FVA as % of gross exports	54.16	54.89	58.36	59.37	60.96
USA	34.80	33.96	20.65	18.25	18.12
China	0.73	1.36	6.99	14.21	15.35
Japan	4.57	3.71	5.85	4.03	3.64
Korea	1.24	1.76	2.92	3.19	3.43
Germany	2.09	1.95	2.90	2.41	2.42
Taiwan	0.94	1.06	2.05	1.85	1.77
Canada	1.15	1.39	1.35	1.45	1.40

Table 8: German Business Services Exports (WIOD sector 30)

Year (1)	TEXP (2)	Backward looking (Share)				Forward looking (Ratio)	
		DVA (% of (2)) (3)	FVA (% of (2)) (4)	RDV (% of (2)) (5)	PDC (% of (2)) (6)	VAX_F (% to (2)) (7)	RVA_F (% to (2)) (8)
1995	14725	93.2	2.7	3.2	0.9	377.3	7.4
2000	19597	91.8	3.8	2.8	1.5	344.0	6.8
2005	43240	92.9	3.8	2.0	1.3	293.2	5.2
2007	58061	92.0	4.0	2.1	1.9	291.1	5.1
2009	59629	92.5	3.4	2.3	1.8	278.7	4.8
2010	59814	93.0	3.9	1.8	1.4	282.8	4.3
2011	62854	92.8	4.0	1.8	1.5	291.6	4.7

Table 9: Structure of DVA US Transport Equipment Exports (WIOD sector 15)

Year (1)	Gross exports (2)	DVA (% of (2)) (3)	DVA_Fin (% of (3)) (4)	DVA_INT (% of (3)) (5)	DVA _INTrex (% of (3)) (6)	DVA_INTrexI1 (% of (6)) (7)	DVA_INTrexF (% of (6)) (8)	DVA_INTrexI2 (% of (6)) (9)
1995	9,715	91.5	45.5	35.7	18.8	43.5	45.4	11.1
2000	13,671	88.0	44.7	30.6	24.7	39.9	46.2	13.9
2005	15,768	85.7	36.6	32.6	30.8	43.1	41.7	15.2
2007	18,003	82.8	34.1	33.0	32.9	43.1	41.2	15.6
2009	12,918	84.7	35.5	36.3	28.2	44.5	41.5	14.0
2010	15,729	83.2	34.8	36.3	28.9	45.3	40.2	14.5
2011	15,185	82.2	33.8	38.6	27.5	46.1	40.2	13.7

Table 10: US-China trade in Electrical and Optical Equipment (WIOD C14)

Year		TEXP	TEXPF	TEXPI	DVA	DVA_Fin	DVA_INT	DVA_Intrex	RDV	MVA	OVA	PDC	VAX_F	VAX_B
		(1) =2a +2b =3+4+5 +6+7	(2a)	(2b)	(3) =3a+3b+3c	(3a)	(3b)	(3c)	(4)	(5)	(6)	(7)	(8)	(9)
China exports to the United States														
1995	**Value**	10,998	7,634	3,364	8,544	5,947	2,046	552	16	314	1,948	176	3,922	9,064
	Share	100	69.4	30.6	77.7	54.1	18.6	5.0	0.1	2.9	17.7	1.6	35.7	82.4
2005	**Value**	87,608	53,492	34,116	53,784	33,399	16,329	4,056	341	3,665	26,332	3,485	25,682	59,923
	Share	100	58.4	41.6	61.4	38.1	18.6	4.6	0.4	4.2	30.1	4.0	29.3	76
2011	**Value**	176,924	104,156	72,769	123,187	74,043	39,801	9,344	1,296	5,581	40,915	5,946	53,078	134,710
	Share	100	58.9	41.1	69.6	41.9	22.5	5.3	0.7	3.2	23.1	3.4	30	76.1
US exports to China														
1995	**Value**	3,400	1,284	2,116	2,691	1,097	1,215	379	182	13	383	130	1,746	3,286
	Share	100	37.8	62.2	79.1	32.3	35.7	11.1	5.4	0.4	11.3	3.8	51.4	96.7
2005	**Value**	16,402	3,845	12,556	11,926	3,264	5,072	3,591	1,777	231	1,251	1,216	8,748	13,766
	Share	100	25.1	74.9	72.7	19.9	30.9	21.9	10.8	1.4	7.6	7.4	68.2	86.5
2011	**Value**	35,059	10,584	24,475	28,314	9,377	12,195	6,742	2,470	718	2,044	1,513	23,754	29,887
	Share	100	30.2	69.8	80.8	26.7	34.8	19.2	7.0	2.0	5.8	4.3	67.8	85.2

Note: (3a) and (3b) equal T1 and T2, (3c) equals the sum of T3 to T5, (4) equals the sum of T6 to T8, (5) equals T11+T13, (6) equals T12+T14 and (7) equals the sum of T9, T10, T15 and T16 in equation (52) of this paper.

Table 11: Patterns of International Production Sharing by Sector

	VS share in gross exports		Numbers of countries in each type of production sharing arrangements in 2011		
Sector	**1995**	**2011**	**National Production**	**Regional Sharing**	**Global Sharing**
c14: Electrical Equipment	28.59	33.53	1	15	24
c15: Transport Equipment	26.8	33.41	0	23	17
c12: Basic Metals	24.7	28.67	3	25	12
c13: Machinery	25.95	28.62	0	23	17
c10: Rubber and Plastics	26.39	28.54	0	23	17
c09: Chemical Products	24.19	26.87	2	23	15
c04: Textiles Products	24.96	25.58	3	20	17
c25: Air Transport	21.01	25.54	4	15	21
c16: Recycling	20.58	24.45	3	24	13
c24: Water Transport	22.33	23.63	6	16	18
c08: Refined Petroleum	23.25	22.93	8	19	13
c07: Paper and Printing	21.05	22.4	4	26	10
c05: Leather and Footwear	21.13	20.87	6	16	18
c06: Wood Products	17.54	19.66	10	23	7
c11: Other Non-Metal	17.25	18.9	8	22	10
c18: Construction	16.5	17.86	6	21	13
c03: Food	13.58	15.92	9	19	12
c17: Electricity, Gas and Water	13.68	15.37	10	17	13
c23: Inland Transport	11.68	15.26	11	16	13
c26: Other Transport	13.17	15.14	13	13	14
c01: Agriculture	10.6	13.48	13	20	7
c19: Sale of Vehicles and Fuel	11.4	13.27	13	17	10
c27:Post and Telecommunications	9.3	12.72	15	9	16
c02: Mining	11.9	12.54	18	17	5
c34: Other Services	10.76	12.05	18	10	12
c20: Wholesale Trade	10.37	11.67	19	9	12
c30: Business services	11.18	11.67	19	6	15
c33: Health and Social Work	9.85	11.6	18	17	5
c22: Hotels and Restaurants	9.64	10.62	22	10	8
c28: Financial Intermediation	7.99	9.87	27	4	9
c21: Retail Trade	8.52	9.51	26	6	8
c31: Public Admin	8.17	8.93	24	6	10
c29: Real Estate	4.18	6.02	34	3	3
c32: Education	4.75	4.91	39	0	1
c35: Private Households	0.3	0.27	40	0	0

Note: VS is sourced from manufacturing and services sector only. National sharing defined as VS <10%; Regional sharing defined as VS > 10%, regional VS > 60% of total VS; Global sharing defined as VS >10%, regional VS<60% of total VS.

Table 12: Production Sharing Patterns in the Transport Equipment Sector (WIOD 15) for Selected Countries in 2011

Country (1)	Gross exports (2)	Top 3 Destinations and its Share (3)	VS (4)	Region % of (4) (5)	Extra-regional % of (4) (6)	Top 3 Suppliers of FVA (7)
Germany	312,488	FRA(13.1),USA(9.0) CHN(8.9)	31.08	62.80	37.20	FRA(8.9),CHN(8.2), ITA(7.8)
USA	198,891	CAN(24.8),MEX(11.5) CHN(7.4)	23.73	25.70	74.30	CHN(16.9),CAN(12.6), JPN(11.3),MEX(11.1)
Japan	178,412	USA(23.6),CHN(11.1) RUS(8.4)	11.00	44.72	55.28	CHN(25.0),USA(13.3) KOR(9.0)
France	127,659	DEU(20.4)ESP(8.7) GBR(5.9)	36.78	65.79	34.21	DEU(27.3),USA(9.6) ITA(7.0)
Korea	121,150	USA(12.0),RUS(9.4) DEU(7.9)	24.63	50.01	49.99	CHN(22.2), JPN(19.5) USA(12.4)
China	96,956	USA(12.4), DEU(8.4) RUS(5.5)	16.68	36.52	63.48	JPN(17.9), USA(13.1) DEU(11.8)
UK	84,809	DEU(16.6), USA(9.4) FRA(7.2)	33.83	56.15	43.85	DEU(18.8),USA(15.1) CHN(7.3)
Canada	75,047	USA(79.7), MEX(2.8) DEU(1.9)	31.51	57.91	42.09	USA(50.5),CHN(8.7) JPN(6.6)
Italy	50,463	DEU(17.5), FRA(9.6)) GBR(7.5)	26.38	62.03	37.97	DEU(19.6),CHN(8.7) FRA(7.9)
Poland	34,410	DEU(25.8), ITA(12.2) GBR(7.8)	42.60	70.85	29.15	DEU(28.4),ITA(8.6) CHN(6.8)
Czech	28,520	DEU(31.5), RUS(8.6) FRA(7.3)	49.18	72.52	27.48	DEU(29.6),POL(6.5) CHN(6.4)
Brazil	24,792	USA(8.2), MEX(5.3) CHN(4.6)	17.01	22.91	77.09	USA(18.4), CHN(11.2) DEU(10.1)
India	21,383	GBR(10.8), USA(6.6) DEU(3.8)	15.52	32.73	67.27	CHN(18.9),USA(12.1) DEU(6.7)
Russia	2,551	POL(3.2), DEU(2.5) FRA(0.9)	30.76	50.07	49.93	DEU(18.1), JPN(16.0) CHN(9.3)

Note：Regional division is defined in “WIOD Country and Region” table in the appendix. VS is sourced from manufacturing and services sector only.

Table 13: Average VS Structure of World Manufacturing Industries

Year	Gross exports	VS share in gross exports	% of VS		
			FVA _final	FVA _INT	FDC
1995	4,020,202	22.5	45.5	34.9	19.5
2000	4,916,605	26.5	45.7	32.2	22.2
2005	7,850,625	29.9	42.3	32.5	25.1
2007	10,472,405	31.6	40.7	32.4	26.9
2009	9,093,710	28.4	43.3	33.4	23.2
2010	10,878,662	30.3	41.7	33.6	24.7
2011	12,458,263	30.8	40.6	34.5	25.0

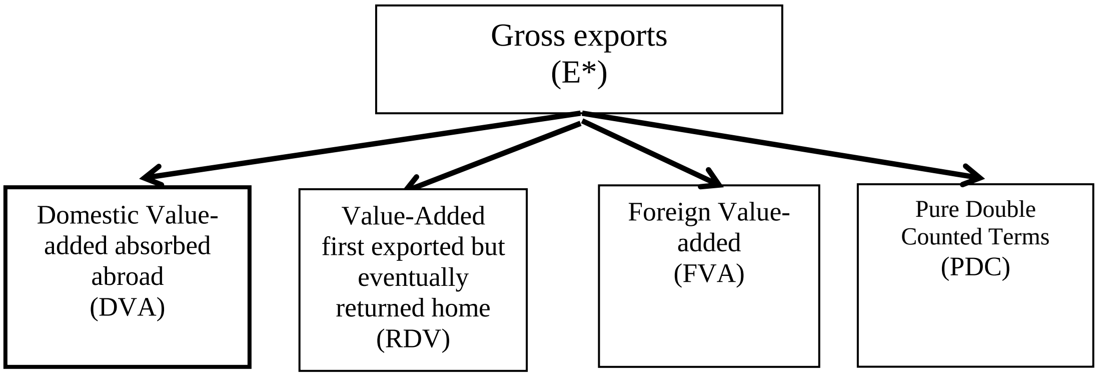
Note: VS is sourced from manufacturing and services sector only.

Table 14: VS Structure of Electric and Optical Equipment Exports for selected Asian Economies

Year	Gross Exports	VS share in Gross Exports	% of VS			Gross Exports	VS share in Gross Exports	% of VS		
			FAV _final	FVA _INT	FDC			FAV _final	FVA _INT	FDC
	China					**Japan**				
1995	34,032	22.1	56.9	27.5	15.6	124,265	6.7	44.6	34.8	20.6
2000	68,998	25.9	54.0	23.9	22.1	136,123	9.5	43.5	29.5	27.0
2005	296,936	37.6	52.3	24.4	23.3	143,324	11.8	35.5	31.4	33.1
2010	638,982	29.3	50.4	27.0	22.7	162,861	14.9	34.0	35.1	30.8
2011	721,417	28.9	50.2	27.7	22.1	166,935	16.0	33.1	37.5	29.4
	India					**Korea**				
1995	1,260	10.9	38.2	40.2	21.6	40,639	27.8	30.0	43.7	26.3
2000	1,927	17.8	41.7	32.2	26.1	60,434	35.1	40.3	30.9	28.7
2005	5,962	20.1	42.3	30.2	27.5	102,595	34.6	31.0	31.2	37.9
2010	23,994	19.0	54.1	24.0	21.9	147,823	36.9	24.8	39.3	36.0
2011	29,470	19.4	52.6	25.3	22.1	159,191	36.8	26.4	40.6	33.0
	Indonesia					**Taiwan**				
1995	2,831	28.7	70.2	19.1	10.7	41,818	43.8	40.2	39.1	20.7
2000	7,637	30.6	53.6	23.3	23.1	77,861	44.8	41.0	31.3	27.6
2005	8,387	29.7	43.6	26.8	29.6	100,957	49.0	22.2	32.8	45.0
2010	11,666	29.0	46.5	28.1	25.3	142,943	49.1	15.8	40.2	44.0
2011	12,558	30.7	48.1	29.1	22.8	147,646	48.2	17.4	41.7	40.9

