5

Laws of production and laws of algebra: Humbug II

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The theoretical basis

Recent debates on capital theory have focused on the notion of capital as a factor of production, which along with labor, *can be used* to *explain the distribution of income in capitalist economy*. Though the intricate point and counterpoint of the controversy often obscure this simple fact, it has become increasingly clear that what is at stake in the current debate is in essence the same issue with which the classical economists, particularly Ricardo, grappled – that of the division of income between wages and profits. The argument thus rages around *descriptive* economic theory, whose aim it is to represent the workings of a competitive capitalist economy. In a sense this is a return to relevance, since much of modern mathematical economics has studiously concerned itself, not with descriptive. but instead with normative theory, such as the study of optimal and efficient growth paths, etc.. (Lancaster, 1968, pp. 9-10).

In neoclassical theory, the model of pure exchange occupies a central position, for it illustrates simply and elegantly the fundamental truths of the paradigm, truths which any more complex representations may modify but certainly cannot undermine.' Thus, in the model of pure exchange. trading begins with selfish individuals each having an arbitrarily determined initial endowment of goods, and proceeds to a final state in which no one individual can improve his or her basket of commodities without making someone else worse off. Such a situation is known as a pareto-optimal allocation, and it implies a set of final exchange ratios between commodities that is, a set of *equilibrium* relative prices. What is more, given the assumption of well-behaved neoclassical utility functions for each individual, the equilibrium prices of the model of pure exchange will be *scarcity prices*: the higher the relative availability of some commodity – other things being equal -the lower its relative price.

The next step in the analysis requires its extention to the case of production. Initial endowments are now assumed to contain not just consumer goods but also means of production, such as land, machines, raw materials, etc.; in addition, since the game cannot continue unless every individual has at least some wealth, it is generally assumed that each and every initial endowment includes potentially saleable labor services. By assumption, the ultimate objective of every individual is consumption: means of production and labor services, however, are not directly consumable. At this point, therefore, production is introduced as a roundabout way of consumption, a process in which inputs are transformed into outputs. In order to translate any given initial endowment into the production possibilities inherent in it. neoclassical economics commonly relies on the assumption of a well behaved neoclassical production function, one for each commodity produced.

Each individual then faces three basic methods of arriving at some preferred final allocation, methods which he or she is free to use in any combination permitted by the initial endowment and consistent with the utility function. First, he can trade any of the consumer goods or means of production in his possession for other goods he desires; second, he may rent out the *services* of the means of production he owns, and/or rent out his labor power: and third, if his

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initial endowment so permits, he may choose to become a producer, renting and/or buying means of production and labor-power and combining these with the elements of his initial endowment to turn out one or more commodities via a well-behaved neoclassical production function. Ruled only by his enlightened self-interest, which dictates that more is better, and constrained only by his native abilities and initial endowment, he is assumed to eventually arrive at some most "efficient" combination of the trader-rentier-producer modes, thereby attaining his personal optimum in the form of some final allocation.

Because preferences (utility functions) and initial endowments are *parameters* of the analysis, the whole structure of equilibrium is ruled by them, so that once again, the forces of consumer sovereignty lead us ineluctably to Pareto-optimality . Equilibrium relative prices are once again *scarcity* prices, a term which now covers the prices of consumption goods, the wage rate for labor services, and the rental and sale prices of means of production (Hershleifer, 1970).

Under carefully fashioned assumptions involving well-behaved utility and production functions, these sorts of models are determinate in the sense that one or more possible equilibria can be shown to exist. But the model, as outlined here, contains no reference to the uniform rate of profit which is supposed to characterize competitive capitalism. The explanation of this rate of profit is what (descriptive) neoclassical capital theory is all about. Moreover, given that the basic parables of the theory have already identified the equilibrium price of every good or service as a scarcity price, one that reflects its individual and social scarcity, the task that confronts the theory is clear: somehow, the rate of profit too must be explained as the scarcity price of some *thing* with both the price and quantity of this thing to be mutually determined in some market. This market, it turns out, is the capital market, in which demand is determined by individual's preferences for present versus future consumption - their "taste for investment" (Dewey, 1965) and supply is determined by the technological structure. The price that supposedly emerges from this interaction is the *rate of interest,* the scarcity index of the quantity of *capital*, and with the addition of a few more convenient assumptions, the rate of profit is made equal to this rate of interest. *If these conditions can be maintained,* then, it is argued, *the distribution of income in a capitalist society is a consequence of the efficient allocation of resources;* in fact, within this wondrous construct, capitalism itself represents the resolution of one of Nature's most problematical gifts - the "natural" selfishness of every individual!

Scarcity pricing parables and the aggregate production function

Traditionally, several models have been used to extend scarcity pricing to the theory of distribution. The simplest, and by far the most widely used in both the theoretical and empirical literature, is the aggregate production function model. Such a model, we are told, is an aggregated version of the general equilibrium model outlined above, constructed as an empirically useful approximation, *and strongly supported by the data.* Even the sophisticates, the so-called high-brows of neoclassical theory, at one time, took this and similar parables seriously:

> . . . In various places I have subjected to detailed analysis certain simplified models involving only a few factors of production . . . [These] simple models or parables do. I think. have considerable heuristic value in giving insights into the fundamentals of interest theory in all of its complexities. (Samuelson, 1962, p. 194)
>
> The originators of the "production function" theory of distribution (in the static sense, where I still think it should be taken fairly seriously) were Wicksteed, Edgeworth, and Pigou. (Hicks, 1965, p. 293, footnote 1)

Though aggregate or surrogate production function models occupy the bulk of the theoretical and empirical literature on the distribution of income in a capitalist society, the essential characteristic of this and all other parables of neoclassical theory concerns their attempt to explain the wage rate and the rate of profit as scarcity prices of labor and capital, respectively, determined in the final analysis by efficiency considerations. It was precisely this technocratic apologia for capitalism which became the target of the neo-Keynesian counterattack of the 1960s, during the so-called Cambridge capital controversies.

One of the most striking, and for neoclassical economics most devastating, results of the above capital controversies was the proof that *any* version of the neoclassical parable, in which the rate of profit varied inversely with the quantity of capital and the wage rate inversely with the quantity of labor (so that each at least behaved like a scarcity price) was valid in static conditions *if and only if prices in all possible* competitive equilibria were proportional to labor values." These results, therefore, apply, *inter alia,* to that particular version of the parable known as the aggregate (or surrogate) produc-

tion function, in which the wage rate and the rate of profit not only move inversely to the quantities of labor and capital, respectively, but are also equal to and determined by their respective marginal products. Considering that the neoclassical parables have their origins in a "conscious counterrevolution *against* the classical school, against Ricardo and Marx in particular" (Dobb, 1970, p. 1), and above all, against the labor theory of value in *any* form, it is gratifying to discover that in the end these parables themselves depend on the *simple* labor theory of value. The irony is inescapable.

These and other inimical results were not lost on the faithful. As awareness of the internal inconsistencies of neoclassical theory began to grow, many were led to abandon it. But for others, hope died hard; and hope, it seems, lay in the data. "As a neoclassical theorist, I can only reply that the relevant question is what is relevant: should we make our predictions on the basis of what Mrs. Robinson has called perverse technical behavior *or on the basis of relations that have been repeatedly observed*?" (Ferguson, 1971, p. 254, emphasis added)

What has been "repeatedly observed," it is argued, is the empirical efficacy of aggregate production functions. In spite of the very strongest theoretical requirements for their existence, the use of such functions flourishes the current justification being that their empirical basis appears strong. In study after study, empirically derived functions appear to strongly support both the constancy of returns to scale and the equality of marginal products with "factor rewards": in particular, for both time-series and cross-section studies (within any one country), the Cobb-Douglas function appears to dominate the field,

For the neoclassical faithful, these results represent their salvation; no matter what those critics from Cambridge say, the "real" world, it would seem, is neoclassical. Or is it? The answer is simple: no. *The so-called empirical strength of aggregate production is an illusion,* due not to some mystical laws of production, but instead, to some rather prosaic laws of algebra. To see why, however, we must first examine how production functions are estimated.