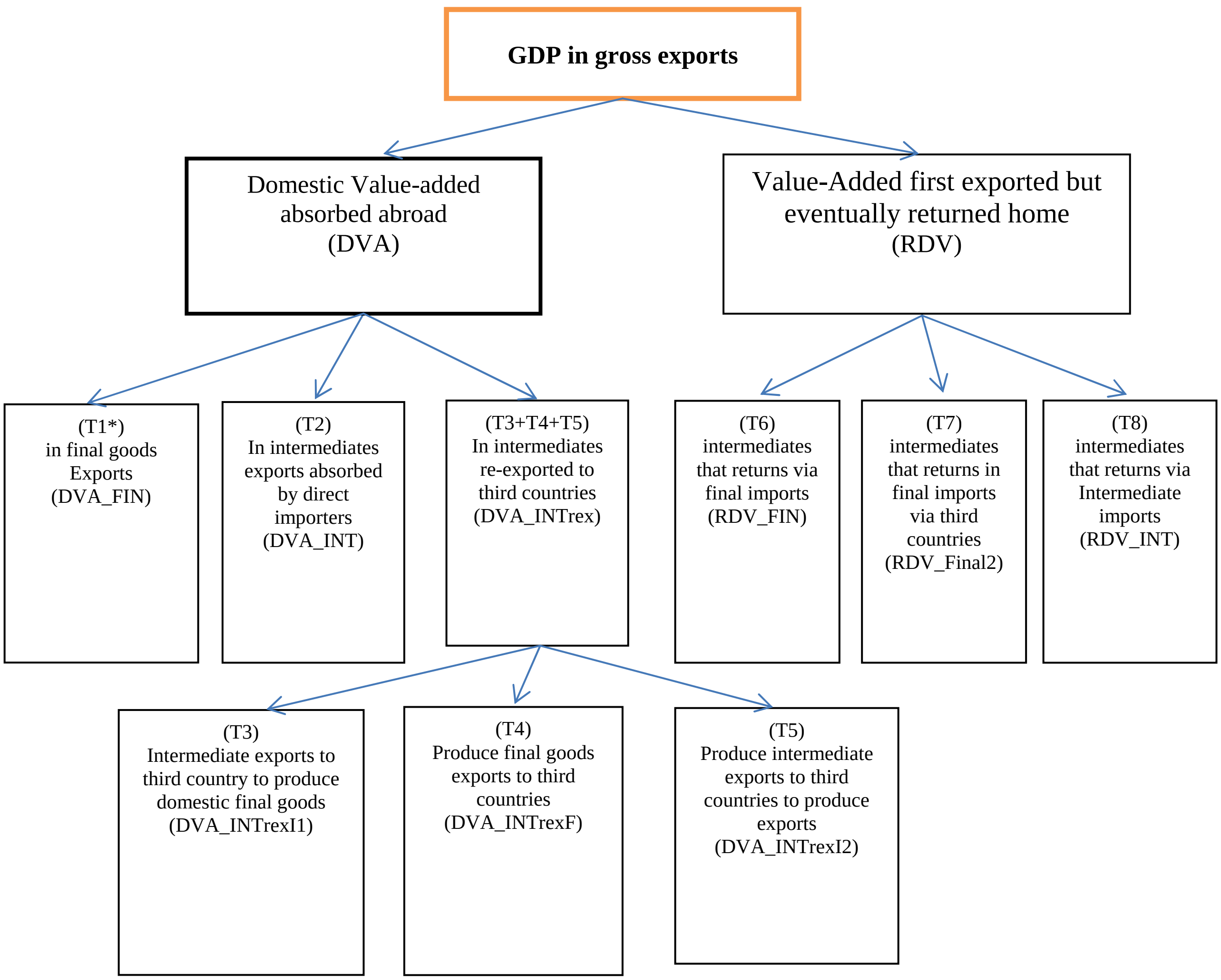
Note: VS is sourced from manufacturing and services sector only.

Figure 1a Gross Exports Accounting: Major Categories



Note: E* can be at country/sector, country aggregate, bilateral /sector or bilateral aggregate; both DVA and RDV are based on backward linkages.

Figure 1b Gross Exports Accounting: Domestic Value-added



Note: *corresponds to terms in equation (52) in the main text.

Figure 1c Gross Exports Accounting: Foreign Value-added

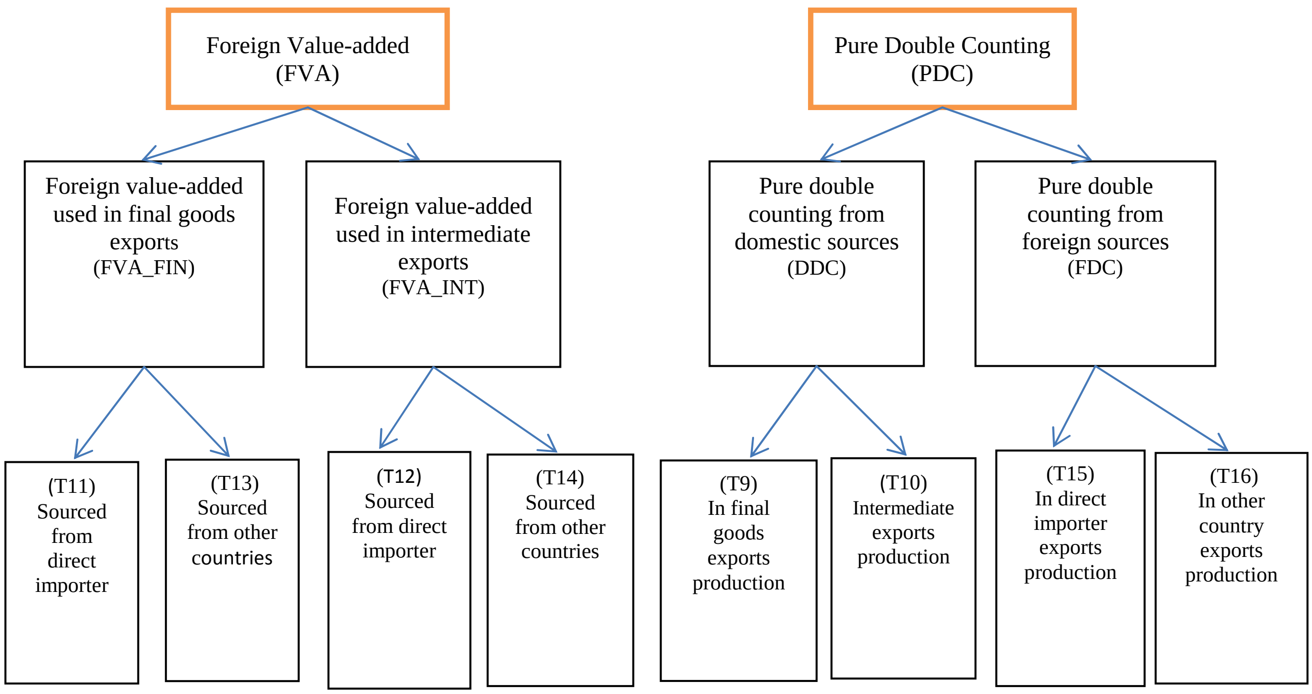


Figure 2a: Structure of US Transport Equipment Exports Decomposition

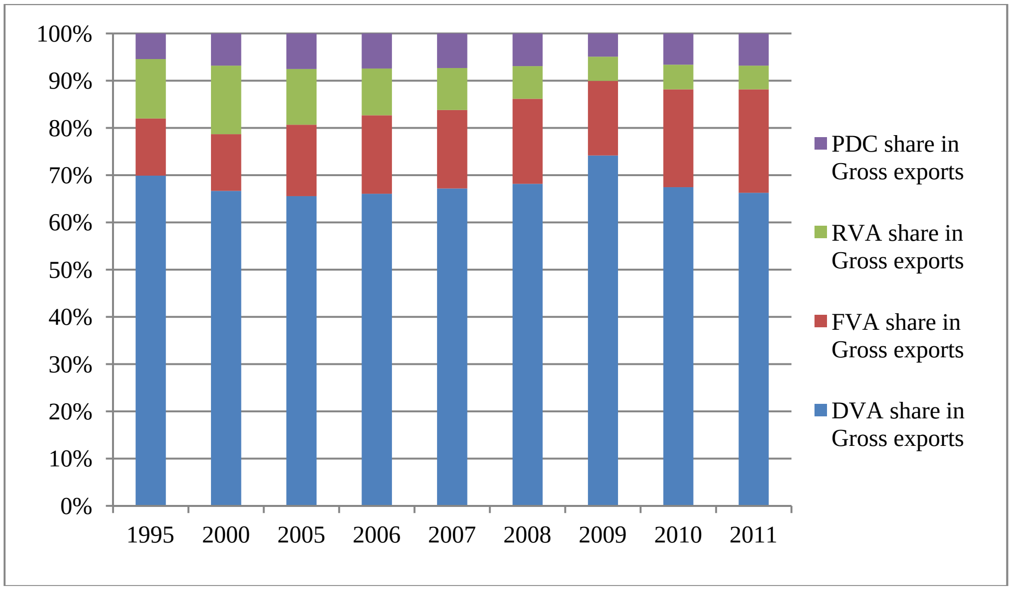


Figure 2b: Structure of Mexico Electrical and Optical Equipment Exports Decomposition

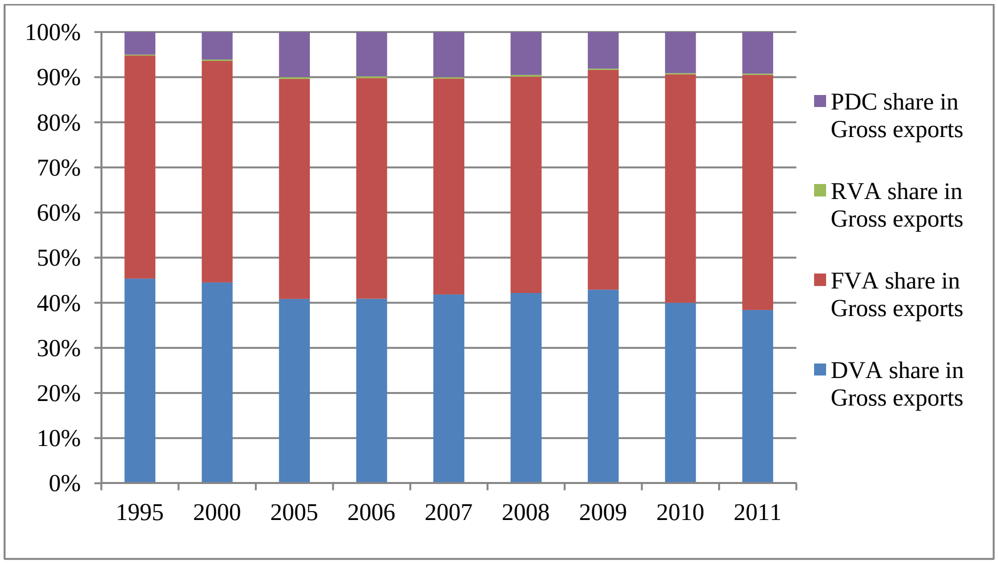


Figure 3a: VS Share by source in US transport equipment exports (Unit %)

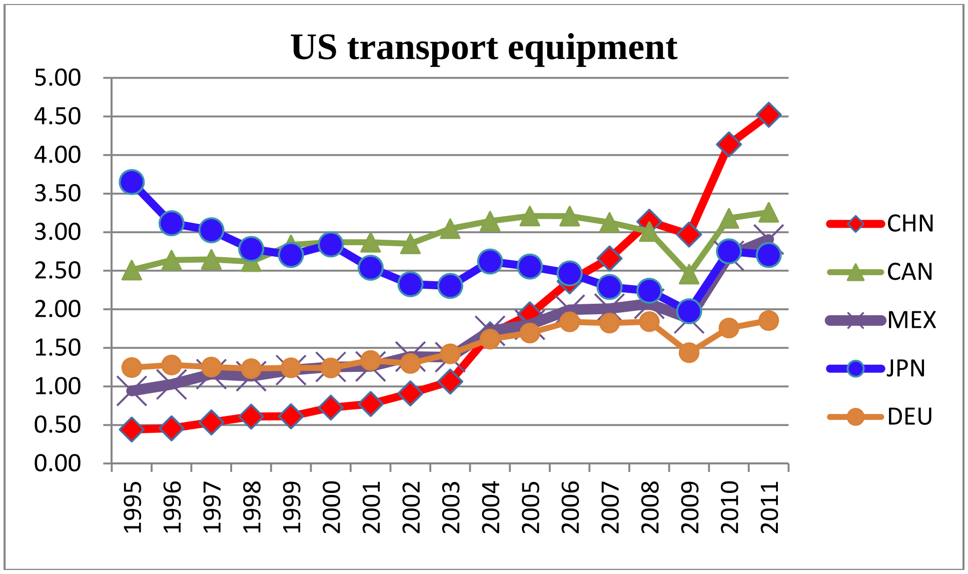


Figure 3b: VS Share by source in DEU transport equipment exports (Unit %)

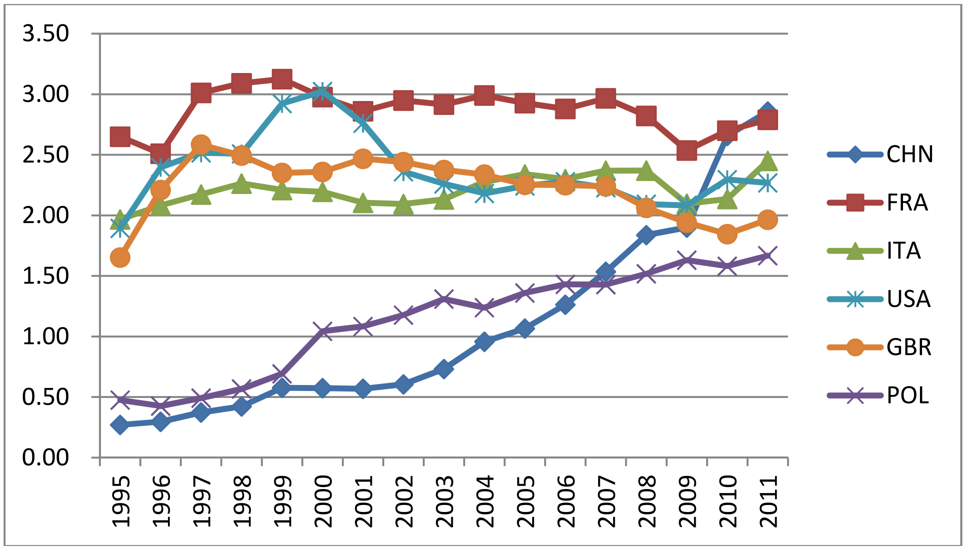


Figure 3c: VS share by source in Mexico's electronics exports (Unit %)