The empirical basis of aggregate production functions

The most popular methods of estimating aggregate production functions have been the single equation least squares method and the factor shares method (Walters, 1963). The former can be *most* generally described as fitting a function of the form[3] $Q(t) = F[K(t), L(t), t]$ to observed *data while the* latter consists of *assuming* that aggregate marginal *products of capital* and labor are equal to their respective unit earnings and then using this assumption to specify structural coefficients. In general, for both time series and cross-section data, the Cobb–Douglas function wins out: "the sum of coefficients usually approximate closely to unity" (thus implying constant returns to scale), with the additional bonus of a close "agreement between the labor exponent and the share of wages in the value of output" (thus supporting aggregate marginal productivity theory) (Walters, 1963, p. 27).

In a recent paper, Franklin Fisher concedes that the requirements "under which the production possibilities of a technically diverse economy can be represented by *an aggregate* production function are far too stringent to be believable" (Fisher, 1971, p. 306). He proposes therefore to investigate the puzzling uniformity of the empirical results by means of a simulation experiment: each of N industries in this simulated economy is assumed to be characterized by a microeconomic Cobb-Douglas production function relating its homogeneous output to its homogeneous labor input and its own *distinct* machine stock. The conditions for theoretical aggregation are studiously violated, and the question *is, how well, and under what circumstances*, does an aggregate Cobb-Douglas function represent the data generated? In such an economy, the aggregate wage share is often variable over time, so that in general an aggregate Cobb-Douglas would not be expected to give a good fit. What seems to surprise Fisher, however, is that when the wage share happens *coincidentally* to be roughly constant, a Cobb–Douglas production function will not only fit the data well but *also* provide a good explanation of wages, "*even though the true relationships are far from yielding an aggregate Cobb–Douglas,*" *suggesting that "the view that the constancy of labor's share is due to the presence of an aggregate Cobb–Douglas production function is mistaken. Causation runs the other way and the apparent success of aggregate Cobb–Douglas production functions is due to the relative constancy of labor's share.*" (Emphasis added.) (Fisher, 1971, p. 306).

It is obvious that so long as aggregate shares are roughly constant, the appropriate econometric test of aggregate neoclassical production and distribution theory requires a Cobb-Douglas function. Such a test would then apparently cast some light on the degree of returns to scale (through the sum of the coefficients), and the applicability of aggregate marginal productivity theory (through the comparison of the labor and capital exponents with the wage and profit shares, respectively). What is not obvious, however, is that so long as aggregate shares are constant, an aggregate Cobb-Douglas function having apparently "constant returns to scale" will always provide an exact fit, for any data whatsoever. *In addition, under fairly reasonable conditions, such a function will seem also to possess "marginal products equal to respective factor rewards," thus seeming to justify neoclassical aggregate distribution theory.* These propositions, it will be shown, are *mathematical* consequences of constant shares, and it will be argued that the puzzling uniformity of the empirical results is due in fact to this law of algebra and not to some mysterious law of production. In fact, in order to emphasize the independence of these results from any laws of production, an illustration is provided in the form of the rather implausible data of the Humbug economy, for even data such as this is perfectly consistent with a Cobb-Douglas function having "constant returns to scale," "neutral technical change," and satisfying "marginal productivity rules," so long as shares are constant.

Laws of algebra

Let us begin by separating the aggregate data in any time period into output data (Q, the value of output), distribution data (W, π, wages and profits, respectively), and input data (K, L, the index numbers for capital and labor, respectively). Then we can write the following aggregate identity for any time t:

$$Q(t) \equiv W(t) + \pi(t) \tag{1}$$

Given *any* index numbers $K(t)$, $L(t)$, we can always write:

$$q(t) \equiv w(t) + r(t)k(t) \tag{2}$$

where $q(t)$ and $k(t)$ are the output-labor and capital-labor ratios, respectively, and $w(t) \equiv W(t)/L(t)$, $r(t) \equiv \pi(t)/K(t)$ are the wage and profit rates, respectively. The above equation is therefore the fundamental identity relating output, distribution, and input data. Defining the share of profits in output as s, and the share of wages as $1 - s$, we can differentiate identity 2 to arrive at identity 3 (time derivatives are denoted by dots, and the time index, *t*, is dropped to simplify notation):

$$\dot{q} = \dot{w} + \dot{r}k + r\dot{k} \equiv w\left(\frac{\dot{w}}{w}\right) + rk\left(\frac{\dot{r}}{r}\right) + rk\left(\frac{\dot{k}}{k}\right)$$