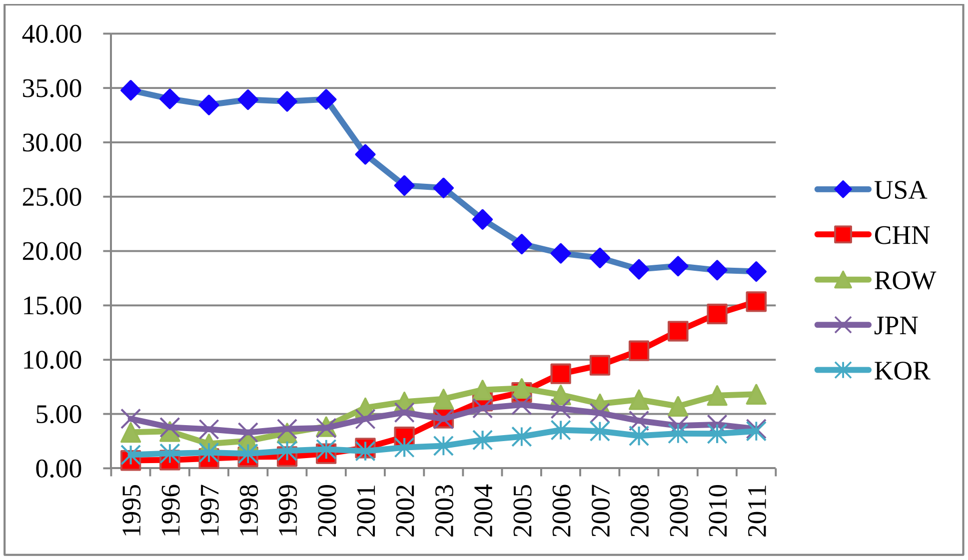


Figure 4: Structure of Germany Business Services Exports Decomposition and VAX ratio

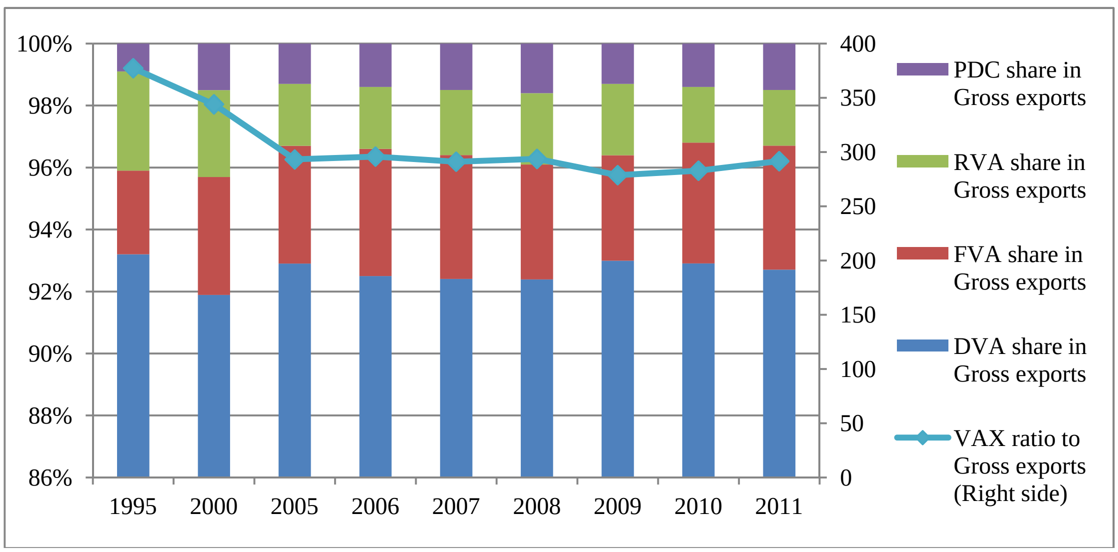


Figure 5a: RCA indexes for electric and optical equipment exports

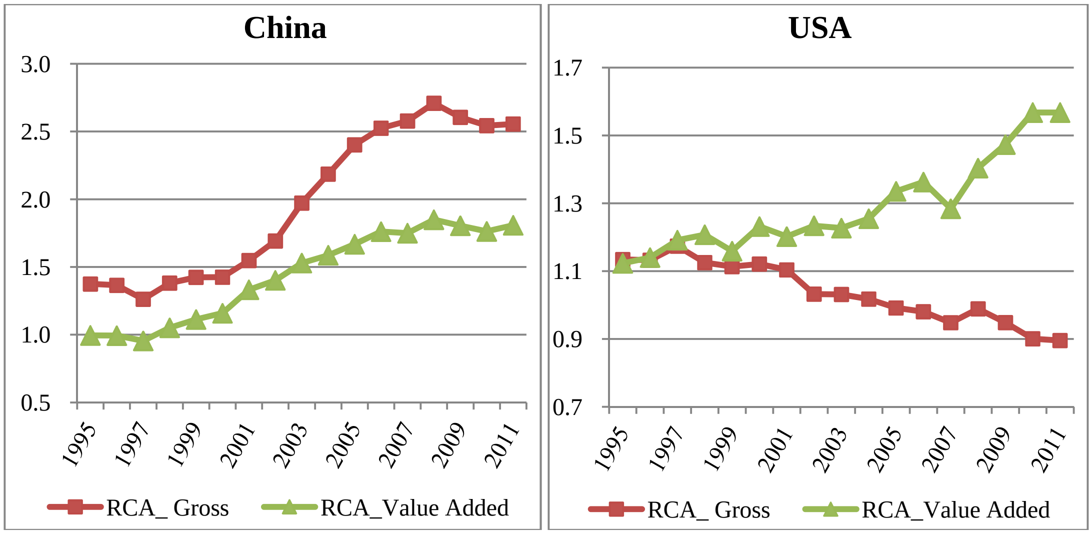


Figure 5b: RCA indexes for business services exports

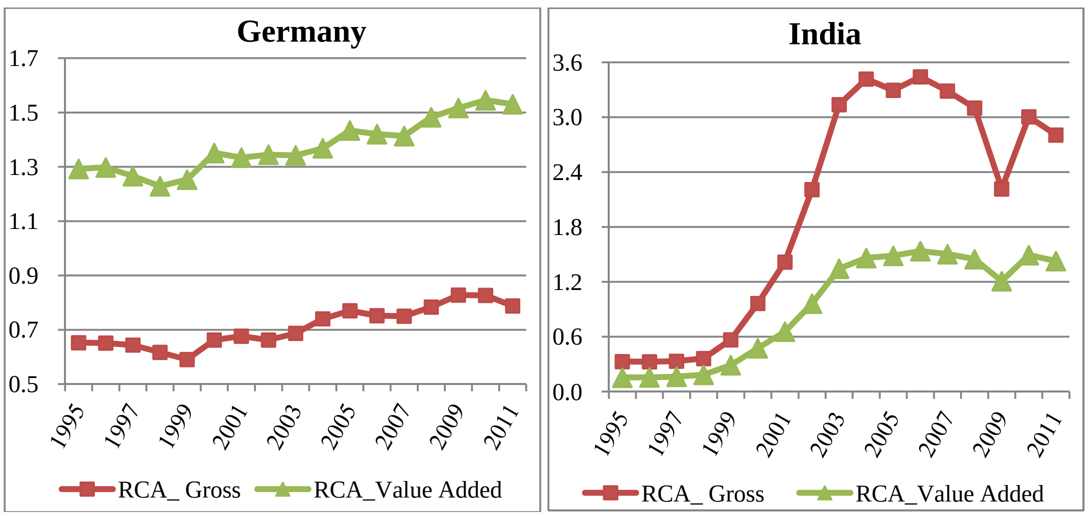


Figure 6a: China and USA trade balance in Electrical and Optical Equipment. Unit: millions USD

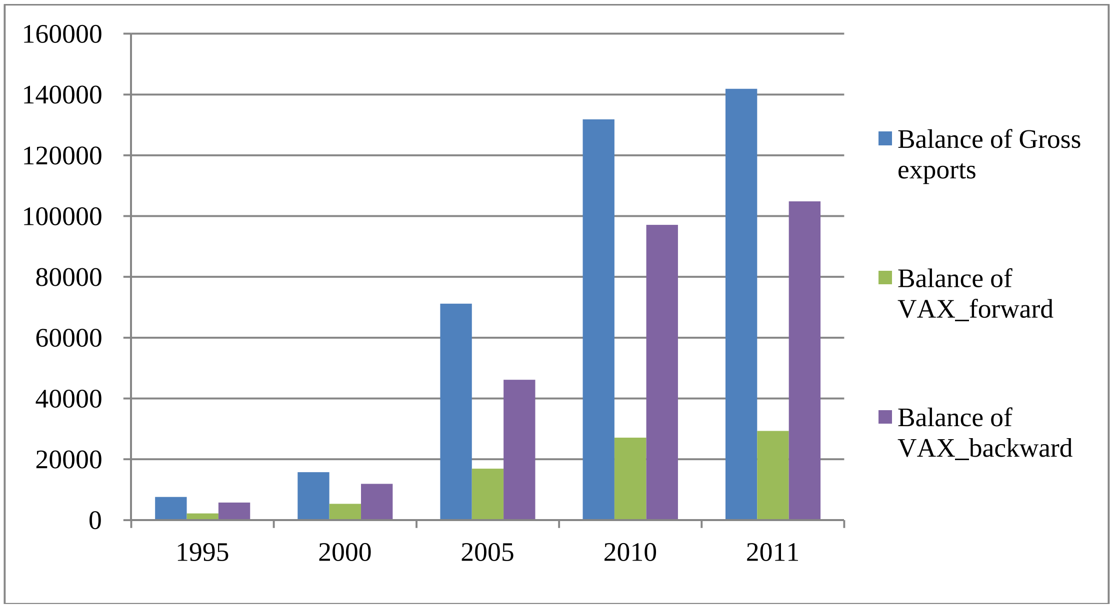


Figure 6b: China and Japan Bilateral trade balance in Rubber and Plastics
Unit: millions USD

Appendix Table A1 WIOD Sectors

Code	NACE	Industry	Description
C01	**AtB**	Agriculture	Agriculture, Hunting, Forestry and Fishing
C02	**C**	Mining	Mining and Quarrying
C03	**15t16**	Food	Food, Beverages and Tobacco
C04	**17t18**	Textiles Products	Textiles and Textile Products
C05	**19**	Leather and Footwear	Leather, Leather and Footwear
C06	**20**	Wood Products	Wood and Products of Wood and Cork
C07	**21t22**	Paper and Printing	Pulp, Paper, Paper, Printing and Publishing
C08	**23**	Refined Petroleum	Coke, Refined Petroleum and Nuclear Fuel
C09	**24**	Chemical Products	Chemicals and Chemical Products
C10	**25**	Rubber and Plastics	Rubber and Plastics
C11	**26**	Other Non-Metal	Other Non-Metallic Mineral
C12	**27t28**	Basic Metals	Basic Metals and Fabricated Metal
C13	**29**	Machinery	Machinery, Nec
C14	**30t33**	Electrical Equipment	Electrical and Optical Equipment
C15	**34t35**	Transport Equipment	Transport Equipment
C16	**36t37**	Recycling	Manufacturing, Nec; Recycling
C17	**E**	Electricity, Gas and Water	Electricity, Gas and Water Supply
C18	**F**	Construction	Construction
C19	**50**	Sale of Vehicles and Fuel	Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel
C20	**51**	Wholesale Trade	Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles
C21	**52**	Retail Trade	Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods
C22	**H**	Hotels and Restaurants	Hotels and Restaurants
C23	**60**	Inland Transport	Inland Transport
C24	**61**	Water Transport	Water Transport
C25	**62**	Air Transport	Air Transport
C26	**63**	Other Transport	Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies
C27	**64**	Post and Telecommunications	Post and Telecommunications
C28	**J**	Financial Intermediation	Financial Intermediation
C29	**70**	Real Estate	Real Estate Activities
C30	**71t74**	Business Activities	Renting of M&Eq and Other Business Activities
C31	**L**	Public Admin	Public Admin and Defense; Compulsory Social Security
C32	**M**	Education	Education
C33	**N**	Health and Social Work	Health and Social Work
C34	**O**	Other Services	Other Community, Social and Personal Services
C35	**P**	Private Households	Private Households with Employed Persons

Appendix Table A2 WIOD Country and Region

Label	Country	Region	Label	Country	Region
AUS	Australia	Asia-Pacific	**IRL**	Ireland	Europe
AUT	Austria	Europe	**ITA**	Italy	Europe
BEL	Belgium	Europe	**JPN**	Japan	Asia-Pacific
BGR	Bulgaria	Europe	**KOR**	South Korea	Asia-Pacific
BRA	Brazil	American	**LTU**	Lithuania	Europe
CAN	Canada	American	**LUX**	Luxembourg	Europe
CHN	China	Asia-Pacific	**LVA**	Latvia	Europe
CYP	Cyprus	Europe	**MEX**	Mexico	American
CZE	Czech Republic	Europe	**MLT**	Malta	Europe
DEU	Germany	Europe	**NLD**	Netherlands	Europe
DNK	Denmark	Europe	**POL**	Poland	Europe
ESP	Spain	Europe	**PRT**	Portugal	Europe
EST	Estonia	Europe	**ROM**	Romania	Europe
FIN	Finland	Europe	**RUS**	Russia	Europe
FRA	France	Europe	**SVK**	Slovak Republic	Europe
GBR	United Kingdom	Europe	**SVN**	Slovenia	Europe
GRC	Greece	Europe	**SWE**	Sweden	Europe
HUN	Hungary	Europe	**TUR**	Turkey	Europe
IDN	Indonesia	Asia-Pacific	**TWN**	Taiwan	Asia-Pacific
IND	India	Asia-Pacific	**USA**	United States	American

Appendix A: Connecting Global and Local Leontief Inverse Matrices

Define local Leontief inverse of country S as:

$$L^{ss}=\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}=\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}$$

And note that it is actually the inverse of the block diagonal matrix in the global IO coefficient matrix A in equation (3) of the main text. Then the following proposition gives the mathematical relation between the block diagonals of global Leontief inverse matrix $B^{ss}=\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}$ and the local Leontief inverse L^{ss}:

Proposition D: if both B^{ss} and L^{ss} exist then

$$B^{ss}\geq L^{ss} \text{ and } B^{ss}-L^{ss}=L^{ss}A^{sr}B^{rs}\geq 0. \qquad \text{(A1)}$$

If and only if $A^{sr}=0$ or $B^{rs}=0$, $B^{ss}=L^{ss}$

This proposition plays an important role in the decomposition of gross exports at the sector level and to the understanding of the difference between trade in value-added estimates from an ICIO table and a national IO table. We give a step by step proof in the 2-country, 2-sector setting bellow and extend it to a N-sector and G-country setting in Appendix G.

Proof:

From the property of inverse matrix:

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}=\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \qquad \text{(A2)}$$

From (A2), we can obtain the relationship between the block diagonals in the global Leontief inverse matrix B and the local (country) Leontief inverse matrix expressed as in the following equation:

$$\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \tag{A3}$$

Rearranging:

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \tag{A4}$$

Because all elements in $\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}$, $\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}$ and $\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}$ are non-negative ,

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \geq 0 .$$

B^{ss} is total output requirement coefficients of country S by one unit increase of its production of final goods, L^{ss} is total output requirement coefficients of country S by one unit increase of its production of final good using domestic intermediate goods, and $B^{ss} - L^{ss}$ is total output requirement coefficients of country S by one unit increase of its production of final goods via its intermediate goods trade. $A^{sr} = 0$ or $B^{rs} = 0$ means there is only one country exporting intermediate goods and only in such condition trade in value-added estimates from an ICIO table will be the same as that from a national IO table.

Appendix B: Derivation of decomposition equation of Country R's gross exports

The gross exports of country R can be decomposed into two parts: final goods exports and intermediate goods exports:

$$E^{rs} = \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} = \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^{s} \\ x_2^{s} \end{bmatrix} \tag{B1}$$

Based on equation (11), country S's gross output can be decomposed as

$$\begin{bmatrix} x_1^{s} \\ x_2^{s} \end{bmatrix} = \begin{bmatrix} x_1^{ss} + x_1^{sr} \\ x_2^{ss} + x_2^{sr} \end{bmatrix} = \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \tag{B2}$$

Inserting equation (B2) into the last term of equation (B1), we can decompose country R's gross intermediate goods exports according to where they are absorbed:

$$\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}$$
$$+ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \quad \text{(B3)}$$

From equation (2), country S's gross output production and use balance condition, we know

$$\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} + \begin{bmatrix} a_{11}^{ss} & a_{12}^{ss} \\ a_{21}^{ss} & a_{22}^{ss} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}^{ss} & a_{12}^{ss} \\ a_{21}^{ss} & a_{22}^{ss} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} + \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \quad \text{(B4)}$$

Re-arrange:

$$\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \quad \text{(B5)}$$

Define $L^{ss} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}$ as local Leontief inverse, then equation (B5) can be re-written as

$$\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \quad \text{(B6)}$$

Therefore, the intermediate goods exports by country R can also be decomposed into two components according to where it is used similar to a single country IO model:

$$\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} = \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} + \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \quad \text{(B7)}$$

From equation (6), we can obtain country R's domestic and foreign value-added multiplier as follows:

$$V^r B^{rr} = \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} = \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} & v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} \tag{B8}$$

$$V^s B^{sr} = \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} = \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} & v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \tag{B9}$$

In a single country IO model, country R's domestic value-added multiplier can be calculate as

$$V^r (I - A^{rr})^{-1} = V^r L^{rr} = \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} = \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} & v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \tag{B10}$$

Using equations (B8)-(B10), and defining "#" as element-wise matrix multiplication operation, the value of country R' gross intermediate exports can be decomposed as

$$\begin{aligned}
\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} &= \left\{ \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} + \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} \right\} \\
&= \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} \right\} + \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} \right\} \\
&= \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} \right\} \\
&+ \left\{ \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} - \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} \right\}
\end{aligned} \tag{B11}$$

Inserting equations (B3) and (B7) into equation (B11), we can obtain the full decomposition of country R's intermediate goods exports:

$$\begin{aligned}
\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} &= \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\
&+ \left\{ \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} - \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^s \\ x_2^s \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix}
\end{aligned} \tag{B12}$$

Finally, based on the Leontief insight, country R's final goods exports can be decomposed into domestic and foreign value-added as follows:

$$\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} = \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \tag{B13}$$

Combining equations (B12) and (B13), we obtain country R's gross exports decomposition equation:

$$
\begin{aligned}
\begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} &= \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \\
&+ \left\{ \begin{bmatrix} v_1^r b_{11}^{rr} + v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr} + v_2^r b_{22}^{rr} \end{bmatrix} - \begin{bmatrix} v_1^r l_{11}^{rr} + v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr} + v_2^r l_{22}^{rr} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} x_1^{s} \\ x_2^{s} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s b_{11}^{sr} + v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr} + v_2^s b_{22}^{sr} \end{bmatrix} \# \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \qquad \text{(B14)}
\end{aligned}
$$

Appendix C: Domestic content of Country S

Since the 2nd-6th terms of equation (24) come from the first 5 terms of equation (22), sum of them equals domestic content of the first and second sector's intermediate exports of country S.

$$
\begin{aligned}
&\begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \right\} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} + \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \right\} \\
&+ \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \qquad \text{(C1)} \\
&= \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} + \left\{ \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} v_1^s l_{11}^{ss} + v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss} + v_2^s l_{22}^{ss} \end{bmatrix} \right\} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\} \\
&= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \right\}
\end{aligned}
$$

Adding the first term of equation (24) into equation (C1), we obtain the domestic contents of country S's gross exports:

$$\begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \left\{ \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \right\}$$

$$= \begin{bmatrix} v_1^s b_{11}^{ss} + v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss} + v_2^s b_{22}^{ss} \end{bmatrix} \# \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} = \begin{bmatrix} \sum_i^2 v_i^s b_{i1}^{ss} e_1^{sr} \\ \sum_i^2 v_i^s b_{i2}^{ss} e_2^{sr} \end{bmatrix} \tag{C2}$$

Appendix D: Consistency between Equation (24) in this paper and Equation (13) in KWW

From equation 24, the decomposition of country S's exports can also be presented in scalar and summation notations:

$$\begin{aligned} e_1^{sr} &= \sum_i^2 v_i^s b_{i1}^{ss} y_1^{sr} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rr} y_k^{rr} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rr} y_k^{rs} \\ &+ \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rs} y_k^{ss} + \sum_i^2 v_i^s l_{i1}^{ss} \sum_j^2 \sum_k^2 a_{1j}^{sr} b_{jk}^{rs} y_k^{sr} + \sum_i^2 v_i^s (b_{i1}^{ss} - l_{i1}^{ss}) \sum_j^2 a_{1j}^{sr} x_j^r \\ &+ \sum_i^2 v_i^r b_{i1}^{rs} y_1^{sr} + \sum_i^2 v_i^r b_{i1}^{rs} \sum_j^2 \sum_k^2 a_{1j}^{sr} l_{jk}^{rr} y_k^{rr} + \sum_i^2 v_i^r b_{i1}^{rs} \sum_j^2 \sum_k^2 a_{1j}^{sr} l_{jk}^{rr} e_k^{rs} \end{aligned} \tag{D1}$$

In matrix notation

$$\begin{aligned} e_1^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\ &+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \\ &+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ 0 \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\ &+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix} \end{aligned} \tag{D2}$$

Similarly, the decomposition of country S's second sector exports to country R can be presented as

$$
\begin{aligned}
e_2^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} 0 \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}
\end{aligned} \tag{D3}
$$

Adding up equation (D2) and (D3), we can get the decomposition of country S's total gross exports:

$$
\begin{aligned}
e_1^{sr} + e_2^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \right\} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \left\{ \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \right\} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^r \\ x_2^r \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}
\end{aligned} \tag{D4}
$$

Based on the definition of Leontief Inverse matrix in equation (3a), the following identity holds:

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} & -a_{11}^{sr} & -a_{12}^{sr} \\ -a_{21}^{ss} & 1-a_{22}^{ss} & -a_{21}^{sr} & -a_{22}^{sr} \\ -a_{11}^{rs} & -a_{12}^{rs} & 1-a_{11}^{rr} & -a_{12}^{rr} \\ -a_{21}^{rs} & -a_{22}^{rs} & -a_{21}^{rr} & 1-a_{22}^{rr} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} & b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{ss} & b_{22}^{ss} & b_{21}^{sr} & b_{22}^{sr} \\ b_{11}^{rs} & b_{12}^{rs} & b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rs} & b_{22}^{rs} & b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \tag{D5}$$

Express in block matrix, equation (D5) becomes

$$\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} - \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix} - \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

With rearrange, we have

$$\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix}$$
$$= \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rr} & b_{12}^{rr} \\ b_{21}^{rr} & b_{22}^{rr} \end{bmatrix} \tag{D6}$$

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1} = \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}$$
$$= \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \tag{D7}$$

$$\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1} = \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} 1-a_{11}^{ss} & -a_{12}^{ss} \\ -a_{21}^{ss} & 1-a_{22}^{ss} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \tag{D8}$$

Combine equation (D7) and (D8):

$$
\begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} - \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} = \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}
$$

$$
= \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \tag{D9}
$$

Insert equation (D6) and (D9) into equation (D4):

$$
\begin{aligned}
e_1^{sr} + e_2^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}
\end{aligned} \tag{D10}
$$

Re-arrange:

$$
\begin{aligned}
e^{sr} = e_1^{sr} + e_2^{sr} &= \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix} + \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^s & v_2^s \end{bmatrix} \begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix} \begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix} \begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix} \begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix} + \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix} \\
&+ \begin{bmatrix} v_1^r & v_2^r \end{bmatrix} \begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix} \begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix} \begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix} \begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}
\end{aligned} \tag{D11}
$$

It is an extension of equation (13) in KWW from a 2-country, 1-sector case into a 2-country,2-sector case.

Appendix E: Difference between global gross exports and global GDP and Consistency between Equation (40) in this paper and Equation (22) in KWW

Subtracting global GDP from global gross exports using equations (B14) and equations (24), (36) and (37) in the main text yields the following:

$$
\begin{aligned}
&E^{sr}+E^{rs}-GDP^{s}-GDP^{r}\\
&=+\begin{bmatrix} v_1^s b_{11}^{sr}+v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr}+v_2^s b_{22}^{sr} \end{bmatrix}\#\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix}+\begin{bmatrix} v_1^s b_{11}^{sr}+v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr}+v_2^s b_{22}^{sr} \end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs} \end{bmatrix}\#\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}+\begin{bmatrix} v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs} \end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^s l_{11}^{ss}+v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss}+v_2^s l_{22}^{ss} \end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} b_{11}^{rs} & b_{12}^{rs} \\ b_{21}^{rs} & b_{22}^{rs} \end{bmatrix}\begin{bmatrix} y_1^{sr} \\ y_2^{sr} \end{bmatrix}\right\}+\left\{\begin{bmatrix} v_1^s b_{11}^{ss}+v_2^s b_{21}^{ss} \\ v_1^s b_{12}^{ss}+v_2^s b_{22}^{ss} \end{bmatrix}-\begin{bmatrix} v_1^s l_{11}^{ss}+v_2^s l_{21}^{ss} \\ v_1^s l_{12}^{ss}+v_2^s l_{22}^{ss} \end{bmatrix}\right\}\#\left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} x_1^{r} \\ x_2^{r} \end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^r l_{11}^{rr}+v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr}+v_2^r l_{22}^{rr} \end{bmatrix}\#\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr} \end{bmatrix}\begin{bmatrix} y_1^{rs} \\ y_2^{rs} \end{bmatrix}+\left\{\begin{bmatrix} v_1^r b_{11}^{rr}+v_2^r b_{21}^{rr} \\ v_1^r b_{12}^{rr}+v_2^r b_{22}^{rr} \end{bmatrix}-\begin{bmatrix} v_1^r l_{11}^{rr}+v_2^r l_{21}^{rr} \\ v_1^r l_{12}^{rr}+v_2^r l_{22}^{rr} \end{bmatrix}\right\}\#\left\{\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} x_1^{s} \\ x_2^{s} \end{bmatrix}\right\}\\
&+\begin{bmatrix} v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs} \\ v_1^r b_{11}^{rs}+v_2^r b_{21}^{rs} \end{bmatrix}\#\left\{\begin{bmatrix} a_{11}^{sr} & a_{12}^{sr} \\ a_{21}^{sr} & a_{22}^{sr} \end{bmatrix}\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} e_1^{rs} \\ e_2^{rs} \end{bmatrix}\right\}+\begin{bmatrix} v_1^s b_{11}^{sr}+v_2^s b_{21}^{sr} \\ v_1^s b_{12}^{sr}+v_2^s b_{22}^{sr} \end{bmatrix}\#\begin{bmatrix} a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs} \end{bmatrix}\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} e_1^{sr} \\ e_2^{sr} \end{bmatrix}\\
&-\begin{bmatrix} v_1^s \\ v_2^s \end{bmatrix}\#\begin{bmatrix} l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss} \end{bmatrix}\begin{bmatrix} y_1^{ss} \\ y_2^{ss} \end{bmatrix}-\begin{bmatrix} v_1^r \\ v_2^r \end{bmatrix}\#\begin{bmatrix} l_{11}^{rr} & l_{12}^{rr} \\ l_{21}^{rr} & l_{22}^{rr} \end{bmatrix}\begin{bmatrix} y_1^{rr} \\ y_2^{rr} \end{bmatrix}\\
&+\begin{bmatrix} v_2^s b_{21}^{ss} y_1^{sr} \\ v_1^s b_{12}^{ss} y_2^{sr} \end{bmatrix}-\begin{bmatrix} v_1^s b_{12}^{ss} y_2^{sr} \\ v_2^s b_{21}^{ss} y_1^{sr} \end{bmatrix}+\begin{bmatrix} v_2^s l_{21}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{sr} b_{jk}^{rr} y_k^{rr} \\ v_1^s l_{12}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{sr} b_{jk}^{rr} y_k^{rr} \end{bmatrix}-\begin{bmatrix} v_1^s l_{12}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{sr} b_{jk}^{rr} y_k^{rr} \\ v_2^s l_{21}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{sr} b_{jk}^{rr} y_k^{rr} \end{bmatrix}\\
&+\begin{bmatrix} v_2^s l_{21}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{sr} b_{jk}^{rr} y_k^{rs} \\ v_1^s l_{12}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{sr} b_{jk}^{rr} y_k^{rs} \end{bmatrix}-\begin{bmatrix} v_1^s l_{12}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{sr} b_{jk}^{rr} y_k^{rs} \\ v_2^s l_{21}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{sr} b_{jk}^{rr} y_k^{rs} \end{bmatrix}+\begin{bmatrix} v_2^s l_{21}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{sr} b_{jk}^{rs} y_k^{ss} \\ v_1^s l_{12}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{sr} b_{jk}^{rs} y_k^{ss} \end{bmatrix}-\begin{bmatrix} v_1^s l_{12}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{sr} b_{jk}^{rr} y_k^{ss} \\ v_2^s l_{21}^{ss}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{sr} b_{jk}^{rs} y_k^{ss} \end{bmatrix}\\
&+\begin{bmatrix} v_2^r b_{21}^{rr} y_1^{rs} \\ v_1^r b_{12}^{rr} y_2^{rs} \end{bmatrix}-\begin{bmatrix} v_1^r b_{12}^{rr} y_2^{rs} \\ v_2^r b_{21}^{rr} y_1^{rs} \end{bmatrix}+\begin{bmatrix} v_2^r l_{21}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{rs} b_{jk}^{ss} y_k^{ss} \\ v_1^r l_{12}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{rs} b_{jk}^{ss} y_k^{ss} \end{bmatrix}-\begin{bmatrix} v_1^r l_{12}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{rs} b_{jk}^{ss} y_k^{ss} \\ v_2^r l_{21}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{rs} b_{jk}^{ss} y_k^{ss} \end{bmatrix}\qquad\text{(E1)}\\
&+\begin{bmatrix} v_2^r l_{21}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{rs} b_{jk}^{ss} y_k^{sr} \\ v_1^r l_{12}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{rs} b_{jk}^{ss} y_k^{sr} \end{bmatrix}-\begin{bmatrix} v_1^r l_{12}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{rs} b_{jk}^{ss} y_k^{sr} \\ v_2^r l_{21}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{rs} b_{jk}^{ss} y_k^{sr} \end{bmatrix}+\begin{bmatrix} v_2^r l_{21}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{rs} b_{jk}^{sr} y_k^{rr} \\ v_1^r l_{12}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{rs} b_{jk}^{sr} y_k^{rr} \end{bmatrix}-\begin{bmatrix} v_1^r l_{12}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{2j}^{rs} b_{jk}^{sr} y_k^{rr} \\ v_2^r l_{21}^{rr}\sum_{j}^{2}\sum_{k}^{2} a_{1j}^{rs} b_{jk}^{sr} y_k^{rr} \end{bmatrix}
\end{aligned}
$$

Multiplying μ =[1,1], the unit vector, with each term on the left hand side of (E1) and cancelling similar terms on the right hand side, we obtain equation (38) in the main text.