Dividing through by q,

$$\frac{\dot{q}}{q} = \frac{w}{q}\left(\frac{\dot{w}}{w}\right) + \frac{rk}{q}\left(\frac{\dot{r}}{r}\right) + \frac{rk}{q}\left(\frac{\dot{k}}{k}\right)$$

By definition, the profit and wage shares, respectively, are

$$s \equiv \frac{rk}{q}, \quad 1 - s \equiv \frac{w}{q}$$

so that we may write,

$$\frac{\dot{q}}{q} = \frac{\dot{B}}{B} + s\,\frac{\dot{k}}{k} \quad \text{where} \quad \frac{\dot{B}}{B} = \left[(1 - s)\frac{\dot{w}}{w} + s\frac{\dot{r}}{r}\right] \tag{3}$$

It is important to note that all relations given so far are *always* true for *any* aggregate data at all, irrespective of production or distribution conditions.

Suppose now we are faced with particular data which for some unspecified reasons exhibit constant shares, so that $s = \beta$ (a constant). Remembering that the dotted variables are time derivatives ($\dot{q} = dq/dt$, etc.), we can immediately integrate the identity (3):

$$\int \frac{\dot{q}}{q}\,dt = \int \frac{\dot{B}}{B}\,dt + \int \beta\frac{\dot{k}}{k}\,dt$$

$$\ln q = \int \frac{\dot{B}}{B}\,dt + \beta \ln k + \ln c_0$$

where for convenience the constant of integration is written as $\ln c_0$. Rewriting, we have,

$$q = \left[\exp\left(\int \frac{\dot{B}}{B}\,dt\right)\right] c_0 k^\beta = [B]c_0 k^\beta \tag{4}$$

where by definition

$$B = \exp\left(\int \frac{\dot{B}}{B}\,dt\right)$$

Equation (4) is strikingly reminiscent of a constant returns to scale aggregate Cobb-Douglas production function with a shift parameter B. But in fact, it is not a *production* function at all, but merely an algebraic relationship which always holds for *any* output-input data Q, K, L, even data which could not conceivably come from any economy, so long as the distribution data exhibits a constant ratio. Furthermore, since the $\dot{B}/B$ term in identity (3) is a weighted average of the *rates of change* of w and r, respectively, it seems empirically reasonable to expect that measures of K, L would give a capital-labor ratio k which is weakly correlated with $\dot{B}/B$. With measures for which the above is true, *$\dot{B}/B$ may be considered to be primarily a function of time*, so that B will also be solely a function of time. Then we can write

$$q = B(t)\,[c_0 k^\beta] \tag{5}$$

and since $q \equiv Q/L$ and $k \equiv K/L$, we get

$$Q = B(t)\,[c_0 K^\beta L^{1-\beta}] \tag{5a}$$

The algebraic relationship just given has several interesting properties. First, it is homogeneous to the first degree in K and L. Second, since $\beta = s \equiv rk/q$, the partial derivatives $\partial Q/\partial K$, $\partial Q/\partial L$ are equal to r, w, respectively. And third, the effect of time is "neutral," as incorporated in the shift parameter B(t). What we have, actually, is *mathematically* identical to a constant returns to scale Cobb-Douglas production function having neutral technical change and satisfying marginal productivity "rules." And yet, as we have seen, *any production data whatsoever can be presented as being "generated" by such a function, so* long as shares are constant and the measures of capital and labor such that k is uncorrelated with $\dot{B}/B$. Therefore, precisely because (5a) is a mathematical relationship, holding true for large classes of data associated with constant shares, it cannot be interpreted as a production function, or any production relation at all. If anything, it is a distributive relation, and sheds little or no light on the underlying production relationships.[4] In fact, since the constancy of shares has been taken as an empirical datum throughout, equation (5a) does not shed much light on any theory of distribution either.