Based on equation (40), the aggregation of domestic value added in the two sectors' exports can be presented as

$$
\begin{aligned}
\mu dv^{sr} &= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_1^s \\ v_2^s\end{bmatrix} \# \begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}\begin{bmatrix}y_1^{sr} \\ y_2^{sr}\end{bmatrix} + \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_1^s \\ v_2^s\end{bmatrix} \# \begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rr} \\ y_2^{rr}\end{bmatrix} \\
&+ \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_1^s \\ v_2^s\end{bmatrix} \# \begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rs} \\ y_2^{rs}\end{bmatrix} + \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}v_1^s \\ v_2^s\end{bmatrix} \# \left\{\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix} - \begin{bmatrix}l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\right\}\begin{bmatrix}y_1^{ss} \\ y_2^{ss}\end{bmatrix} \\
&= \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}\begin{bmatrix}y_1^{sr} \\ y_2^{sr}\end{bmatrix} + \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rr} \\ y_2^{rr}\end{bmatrix} \\
&+ \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rs} \\ y_2^{rs}\end{bmatrix} + \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\left\{\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix} - \begin{bmatrix}l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\right\}\begin{bmatrix}y_1^{ss} \\ y_2^{ss}\end{bmatrix}
\end{aligned} \tag{E2}
$$

Insert equation (D9) into (E1)

$$
\begin{aligned}
\mu dv^{sr} &= \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{ss} & b_{12}^{ss} \\ b_{21}^{ss} & b_{22}^{ss}\end{bmatrix}\begin{bmatrix}y_1^{sr} \\ y_2^{sr}\end{bmatrix} + \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rr} \\ y_2^{rr}\end{bmatrix} \\
&+ \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}y_1^{rs} \\ y_2^{rs}\end{bmatrix} + \begin{bmatrix}v_1^s & v_2^s\end{bmatrix}\begin{bmatrix}b_{11}^{sr} & b_{12}^{sr} \\ b_{21}^{sr} & b_{22}^{sr}\end{bmatrix}\begin{bmatrix}a_{11}^{rs} & a_{12}^{rs} \\ a_{21}^{rs} & a_{22}^{rs}\end{bmatrix}\begin{bmatrix}l_{11}^{ss} & l_{12}^{ss} \\ l_{21}^{ss} & l_{22}^{ss}\end{bmatrix}\begin{bmatrix}y_1^{ss} \\ y_2^{ss}\end{bmatrix}
\end{aligned} \tag{E3}
$$

It is easy to show that equation (E3) is the extension of equation (22) in KWW into a 2-country, 2-sector case.

Appendix F: Numerical Example: the 2-country, 2-sector case

The 2-country, 2-sector ICIO table:

Country	Sector	S1	S2	R1	R2	YS	YR	Gross Output
		S		R				
S	S1	1	1	0	0	1	0	3
	S2	0	1	0	1	0	1	3
R	R1	1	0	1	0	0	1	3
	R2	0	0	1	1	1	0	3
Value-added		1	1	1	1			
Total input		3	3	3	3			

Gross intermediate and final good exports matrix is:

$$E = EI + EF = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

The direct input coefficient matrix A, Global Leontief inverse Matrix B and Local Leontief inverse matrix L and direct value-added coefficient vector V are

$$A = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 \end{bmatrix} \qquad B = \begin{bmatrix} 8/5 & 4/5 & 1/5 & 2/5 \\ 1/5 & 8/5 & 2/5 & 4/5 \\ 4/5 & 2/5 & 8/5 & 1/5 \\ 2/5 & 1/5 & 4/5 & 8/5 \end{bmatrix}$$

$$L = \begin{bmatrix} 3/2 & 3/4 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 3/2 & 0 \\ 0 & 0 & 3/4 & 3/2 \end{bmatrix} \qquad V = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

The block matrixes are defined below：

$$E^{sr} = EI^{sr} + EF^{sr} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad E^{rs} = EI^{rs} + EF^{rs} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$Y^{ss} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Y^{rr} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Y^{sr} = EF^{sr} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Y^{rs} = EF^{rs} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A^{ss} = \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1/3 \end{bmatrix}, \quad A^{sr} = \begin{bmatrix} 0 & 0 \\ 0 & 1/3 \end{bmatrix}, \quad A^{rs} = \begin{bmatrix} 1/3 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^{rr} = \begin{bmatrix} 1/3 & 0 \\ 1/3 & 1/3 \end{bmatrix},$$

$$B^{ss} = \begin{bmatrix} 8/5 & 4/5 \\ 1/5 & 8/5 \end{bmatrix}, \quad B^{sr} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}, \quad B^{rs} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix}, \quad B^{rr} = \begin{bmatrix} 8/5 & 1/5 \\ 4/5 & 8/5 \end{bmatrix},$$

$$L^{ss} = \begin{bmatrix} 3/2 & 3/4 \\ 0 & 3/2 \end{bmatrix}, \quad L^{rr} = \begin{bmatrix} 3/2 & 0 \\ 3/4 & 3/2 \end{bmatrix}, \quad V^{s} = \begin{bmatrix} 1/3 & 1/3 \end{bmatrix}, V^{r} = \begin{bmatrix} 1/3 & 1/3 \end{bmatrix}$$

Based on equation (18)-(20), the total value added coefficients can be computed as

$$\left(V^{s} B^{ss}\right)^{T} = \left\{ \begin{bmatrix} 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8/5 & 4/5 \\ 1/5 & 8/5 \end{bmatrix} \right\}^{T} = \begin{bmatrix} 3/5 & 4/5 \end{bmatrix}^{T} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$\left(V^{s} B^{sr}\right)^{T} = \left\{ \begin{bmatrix} 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} \right\}^{T} = \begin{bmatrix} 1/5 & 2/5 \end{bmatrix}^{T} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$$

$$\left(V^r B^{rs}\right)^T = \left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}4/5 & 2/5\\ 2/5 & 1/5\end{bmatrix}\right\}^T = \begin{bmatrix}2/5 & 1/5\end{bmatrix}^T = \begin{bmatrix}2/5\\ 1/5\end{bmatrix}$$

$$\left(V^r B^{rr}\right)^T = \left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}8/5 & 1/5\\ 4/5 & 8/5\end{bmatrix}\right\}^T = \begin{bmatrix}4/5 & 3/5\end{bmatrix}^T = \begin{bmatrix}4/5\\ 3/5\end{bmatrix}$$

$$\left(V^s L^{ss}\right)^T = \left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}3/2 & 3/4\\ 0 & 3/2\end{bmatrix}\right\}^T = \begin{bmatrix}1/2 & 3/4\end{bmatrix}^T = \begin{bmatrix}1/2\\ 3/4\end{bmatrix}$$

$$\left(V^r L^{rr}\right)^T = \left\{\begin{bmatrix}1/3 & 1/3\end{bmatrix}\begin{bmatrix}3/2 & 0\\ 3/4 & 3/2\end{bmatrix}\right\}^T = \begin{bmatrix}3/4 & 1/2\end{bmatrix}^T = \begin{bmatrix}3/4\\ 1/2\end{bmatrix}$$

Based on equation (13), country S’s intermediate exports to country R can be split as

$$A^{sr}B^{rr}Y^{rr} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}8/5 & 1/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 4/15\end{bmatrix}$$

$$A^{sr}B^{rr}Y^{rs} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}8/5 & 1/5\\ 4/5 & 8/5\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 8/15\end{bmatrix}$$

$$A^{sr}B^{rs}Y^{ss} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}4/5 & 2/5\\ 2/5 & 1/5\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 2/15\end{bmatrix}$$

$$A^{sr}B^{rs}Y^{sr} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}4/5 & 2/5\\ 2/5 & 1/5\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 1/15\end{bmatrix}$$

Adding up the four ABY above, we can get the country S’s intermediate exports to country R $EI^{sr} = \begin{bmatrix}0\\ 1\end{bmatrix}$.

Country S’s intermediate exports to country R can be also split as

$$A^{sr}L^{rr}Y^{rr} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}3/2 & 0\\ 3/4 & 3/2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ 1/4\end{bmatrix}$$

$$A^{sr}L^{rr}E^{rs} = \begin{bmatrix}0 & 0\\ 0 & 1/3\end{bmatrix}\begin{bmatrix}3/2 & 0\\ 3/4 & 3/2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 3/4\end{bmatrix}$$

Similarly, country R’s intermediate exports to country S can be split as

$$A^{rs}B^{ss}Y^{ss} = \begin{bmatrix}8/15\\ 0\end{bmatrix}, A^{rs}B^{ss}Y^{sr} = \begin{bmatrix}4/15\\ 0\end{bmatrix},$$

$$A^{rs}B^{sr}Y^{rr} = \begin{bmatrix} 1/15 \\ 0 \end{bmatrix}, A^{rs}B^{sr}Y^{rs} = \begin{bmatrix} 2/15 \\ 0 \end{bmatrix},$$

$$A^{rs}L^{ss}Y^{ss} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, A^{rs}L^{ss}E^{sr} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

Using decomposition equation (24), we can fully decompose country S and R's gross exports into the nine value-added and double counted components as reported in table 3. Detailed computation is listed below:

$$T_1^{sr} = \left(V^s B^{ss}\right)^T \# Y^{sr} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \# \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/5 \end{bmatrix}$$

$$T_2^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr} B^{rr} Y^{rr}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 4/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$T_3^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr} B^{rr} Y^{rs}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 8/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \end{bmatrix}$$

$$T_4^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr} B^{rs} Y^{ss}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 2/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/10 \end{bmatrix}$$

$$T_5^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr} B^{rs} Y^{sr}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 1/15 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/20 \end{bmatrix}$$

$$T_6^{sr} = \left(V^s B^{ss} - V^s L^{ss}\right)^T \# \left(EI^{sr}\right) = \left\{ \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \right\} \# \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/20 \end{bmatrix}$$

$$T_7^{sr} = \left(V^r B^{rs}\right)^T \# Y^{sr} = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix} \# \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/5 \end{bmatrix}$$

$$T_8^{sr} = \left(V^r B^{rs}\right)^T \# \left(A^{rs} L^{ss} Y^{ss}\right) = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix} \# \begin{bmatrix} 0 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/20 \end{bmatrix}$$

$$T_9^{sr} = \left(V^r B^{rs}\right)^T \# \left(A^{rs} L^{ss} E^{sr}\right) = \begin{bmatrix} 2/5 \\ 1/5 \end{bmatrix} \# \begin{bmatrix} 0 \\ 3/4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/20 \end{bmatrix}$$

Adding up the nine components above, we can get the country S's sectoral exports to country R, $E^{sr} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Similarly, country R's sectoral exports to country S can be fully decomposed as

$$T_1^{rs}=\begin{bmatrix}0\\3/5\end{bmatrix},\ T_2^{rs}=\begin{bmatrix}2/5\\0\end{bmatrix},\ T_3^{rs}=\begin{bmatrix}1/5\\0\end{bmatrix},\ T_4^{rs}=\begin{bmatrix}1/20\\0\end{bmatrix},\ T_5^{rs}=\begin{bmatrix}1/10\\0\end{bmatrix},$$

$$T_6^{rs}=\begin{bmatrix}1/20\\0\end{bmatrix},\ T_7^{rs}=\begin{bmatrix}0\\2/5\end{bmatrix},\ T_8^{rs}=\begin{bmatrix}1/10\\0\end{bmatrix},\ T_9^{rs}=\begin{bmatrix}1/10\\0\end{bmatrix}$$

Adding up the nine components above, we can get country R's sectoral exports to country S, $E^{rs}=\begin{bmatrix}1\\1\end{bmatrix}$.

Based on equation (25) and (26), we can estimate country S's VAX to country R at forward linkage.

$$VAX\_F^{sr}=\begin{bmatrix}1/3\\1/3\end{bmatrix}\#\left\{\begin{bmatrix}8/5 & 4/5\\1/5 & 8/5\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+\begin{bmatrix}1/5 & 2/5\\2/5 & 4/5\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right\}=\begin{bmatrix}1/3\\2/3\end{bmatrix}$$

$$VAX\_B^{sr}=T_1^{sr}+T_2^{sr}=\begin{bmatrix}0\\1\end{bmatrix}$$

So the VAX ratio for country S's exports to country R can be estimated as

$$VAX\_F^{sr}\ \ Ratio=\begin{bmatrix}\infty\\1/3\end{bmatrix}\qquad\qquad VAX\_B^{sr}\ \ Ratio=\begin{bmatrix}0\\1/2\end{bmatrix}$$

Appendix G: Numerical Example: the 3-country, 2-sector case

The 3-country, 2-sector ICIO table

Country	Sector	S		R		T		YS	YR	YT	Gross output
		S1	S2	R1	R2	T1	T2				
S	S1	1	1	0	0	0	0	9/10	1/10	0	3
	S2	0	1	0	1	0	0	1	0	0	3
R	R1	0	0	1	1	0	0	0	1	0	3
	R2	0	0	1	1	0	0	0	1	1	4
T	T1	1	0	0	0	1	0	1	0	0	3
	T2	0	0	0	0	1	1	0	0	1	3
Value-added		1	1	1	1	1	2				
Total input		3	3	3	4	3	3				

Gross intermediate and final good exports matrix is

$$E = EI + EF = \begin{bmatrix} 0 & 1/10 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/10 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The direct input coefficient matrix A, Global Leontief inverse Matrix B and Local Leontief inverse matrix L and direct value-added coefficient vector V can be easily computed as

$$A = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 1/3 & 1/4 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix} \qquad B = \begin{bmatrix} 3/2 & 3/4 & 3/20 & 3/10 & 0 & 0 \\ 0 & 3/2 & 3/10 & 3/5 & 0 & 0 \\ 0 & 0 & 9/5 & 3/5 & 0 & 0 \\ 0 & 0 & 4/5 & 8/5 & 0 & 0 \\ 3/4 & 3/8 & 3/40 & 3/20 & 3/2 & 0 \\ 3/8 & 3/16 & 3/80 & 3/40 & 3/4 & 3/2 \end{bmatrix}$$

$$L = \begin{bmatrix} 3/2 & 3/4 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9/5 & 3/5 & 0 & 0 \\ 0 & 0 & 4/5 & 8/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 3/2 \end{bmatrix} \qquad V = \begin{bmatrix} 1/3 & 1/3 & 1/3 & 1/4 & 1/3 & 2/3 \end{bmatrix}$$

The direct input-output coefficients

Name	A^{ss}		A^{sr}		A^{st}	
Block	1/3	1/3	0	0	0	0
matrix	0	1/3	0	1/4	0	0
Name	A^{rs}		A^{rr}		A^{rt}	
Block	0	0	1/3	1/4	0	0
matrix	0	0	1/3	1/4	0	0
Name	A^{ts}		A^{tr}		A^{tt}	
Block	1/3	0	0	0	1/3	0
matrix	0	0	0	0	1/3	1/3

The global Leontief inverse

Name	B^{ss}		B^{sr}		B^{st}	
Block	3/2	3/4	3/20	3/10	0	0
matrix	0	3/2	3/10	3/5	0	0
Name	B^{rs}		B^{rr}		B^{rt}	
Block	0	0	9/5	3/5	0	0

matrix	0	0	4/5	8/5	0	0
Name	B^{ts}		B^{tr}		B^{tt}	
Block	3/4	3/8	3/40	3/20	3/2	0
matrix	3/8	3/16	3/80	3/40	3/4	3/2

The Local Leontief inverse

Name	L^{ss}		L^{rr}		L^{tt}	
Block	3/2	3/4	9/5	3/5	3/2	0
matrix	0	3/2	4/5	8/5	3/4	3/2

The Value Added Coefficients Vectors

Name	V^s		V^r		V^t	
Vectors	1/3	1/3	1/3	1/4	1/3	2/3

Based on equation (18)-(20) and (47), the total value added coefficients can be computed as

Total Value Added Coefficients Vectors

Name	V^sB^{ss}		V^rB^{rr}		V^tB^{tt}	
Vectors	1/2	3/4	4/5	3/5	1	1
Name	V^sL^{ss}		V^rL^{rr}		V^tL^{tt}	
Vectors	1/2	3/4	4/5	3/5	1	1
Name	V^rB^{rs}		V^tB^{tr}		V^sB^{st}	
Vectors	0	0	1/20	1/10	0	0
Name	V^tB^{ts}		V^sB^{sr}		V^rB^{rt}	
Vectors	1/2	1/4	3/20	3/10	0	0

Based on equation (43), country S's intermediate exports to country R can be split as

$$A^{sr}B^{rr}Y^{rr} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 9/5 & 3/5 \\ 4/5 & 8/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/5 \end{bmatrix}$$

$$A^{sr}B^{rt}Y^{tt} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{sr}B^{rr}Y^{rt} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 9/5 & 3/5 \\ 4/5 & 8/5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \end{bmatrix}$$

$$A^{sr}B^{rt}Y^{tr} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{sr}B^{rr}Y^{rs} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 9/5 & 3/5 \\ 4/5 & 8/5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{sr}B^{rt}Y^{ts} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{sr}B^{rs}Y^{ss} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 9/10 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^{sr}B^{rs}(Y^{sr} + Y^{st}) = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Adding up the four ABY above, we can get country S's intermediate exports to country R $EI^{sr} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Country S's intermediate exports to country R can be also split as

$$A^{sr}L^{rr}Y^{rr} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 9/5 & 3/5 \\ 4/5 & 8/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/5 \end{bmatrix}$$

$$A^{sr}L^{rr}E^{r*} = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 9/5 & 3/5 \\ 4/5 & 8/5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/5 \end{bmatrix}$$

Applying decomposition equation (52), we can fully decompose each of the three countries' gross bilateral exports into the 16 value-added and double counted components as reported in table 5. Detailed computation is listed below:

$$T_1^{sr} = \left(V^s B^{ss}\right)^T \# Y^{sr} = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/20 \\ 0 \end{bmatrix}$$

$$T_2^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr}B^{rr}Y^{rr}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 9/20 \end{bmatrix}$$

$$T_3^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr}B^{rt}Y^{tt}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_4^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr}B^{rr}Y^{rt}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/10 \end{bmatrix}$$

$$T_5^{sr} = \left(V^s L^{ss}\right)^T \# \left(A^{sr}B^{rt}Y^{tr}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_6^{sr} = \left(V^s L^{ss}\right)^T \#\left(A^{sr} B^{rr} Y^{rs}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_7^{sr} = \left(V^s L^{ss}\right)^T \#\left(A^{sr} B^{rt} Y^{ts}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_8^{sr} = \left(V^s L^{ss}\right)^T \#\left(A^{sr} B^{rs} Y^{ss}\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_9^{sr} = \left(V^s L^{ss}\right)^T \#\left(A^{sr} B^{rs} (Y^{sr} + Y^{st})\right) = \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_{10}^{sr} = \left(V^s B^{ss} - V^s L^{ss}\right)^T \#\left(EI^{sr}\right) = \left\{ \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 3/4 \end{bmatrix} \right\} \# \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_{11}^{sr} = \left(V^r B^{rs}\right)^T \# Y^{sr} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \# \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_{12}^{sr} = \left(V^r B^{rs}\right)^T \#\left(A^{rs} L^{ss} Y^{ss}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \# \begin{bmatrix} 0 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_{13}^{sr} = \left(V^r B^{rs}\right)^T \#\left(A^{rs} L^{ss} E^{s*}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \# \begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_{14}^{sr} = \left(V^t B^{ts}\right)^T \# Y^{sr} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \# \begin{bmatrix} 1/10 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/20 \\ 0 \end{bmatrix}$$

$$T_{15}^{sr} = \left(V^t B^{ts}\right)^T \#\left(A^{rs} L^{ss} Y^{ss}\right) = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/20 \end{bmatrix}$$

$$T_{16}^{sr} = \left(V^t B^{ts}\right)^T \#\left(A^{rs} L^{ss} E^{s*}\right) = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \# \begin{bmatrix} 0 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/10 \end{bmatrix}$$

Adding up the nine components above, we can get country S's sectoral exports to country R $E^{sr} = \begin{bmatrix} 1/10 \\ 1 \end{bmatrix}$.

Similarly, other bilateral trade can be fully decomposed as in Table 5.

Appendix H: The General Case of G Countries and N Sectors

This appendix specifies the general case with any arbitrary number of countries and sectors. The ICIO model, the gross output decomposition matrix based on the

Leontief insight, and the value-added multiplier or value added share by source matrix can be specified as follows:

$$\begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^G \end{bmatrix} = \begin{bmatrix} A^{11} & A^{12} & \cdots & A^{1G} \\ A^{21} & A^{22} & \cdots & A^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ A^{G1} & A^{G2} & \cdots & A^{GG} \end{bmatrix} \begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^G \end{bmatrix} + \begin{bmatrix} Y^{11} & Y^{12} & \cdots & Y^{1G} \\ Y^{21} & Y^{22} & \cdots & Y^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{G1} & Y^{G2} & \cdots & Y^{GG} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \tag{H1}$$

$$\begin{bmatrix} X^{11} & X^{12} & \cdots & X^{1G} \\ X^{21} & X^{22} & \cdots & X^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ X^{G1} & X^{G2} & \cdots & X^{GG} \end{bmatrix} = \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1G} \\ B^{21} & B^{22} & \cdots & B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ B^{G1} & B^{G2} & \cdots & B^{GG} \end{bmatrix} \begin{bmatrix} Y^{11} & Y^{12} & \cdots & Y^{1G} \\ Y^{21} & Y^{22} & \cdots & Y^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{G1} & Y^{G2} & \cdots & Y^{GG} \end{bmatrix} \tag{H2}$$

$$VB = \begin{bmatrix} V^1 & 0 & \cdots & 0 \\ 0 & V^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V^G \end{bmatrix} \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1G} \\ B^{21} & B^{22} & \cdots & B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ B^{G1} & B^{G2} & \cdots & B^{GG} \end{bmatrix} = \begin{bmatrix} V^1B^{11} & V^1B^{12} & \cdots & V^1B^{1G} \\ V^2B^{21} & V^2B^{22} & \cdots & V^2B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ V^GB^{G1} & V^GB^{G2} & \cdots & V^GB^{GG} \end{bmatrix}^T \tag{H3}$$

The sum of value added share from all countries in country S's production equals to unity.

$$\sum_{t}^{G} V^t B^{ts} = \mu \tag{H4}$$

With G countries and N sectors, **A**, and **B** are GN×GN matrices. $\mathbf{A}_{sr}$ is an N×N block input-output coefficient matrix, and $\mathbf{B}_{sr}$ denotes the N×N block Leontief (global) inverse matrix, which is the total requirement matrix that describes the amount of gross output in producing country S required for a one-unit increase in the final demand in destination country R. $\mathbf{V}_s$ is a 1 by N vector of direct value-added coefficients of country S. $\mathbf{X}_{sr}$ is an N×1 gross output vector that gives gross output produced in S and absorbed in R. $\mathbf{X}_s = \sum_r^G \mathbf{X}_{sr}$ is also an N×1 vector that gives country S' total gross output. $\mathbf{Y}_{sr}$ is an N×1 vector gives final goods produced in s and consumed in R. $\mathbf{Y}_s = \sum_r^G \mathbf{Y}_{sr}$ is also an N×1 vector that gives the global use of S' final goods. The final demand matrix **Y** in equation (H1), the gross output decomposition matrix **X** in equation (H2) and the value-added multiplier matrix **V** are all GN×G matrices.

Country S's gross exports to country R include intermediate and final goods exports:

$$E^{sr} = Y^{sr} + A^{sr} X^{r} \tag{H5}$$

Where E^{sr} is an N by1 vector of country S's gross exports to country R.

Based on equation (H3), country R's gross output can be decomposed as

$$\begin{aligned} X^{r} &= \sum_{t}^{G} X^{rt} = \sum_{t}^{G}\sum_{u}^{G} B^{rt} Y^{tu} \\ &= B^{rr}Y^{rr} + \sum_{t\neq s,r}^{G} B^{rt}Y^{tt} + B^{rr}\sum_{t\neq s,r}^{G} Y^{rt} + \sum_{t\neq s,r}^{G}\sum_{u\neq s,t}^{G} B^{rt}Y^{tu} \\ &+ B^{rr}Y^{rs} + \sum_{t\neq s,r}^{G} B^{rt}Y^{ts} + B^{rs}Y^{ss} + \sum_{t\neq s}^{G} B^{rs}Y^{st} \end{aligned} \tag{H6}$$

Inserting equation (H6) in to country S's intermediate exports to country R, the last term in equation (H5) can be expressed as:

$$\begin{aligned} A^{sr}X^{r} &= A^{sr}B^{rr}Y^{rr} + A^{sr}\sum_{t\neq s,r}^{G} B^{rt}Y^{tt} + A^{sr}B^{rr}\sum_{t\neq s,r}^{G} Y^{rt} + A^{sr}\sum_{t\neq s,r}^{G}\sum_{u\neq s,t}^{G} B^{rt}Y^{tu} \\ &+ A^{sr}B^{rr}Y^{rs} + A^{sr}\sum_{t\neq s,r}^{G} B^{rt}Y^{ts} + A^{sr}B^{rs}Y^{ss} + A^{sr}\sum_{t\neq s}^{G} B^{rs}Y^{st} \end{aligned} \tag{H7}$$

On the right-hand side of equation H7, country S's intermediate exports are split into eight terms, similar to equation (43) in the three country model. The 1st term ($A^{sr}B^{rr}Y^{rr}$), 5th term ($A^{sr}B^{rr}Y^{rs}$), and 7th term($A^{sr}B^{rs}Y^{ss}$) are country S's intermediate exports which are direct absorbed by the importing country to produce its domestic consumed final goods; used by the direct importing country to produce its final goods exports and shipped back to the source country; and used by the direct importing country to produce intermediate goods exports and shipped back to the source country for production of source country's final goods for domestic consumption, respectively, the same as the three terms in equation (43), but without giving detailed sector elements in each of the three related block (A, B, Y) matrixes. The 2nd term ($A^{sr}\sum_{t\neq s,r}^{G} B^{rt}Y^{tt}$), 3rd term ($A^{sr}B^{rr}\sum_{t\neq s,r}^{G} Y^{rt}$), 4th term ($A^{sr}\sum_{t\neq s,r}^{G}\sum_{u\neq s,t}^{G} B^{rt}Y^{tu}$), and the 6th term ($A^{sr}\sum_{t\neq s,r}^{G}B^{rt}Y^{ts}$) are country S's intermediate exports which are used by the direct importing country to produce intermediate exports to the third country in production of its domestic consumed final goods; used by direct importing country to produce its final exports to the third country (but do not return back to the source country); used by the direct importing country to produce intermediate exports to the third country T for production of final exports shipped to other countries including those returning back to the direct importer (country R); used by the direct importing country to produce intermediate exports to the third country for production of final exports that return back to the source country respectively. Although with a very similar economic interpretation as those terms in equation (43), all of these four third-country terms in equation (H7) include all other G-2 counties, not only a single third country T as that in equation (43).This means all other counties besides the two partner countries that are the final destinations of the source country S's intermediate exports , are aggregated together as one group in equation (H7). The final term, $A^{sr}\sum_{t\neq s}^{G}B^{rs}Y^{st}$, is country S's intermediate exports used by the direct importing country to produce intermediate goods exports that are shipped back to source country for production of its own total final goods exports, similar to the last term in equation (43) for the three-country model.

Based on equation (34), we can decompose Country S's intermediate exports to country R into two parts by using the gross output use identity:

$$A^{sr}X^{r} = A^{sr}L^{rr}Y^{rr} + A^{sr}L^{rr}E^{r*} \tag{H8}$$

Where L^{rr} is the N by N local Leontief inverse matrix, and E^{r*} is an N by 1 vector of total gross exports by country R. $E^{r*}=\sum_{t\neq r}^{G}E^{rt}$.