I emphasized earlier that the theoretical basis of aggregate production function analysis was extremely weak. It would seem now that its apparent empirical strength is no strength at all, but merely a statistical reflection of an algebraic relationship. For the neoclassical old guard, the retreat to data is really a rout.

Applications

It is obvious that one can apply Equation (5a) in many ways. The section that follows will reexamine Solow's famous paper on measuring technical change. The "humbug production function" section will present a numerical example to illustrate the generality of Equation (5a). The section on Fisher's simulation experiments will extend the preceding analysis; and the final section will touch briefly on cross-section production function studies.

Technical change and the aggregate production function: Solow. In what is considered a "seminal paper" (Solow, 1957), Robert Solow introduced in 1957 a novel method for measuring the contribution of technical change to economic growth. Since that time several refinements of Solow's original calculations have been established, all aimed at providing better measures of labor and capital by taking account of education, vintages of machines, etc., but the basic approach has remained unchanged.

Solow's approach is by now a familiar one. Equation (6) expresses the assumption of a constant returns to scale aggregate production function, with the parameter $A(t)$ expressing the assumption of neutral technical change.

$$q - A(t)f(k) \tag{6}$$

For such a function, the marginal product of capital is $dq/dk = A(t)\ [df/dk] = [q/f]\ [df/dk]$, since $A(t) = q/f$. By assumption, this marginal product is equal to the rate of profit r:

$$\frac{dq}{dk} = \frac{q}{f}\frac{df}{dk} = r$$

and by rewriting, we can express this in terms of the profit share s:

$$\frac{df}{dk}\frac{k}{f} = \frac{rk}{q} = s \quad \text{share of profit in output} \tag{7}$$

Solow's expressed purpose was to distinguish between shifts of the assumed production function (due to "technical change") and movements along it (due to changes in the capital-labor ratio, k).[6]

Figure 5.1 illustrates the geometric assumption implicit in Solow's paper. Points A, and B_1 are observed points, at times t_0 and t_1, respectively, while B_0 represents the "adjusted" point after "neutral technical change" has been removed. Thus points A_0 and B_0 lie on the "underlying production function."

Algebraically, in terms of Equation (6), the aim of his procedure is to partition output per worker q into A, the technical change shift parameter, and $f(k)$, the "underlying production function" to which 1 just referred. In order to do this, Solow first differentiates Equation (6):

$$\dot{q} = \dot{A}f(k) + A\frac{df(k)}{dt} = \dot{A}f(k) + A\frac{df(k)}{dk}\dot{k}$$

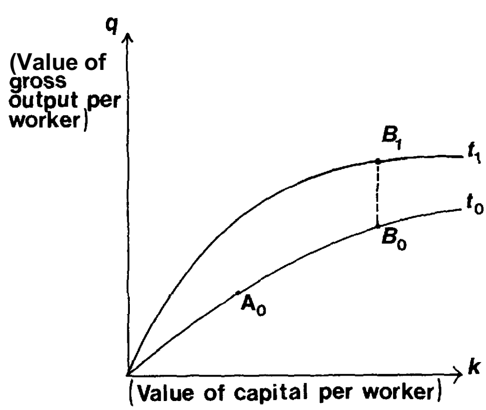

Figure 5.1

Rearranging,

$$\dot{q} = \frac{\dot{A}}{A} Af(k) + Af'(k) \left[\frac{df(k)}{dk} \cdot \frac{k}{f(k)}\right] \frac{\dot{k}}{k}$$

Since from (6), $q = Af(k)$, and from (7),

$$\frac{df(k)}{dk} \frac{k}{f(k)} = \frac{rk}{q} = s$$

s being the share of profit in gross output, we can write

$$\frac{\dot{q}}{q} = \frac{\dot{A}}{A} + s \frac{\dot{k}}{k} \tag{8}$$

Equation (8) is derived from the *assumptions* of a constant returns to scale aggregate production function, with distribution determined by marginal productivity rules. Equation (3), derived earlier from an *identity* and therefore always true for any production and distribution behavior, is mathematically identical to (8). It follows therefore that $\dot{A}/A = \dot{B}/B \equiv [(1 - s)\ \dot{w}/w + (s)\dot{r}/r]$; that is, Solow's measure of technical change is merely a weighted average of the growth rates of the wage rate, w and the rate of profit, r.