From equation (H3), we can obtain the total value-added multiplier for every country:

$$V^sB^{ss}=\begin{bmatrix}\sum_i^N v_i^s b_{i1}^{ss}\\ \sum_i^N v_i^s b_{i2}^{ss}\\ \vdots\\ \sum_i^N v_i^s b_{iN}^{ss}\end{bmatrix}\quad V^rB^{rs}=\begin{bmatrix}\sum_i^N v_i^r b_{i1}^{rs}\\ \sum_i^N v_i^r b_{i2}^{rs}\\ \vdots\\ \sum_i^N v_i^r b_{iN}^{rs}\end{bmatrix}\quad V^tB^{ts}=\begin{bmatrix}\sum_i^N v_i^t b_{i1}^{ts}\\ \sum_i^N v_i^t b_{i2}^{ts}\\ \vdots\\ \sum_i^N v_i^t b_{iN}^{ts}\end{bmatrix} \tag{H9}$$

Using equations (H7) to (H9) and add the decomposition of country S's final goods exports to country R based on the Leontief insight directly we obtain the decomposition equation of gross bilateral exports from country S to country R in the most general G-country N-sector case as follows:

$$\begin{aligned}
E^{sr} &= V^sB^{ss}\#Y^{sr} + V^sL^{ss}\#A^{sr}X^r + (V^sB^{ss}-V^sL^{ss})\#A^{sr}X^r\\
&+V^rB^{rs}\#Y^{sr} + V^rB^{rs}\#A^{sr}X^r + \sum_{t\neq s,r}^{G}V^tB^{ts}\#Y^{sr} + \sum_{t\neq s,r}^{G}V^tB^{ts}\#A^{sr}X^r\\
&= V^sB^{ss}\#Y^{sr} + V^sL^{ss}\#A^{sr}B^{rr}Y^{rr} + V^sL^{ss}\#A^{sr}\sum_{t\neq s,r}^{G}B^{rt}Y^{tt}\\
&+V^sL^{ss}\#A^{sr}B^{rr}\sum_{t\neq s,r}^{G}Y^{rt} + V^sL^{ss}\#A^{sr}\sum_{t\neq s,r}^{G}\sum_{u\neq s,t}^{G}B^{rt}Y^{tu}\\
&+V^sL^{ss}\#A^{sr}B^{rr}Y^{rs} + V^sL^{ss}\#A^{sr}\sum_{t\neq s,r}^{G}B^{rt}Y^{ts} + V^sL^{ss}\#A^{sr}B^{rs}Y^{ss}\\
&+V^sL^{ss}\#A^{sr}\sum_{t\neq s}^{G}B^{rs}Y^{st} + (V^sB^{ss}-V^sL^{ss})\#A^{sr}X^r\\
&+V^rB^{rs}\#Y^{sr} + V^rB^{rs}\#A^{sr}L^{rr}Y^{rr} + V^rB^{rs}\#A^{sr}L^{rr}E^{r*}\\
&+\sum_{t\neq s,r}^{G}V^tB^{ts}\#Y^{sr} + \sum_{t\neq s,r}^{G}V^tB^{ts}\#A^{sr}L^{rr}Y^{rr} + \sum_{t\neq s,r}^{G}V^tB^{ts}\#A^{sr}L^{rr}E^{r*}
\end{aligned} \tag{H10}$$

The economic interpretations for the 16 terms in equations (H10) are similar to equation (52), so we do not repeat here to save space. The only difference is that all the third-country related terms become a sum of G-2 countries except for the two trading partner countries, instead of just one third Country T, as in equation (52).

Summing up all the G-1 trading partners, we obtain the decomposition equation of country S's gross exports to the world:

$$
\begin{aligned}
E^{s*} &= \sum_{r\neq s}^{G} E^{sr} = V^s B^{ss} \# \sum_{r\neq s}^{G} Y^{sr} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} Y^{rr} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} B^{rt} Y^{tt} \\
&+ V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} \sum_{t\neq s,r}^{G} Y^{rt} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} \sum_{u\neq s,t}^{G} B^{rt} Y^{tu} \\
&+ V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} Y^{rs} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} B^{rt} Y^{ts} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rs} Y^{ss} \\
&+ V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s}^{G} B^{rs} Y^{st} + (V^s B^{ss} - V^s L^{ss}) \# \sum_{r\neq s}^{G} A^{sr} X^{r} \\
&+ \sum_{r\neq s}^{G} (V^r B^{rs} \# Y^{sr} + \sum_{t\neq s,r}^{G} V^t B^{ts} \# Y^{sr}) + \sum_{r\neq s}^{G} (V^r B^{rs} \# A^{sr} L^{rr} Y^{rr} + \sum_{t\neq s,r}^{G} V^t B^{ts} \# A^{sr} L^{rr} Y^{rr}) \\
&+ \sum_{r\neq s}^{G} (V^r B^{rs} \# A^{sr} L^{rr} E^{r*} + \sum_{t\neq s,r}^{G} V^t B^{ts} \# A^{sr} L^{rr} E^{r*}) \\
&= V^s B^{ss} \# \sum_{r\neq s}^{G} Y^{sr} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} Y^{rr} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} B^{rt} Y^{tt} \\
&+ V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} \sum_{t\neq s,r}^{G} Y^{rt} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} \sum_{u\neq s,t}^{G} B^{rt} Y^{tu} \\
&+ V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} Y^{rs} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} B^{rt} Y^{ts} + V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rs} Y^{ss} \\
&+ V^s L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s}^{G} B^{rs} Y^{st} + (V^s B^{ss} - V^s L^{ss}) \# \sum_{r\neq s}^{G} A^{sr} X^{r} \\
&+ \sum_{r\neq s}^{G} \sum_{t\neq s}^{G} V^t B^{ts} \# Y^{sr} + \sum_{r\neq s}^{G} \sum_{t\neq s}^{G} V^t B^{ts} \# A^{sr} L^{rr} Y^{rr} + \sum_{r\neq s}^{G} \sum_{t\neq s}^{G} V^t B^{ts} \# A^{sr} L^{rr} E^{r*}
\end{aligned}
\tag{H11}
$$

As a sum of domestic value-added in gross exports to all other G-1 countries, the first 10 terms that decompose country S's domestic value-added in exports have the same economic interpretations as the first 10 terms in equation (H10). However, the 6 terms that decompose foreign content in bilateral gross exports are summed to three terms with no distinction between direct importing country and all other countries.

Summing up equation (H11) by sectors, we can obtain a decomposition equation for total gross exports of country S, which is exactly the same as equation (36) in KWW. Detailed math proof is given below.

Consistency between Equation (H11) and Equation (36) in KWW

Based on equation (H9), we can derive the total value-added multiplier for every country by sector as

$$
\begin{aligned}
\mu V^{s} B^{ss} &= \sum_{i}^{N}\sum_{j}^{N} v_i^s b_{ij}^{ss} = \overline{V}^{s} B^{ss} \mu^{T} \quad & \mu V^{s} L^{ss} &= \sum_{i}^{N}\sum_{j}^{N} v_i^s l_{ij}^{ss} = \overline{V}^{s} L^{ss} \mu^{T} \\
\mu V^{r} B^{rs} &= \sum_{i}^{N}\sum_{j}^{N} v_i^r b_{ij}^{rs} = \overline{V}^{r} B^{rs} \mu^{T} \quad & \mu V^{t} B^{ts} &= \sum_{i}^{N}\sum_{j}^{N} v_i^t b_{ij}^{ts} = \overline{V}^{t} B^{ts} \mu^{T}
\end{aligned}
\tag{H12}
$$

where $\overline{V}^{s}$, $\overline{V}^{r}$ and $\overline{V}$ are row vectors of direct value added coefficients of country S, R and T, respectively. μ^{T} is the transpose of μ.

Summing up equation (H12) by sectors, we can obtain a decomposition equation for total gross exports of country S

$$
\begin{aligned}
\mu E^{s*} &= \mu V^{s} B^{ss} \# \sum_{r\neq s}^{G} Y^{sr} + \mu V^{s} L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} Y^{rr} + \mu V^{s} L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} B^{rt} Y^{tt} \\
&+ \mu V^{s} L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} \sum_{t\neq s,r}^{G} Y^{rt} + \mu V^{s} L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} \sum_{u\neq s,t}^{G} B^{rt} Y^{tu} \\
&+ \mu V^{s} L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rr} Y^{rs} + \mu V^{s} L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} B^{rt} Y^{ts} + \mu V^{s} L^{ss} \# \sum_{r\neq s}^{G} A^{sr} B^{rs} Y^{ss} \\
&+ \mu V^{s} L^{ss} \# \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s}^{G} B^{rs} Y^{st} + \mu (V^{s} B^{ss} - V^{s} L^{ss}) \# \sum_{r\neq s}^{G} A^{sr} X^{r} \\
&+ \mu \sum_{r\neq s}^{G}\sum_{t\neq s}^{G} V^{t} B^{ts} \# Y^{sr} + \mu \sum_{r\neq s}^{G}\sum_{t\neq s}^{G} V^{t} B^{ts} \# A^{sr} L^{rr} Y^{rr} + \mu \sum_{r\neq s}^{G}\sum_{t\neq s}^{G} V^{t} B^{ts} \# A^{sr} L^{rr} E^{r*}
\end{aligned}
\tag{H13}
$$

Inserting equation (H12) into (H13):

$$
\begin{aligned}
\mu E^{s*} &= \overline{V}^{s} B^{ss} \sum_{r\neq s}^{G} Y^{sr} + \overline{V}^{s} L^{ss} \sum_{r\neq s}^{G} A^{sr} B^{rr} Y^{rr} + \overline{V}^{s} L^{ss} \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} B^{rt} Y^{tt} \\
&+ \overline{V}^{s} L^{ss} \sum_{r\neq s}^{G} A^{sr} B^{rr} \sum_{t\neq s,r}^{G} Y^{rt} + \overline{V}^{s} L^{ss} \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} \sum_{u\neq s,t}^{G} B^{rt} Y^{tu} \\
&+ \overline{V}^{s} L^{ss} \sum_{r\neq s}^{G} A^{sr} B^{rr} Y^{rs} + \overline{V}^{s} L^{ss} \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s,r}^{G} B^{rt} Y^{ts} + \overline{V}^{s} L^{ss} \sum_{r\neq s}^{G} A^{sr} B^{rs} Y^{ss} \\
&+ \overline{V}^{s} L^{ss} \sum_{r\neq s}^{G} A^{sr} \sum_{t\neq s}^{G} B^{rs} Y^{st} + (\overline{V}^{s} B^{ss} - \overline{V}^{s} L^{ss}) \sum_{r\neq s}^{G} A^{sr} X^{r} \\
&+ \sum_{r\neq s}^{G}\sum_{t\neq s}^{G} \overline{V}^{t} B^{ts} Y^{sr} + \mu \sum_{r\neq s}^{G}\sum_{t\neq s}^{G} \overline{V}^{t} B^{ts} A^{sr} L^{rr} Y^{rr} + \sum_{r\neq s}^{G}\sum_{t\neq s}^{G} \overline{V}^{t} B^{ts} A^{sr} L^{rr} E^{r*}
\end{aligned}
\tag{H14}
$$

Re-arranging:

$$\mu E^{s*} = \overline{V}^s B^{ss} \sum_{r \neq s}^{G} Y^{sr} + \overline{V}^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} Y^{tt} + \overline{V}^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} \sum_{u \neq s,t}^{G} Y^{tu}$$
$$+ \overline{V}^s L^{ss} \sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} Y^{ts} + \overline{V}^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss}$$
$$+ \overline{V}^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} \sum_{t \neq s}^{G} Y^{st} + \overline{V}^s (B^{ss} - L^{ss}) \sum_{r \neq s}^{G} A^{sr} X^r \tag{H15}$$
$$+ \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^t B^{ts} Y^{sr} + \mu \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^t B^{ts} A^{sr} L^{rr} Y^{rr} + \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^t B^{ts} A^{sr} L^{rr} E^{r*}$$

By doing the following manipulations,

$$\sum_{r \neq s}^{G} A^{sr} \sum_{t \neq s}^{G} B^{rt} = \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr}$$

equation (H15) can be re-arranged as

$$\mu E^{s*} = \overline{V}^s B^{ss} \sum_{r \neq s}^{G} Y^{sr} + \overline{V}^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} Y^{rr} + \overline{V}^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} \sum_{u \neq s,r}^{G} Y^{ru}$$
$$+ \overline{V}^s L^{ss} \sum_{t \neq s}^{G} A^{st} \sum_{r \neq s}^{G} B^{tr} \sum_{r \neq s}^{G} Y^{rs} + \overline{V}^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss}$$
$$+ \overline{V}^s L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} \sum_{t \neq s}^{G} Y^{st} + \overline{V}^s (B^{ss} - L^{ss}) \sum_{r \neq s}^{G} A^{sr} X^r \tag{H16}$$
$$+ \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^t B^{ts} Y^{sr} + \mu \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^t B^{ts} A^{sr} L^{rr} Y^{rr} + \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^t B^{ts} A^{sr} L^{rr} E^{r*}$$

Based on the definition of global Leontief Inverse matrix, the following identity holds:

$$\begin{bmatrix} I - A^{11} & -A^{12} & \cdots & -A^{1G} \\ -A^{21} & I - A^{22} & \cdots & -A^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ -A^{G1} & -A^{G2} & \cdots & I - A^{GG} \end{bmatrix} \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1G} \\ B^{21} & B^{22} & \cdots & B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ B^{G1} & B^{G2} & \cdots & B^{GG} \end{bmatrix} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$$
$$= \begin{bmatrix} B^{11} & B^{12} & \cdots & B^{1G} \\ B^{21} & B^{22} & \cdots & B^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ B^{G1} & B^{G2} & \cdots & B^{GG} \end{bmatrix} \begin{bmatrix} I - A^{11} & -A^{12} & \cdots & -A^{1G} \\ -A^{21} & I - A^{22} & \cdots & -A^{2G} \\ \vdots & \vdots & \ddots & \vdots \\ -A^{G1} & -A^{G2} & \cdots & I - A^{GG} \end{bmatrix} \tag{H17}$$

From (H17) we can obtain the following two equations:

$$(I - A^{ss}) B^{sr} - \sum_{t \neq s}^{G} A^{st} B^{tr} = 0 \tag{H18}$$

$$(I - A^{ss}) B^{ss} - \sum_{r \neq s}^{G} A^{sr} B^{rs} = I = B^{ss} (I - A^{ss}) - \sum_{r \neq s}^{G} B^{sr} A^{rs} \tag{H19}$$

Re-arranging equation (H18) and (H19):

$$B^{sr} = (I - A^{ss})^{-1} \sum_{r \neq s}^{G} A^{sr} B^{rr} = L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rr} \tag{H20}$$

$$L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} = B^{ss} - L^{ss} = \sum_{r \neq s}^{G} B^{sr} A^{rs} L^{ss} \tag{H21}$$

Inserting equation (H20) and (H21) into equation (H13):

$$\begin{aligned} \mu E^{s*} &= \overline{V}^{s} B^{ss} \sum_{r \neq s}^{G} Y^{sr} + \overline{V}^{s} L^{ss} \sum_{r \neq s}^{G} B^{sr} Y^{rr} + \overline{V}^{s} \sum_{r \neq s}^{G} B^{sr} \sum_{u \neq s,r}^{G} Y^{ru} + \overline{V}^{s} \sum_{r \neq s}^{G} B^{sr} Y^{rs} \\ &+ \overline{V}^{s} L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss} + \overline{V}^{s} \sum_{r \neq s}^{G} B^{sr} A^{rs} L^{ss} \sum_{t \neq s}^{G} Y^{st} + \overline{V}^{s} \sum_{r \neq s}^{G} B^{sr} A^{rs} L^{ss} \sum_{t \neq s}^{G} A^{st} X^{t} \\ &+ \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^{t} B^{ts} Y^{sr} + \mu \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^{t} B^{ts} A^{sr} L^{rr} Y^{rr} + \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^{t} B^{ts} A^{sr} L^{rr} E^{r*} \end{aligned} \tag{H22}$$

Re-arranging:

$$\begin{aligned} \mu E^{s*} &= \overline{V}^{s} B^{ss} \sum_{r \neq s}^{G} Y^{sr} + \overline{V}^{s} L^{ss} \sum_{r \neq s}^{G} B^{sr} Y^{rr} + \overline{V}^{s} \sum_{r \neq s}^{G} B^{sr} \sum_{u \neq s,r}^{G} Y^{ru} + \overline{V}^{s} \sum_{r \neq s}^{G} B^{sr} Y^{rs} \\ &+ \overline{V}^{s} L^{ss} \sum_{r \neq s}^{G} A^{sr} B^{rs} Y^{ss} + \overline{V}^{s} \sum_{r \neq s}^{G} B^{sr} A^{rs} L^{ss} E^{s*} \\ &+ \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^{t} B^{ts} Y^{sr} + \mu \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^{t} B^{ts} A^{sr} L^{rr} Y^{rr} + \sum_{r \neq s}^{G} \sum_{t \neq s}^{G} \overline{V}^{t} B^{ts} A^{sr} L^{rr} E^{r*} \end{aligned} \tag{H23}$$

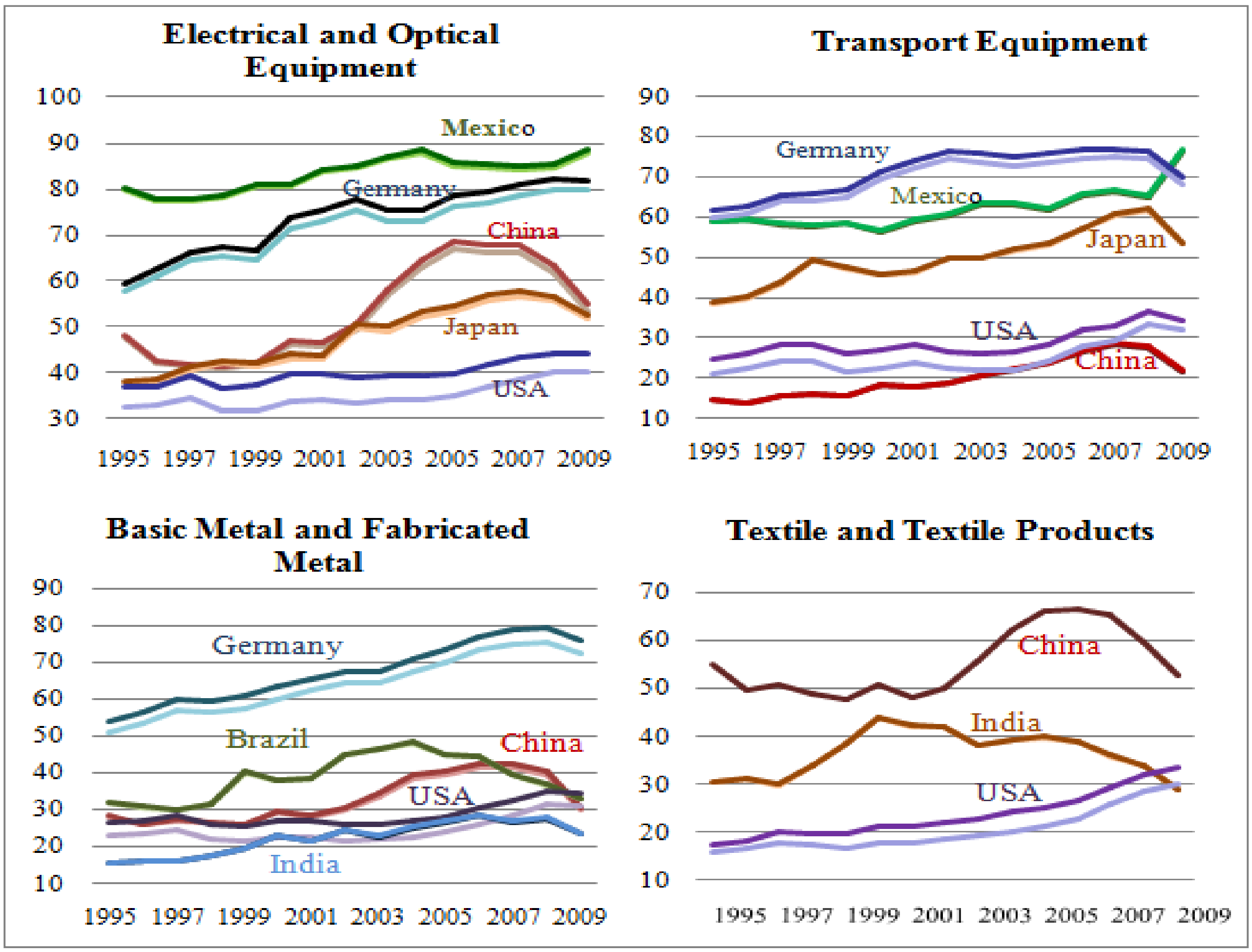
It is the same as equation (36) in KWW.

Appendix I: The difference between Value-added exports and GDP by Industry in Gross Exports at the Country-sector Level

As pointed out in KWW, domestic value-added in a country’s exports and value-added exports are, in general, not equal to each other. They are related but different concepts. The former only looks where the value added is originated regardless where it is ultimately absorbed. While a country’s “value added exports” refers to a subset of “domestic value added in a country’s exports” that is ultimately absorbed abroad.

Figure I1 plots the time trend of “value-added exports”(VAX_F) and “domestic value-added” in exports to GDP ratios (both of them are based on the forward-looking linkage) for four selected industries based on estimates from WIOD. These graphs show clearly domestic value-added in exports to GDP ratios are constantly higher than sector value-added exports to GDP ratios, especially for advanced economies. For instance, the difference between these two ratios is around 4%, 5% and 4% of sector total value-added for the United States, and 3.5%, 2.5% and 2% for Germany in basic mental, electric and optical equipment, and transportation equipment industries, respectively, during the 15 years. Even in the textile and textile industries, there is also a 2-3% difference consistently between these two ratios for the U.S. and Germany during the same period. While the difference between these two ratios for most developing countries is generally tiny.

Figure I1 The Difference between Value-added exports to GDP and Domestic Value-added in exports to GDP ratio



Appendix J: Notations and Important Decomposition Relations

1. At the country aggregate level

(1) $E^s = DVA^s + FVA^s + RDV^s + PDC^s$

(2) $DVA^s = VAX\_F^s = VAX\_B^s$

2. At the country-sector level

(3) $E_j^s = DVA_j^s + FVA_j^s + RDV_j^s + PDC_j^s$

(4) $DVA_j^s = VAX\_B_j^s \neq VAX\_F_j^s$

(5) $GDPinE_j^s = VAX\_F_j^s + RDV\_F_j^s \neq DVA_j^s + RDV_j^s$

3. At the bilateral aggregate level

(6) $E^{sr} = DVA^{sr} + FVA^{sr} + RDV^{sr} + PDC^{sr}$

(7) $DVA^{sr} \neq VAX\_B^{sr} = VAX\_F^{sr}$

4. At the bilateral-sector level

(8) $E_j^{sr} = DVA_j^{sr} + FVA_j^{sr} + RDV_j^{sr} + PDC_j^{sr}$

(9) $DVA_j^{sr} \neq VAX\_B_j^{sr} \neq VAX\_F_j^{sr}$

where E^s is country s's gross exports. (time subscript is omitted for simplicity.); DVA^s is domestic value-added that is exported by country S and ultimately absorbed abroad; FVA^s is foreign value-added in country S's exports; RDV^s is returned domestic value-added in country S's exports, or domestic value added that is initially exported by country S but eventually returned and is consumed at home; PDC^s is pure double counted component due to double counting of the previous terms in some countries' exports (or back-and-forth intermediate goods trade).

VAX_F^s is forward-linkages based value added exports, equaling the sum of $VAX\_F_j^s$ across all sectors; $RDV\_F_j^s$ is forward-linkages based domestic value-added that is first exported but finally returns and is consumed at home; VAX_B^s is backward-linkages based value added in exports of country S, equaling sum of $VAX\_B_j^s$ across all sectors.

E_j^s is total exports of sector j from country S; DVA_j^s, FVA_j^s, RDV_j^s, and PDC_j^s are the four major components of sector j's gross exports, backward-linkage based; $GDPinE_j^s$ is GDP by industry in exports. This concept of value-added created by production factors (labor, capital) employed in sector j of country S and embed in the sector's gross exports, is only concerned with where the value-added is created, but not where it is absorbed;
$VAX\_F_j^s$ is forward-linkages based value added exports of sector j from country S, which is sector j's value added embedded in all sectors gross exports from country S (including indirect exports of sector j's value added through gross exports of country S's other sectors); $VAX\_B_j^s$ is backward-linkage based value added exports of sector j of country S, which is value added from all sectors in country S that is embedded in its sector j's gross exports.

5. **Finer Decompositions**:

(10) $DVA_j^{sr} = DVA\_Fin_j^{sr} + DVA\_Int_j^{sr} + DVA\_Intrex_j^{sr}$

(11) $VAX\_F_j^{sr} = VAX\_F\_Fin_j^{sr} + VAX\_F\_Int_j^{sr} + VAX\_F\_Intrex_j^{sr}$

$$VAX\_F_j^{sr} = \sum_{r \neq s}^{G} V_j^s B^{ss} Y^{sr} + \sum_{r \neq s}^{G} V_j^s B^{sr} Y^{rr} + \sum_{r \neq s}^{G} V_j^s B^{sr} \sum_{t \neq s,r}^{G} Y^{rt}$$

Where $V_j^s = \begin{bmatrix} 0 & \cdots & v_j^s & \cdots & 0 \end{bmatrix}$

(12) $VAX\_B_j^{sr} = VAX\_B\_Fin_j^{sr} + VAX\_B\_Int_j^{sr} + VAX\_B\_Intrex_j^{sr}$

$$VAX\_B_j^{sr} = \sum_{r \neq s}^{G} V^s B^{ss} Y_j^{sr} + \sum_{r \neq s}^{G} V^s B^{sr} Y_j^{rr} + \sum_{r \neq s}^{G} V_j^s B^{sr} \sum_{t \neq s,r}^{G} Y_j^{rt}$$

(13) $PDC_j^{sr} = DDC_j^{sr} + FDC_j^{sr} = DDC\_Fin_j^{sr} + DDC\_Int_j^{sr} + MDC_j^{sr} + ODC_j^{sr}$

(14) $FVA_j^{sr} = FVA\_Fin_j^{sr} + FVA\_Int_j^{sr} = MVA\_Fin_j^{sr} + MVA\_Int_j^{sr} + OVA\_Fin_j^{sr} + OVA\_Int_j^{sr}$

where

$DVA\_Fin_j^{sr}$ is domestic value-added in final goods exports consumed by direct importers; $DVA\_Int_j^{sr}$ is domestic value-added in intermediate goods exports absorbed by direct importers; $DVA\_Intrex_j^{sr}$ is domestic value-added in intermediate goods re-exported to third countries.

Similar to the three sub-components for $VAX\_F_j^{sr}$ and $VAX\_B_j^{sr}$, we have the following sub-components: DDC_j^{sr} is domestic value-added pure double counting in production of exports; FDC_j^{sr} is foreign value-added pure double counting in production of exports; MVA_j^{sr} is foreign value-added sourced from the direct importer; OVA_j^{sr} is foreign value-added sourced from third countries; MDC_j^{sr} is the direct importer's VA double counted in exports production; ODC_j^{sr} is third countries' VA double counted in exports production.

<u>At the country aggregate level</u>

$DVA\_Fin^s = VAX\_B\_Fin^s = VAX\_F\_Fin^s$,

$$DVA\_Int^{s} = VAX\_B\_Int^{s} = VAX\_F\_Int^{s}$$

$$DVA\_Intrex^{s} = VAX\_B\_Intrex^{s} = VAX\_F\_Intrex^{s}$$

$$RDV\_Fin^{s} = RDV\_B\_Fin^{s} = RDV\_F\_Fin^{s}$$

$$RDV\_Int^{s} = RDV\_B\_Int^{s} = RDV\_F\_Int^{s}$$

<u>At the country-sector level</u>

$$DVA\_Fin_{j}^{s} = VAX\_B\_Fin_{j}^{s} \neq VAX\_F\_Fin_{j}^{s},$$

$$DVA\_Int_{j}^{s} = VAX\_B\_Int_{j}^{s} \neq VAX\_F\_Int_{j}^{s}$$

$$DVA\_Intrex_{j}^{s} = VAX\_B\_Intrex_{j}^{s} \neq VAX\_F\_Intrex_{j}^{s}$$

$$RDV\_Fin_{j}^{s} = RDV\_B\_Fin_{j}^{s} \neq RDV\_F\_Fin_{j}^{s}$$

$$RDV\_Int_{j}^{s} = RDV\_B\_Int_{j}^{s} \neq RDV\_F\_Int_{j}^{s}$$

<u>At the bilateral-sector level</u>

$$DVA\_Fin_{j}^{sr} \neq VAX\_B\_Fin_{j}^{sr} \neq VAX\_F\_Fin_{j}^{sr}$$

$$DVA\_Int_{j}^{sr} \neq VAX\_B\_Int_{j}^{sr} \neq VAX\_F\_Int_{j}^{sr}$$

$$DVA\_Intrex_{j}^{sr} \neq VAX\_B\_Intrex_{j}^{sr} \neq VAX\_F\_Intrex_{j}^{sr}$$