Solow's data provide him with a series for gross output per worker q, capital per worker k, and profit share s, for the United States from 1909-1949. From this data, he calculates the rates of change $\dot{q}/q$ and $\dot{k}/k$, and using these rates along with the data for the profit share s, he derives a series for $\dot{A}/A = \dot{q}/q - s\dot{k}/k$.

To Solow, the series for $\dot{A}/A$ represents the rate of change of technology; since a scatter diagram of $\dot{A}/A$ on k shows no apparent correlation, he concludes that technical change is essentially neutral. By setting $A(0) = 1$, he is able to translate the rate of technical change $\dot{A}/A$ into a series for $A(t)$, the shift parameter.' Finally, since by definition $q = A(t)\,f(k)$, he is able to combine his derived series for $A(t)$ with his given series on q to derive the underlying production function $f(k) = q/A(t)$.

Plotting $f(k)$ versus k, Solow gets a diagram with noticeable curvature, and notes with obvious satisfaction that the data "gives a distinct impression of diminishing returns" (Solow, 1957, p. 380). In fact, Solow finds this underlying production function to be extremely well represented by a Cobb-Douglas function:

$$\ln \hat{f}(k) = -.729 + .353 \ln k \quad (R^2 = .9992) \tag{9}$$

Given our preceding analysis in the section on laws of algebra, it is not difficult to see why Solow's results turn out so nicely. We know for instance that his data exhibit roughly constant shares, and the residual term $\dot{B}/B = \dot{A}/A$ is uncorrelated with k. From purely algebraic considerations, therefore, one would expect the data to be well represented by the functional form in (5), $q = B(t)\ c_0 k^{\beta}$, a form which is mathematically identical to a constant returns to scale Cobb-Douglas function, with neutral technical change and "marginal products equal to factor rewards." In fact, the *algebra* indicates that Solow's underlying production function should be of the form:

$$f(k) = c_0 k^{\beta} \tag{10}$$

$$\ln f(k) = \ln c_0 + \beta \ln k \tag{10a}$$

β is of course the (roughly) constant share and c_0 is a constant of integration which depends *only* on the initial points q_0, k_0, of the data. Solow uses the years 1909-1942 in his regressions, and for these years the average profit share $s \cong \beta = .35$.[8] Moreover, since in any period t, $q_t = B(t)\ c_0 k_t^{\beta}$ from Equation (6), in period $t = t_0$ we may write $q_0 = B(0)\ c_0 k_0^{\beta}$, which gives us $\ln c_0 = \ln q_0 - \ln B(0) - \beta \ln k_0$. For Solow, this residual B(t) represents the shift parameter A(t) (compare Equations (3) derived from an identity, and Equation (8) derived from Solow's assumptions), so that $B(0) = A(0)$; as mentioned earlier, he takes $A(0) = 1$. From Table 1, p. 315 of his article, we get $q_0 = .623$, $k_0 = 2.06$, which when combined with $B(0) = A(0) = 1$, gives $\ln c_0 \cong -0.725$.

Thus, on purely algebraic considerations one would expect Solow's underlying production function to be characterized by

$$\ln f(k) = .725 + .35 \ln k \tag{11}$$

This, of course, is virtually identical to Solow's regression result, equation (9), as it should be, *for it is a law of algebra, not a law of production!*

The humbug production function. The analysis of the laws of algebra led to the conclusion that *any production data series q, k whatsoever, can be represented as being generated by a Cobb-Douglas production function* having neutral technical change and satisfying marginal productivity "rules," so long as shares are constant and the measures of capital and labor such that k is uncorrelated with $\dot{B}/B$. It is possible to illustrate the generality of the above result by means of a numerical example. Consider, for example, an economy with the output-input data illustrated in Figure 5.2 and having the same profit share as in Solow's data for the United States.

The Humbug data set gives us a series for q, k, and s, from which we can calculate rates of change $\dot{q}/q$ and $\dot{k}/k$. From these, in turn, we derive $\dot{B}/B = \dot{q}/q - s(\dot{k}/k)$. (The calculations appear in Figure 5.5 .)

Plotting B/B on k gives us a scatter diagram

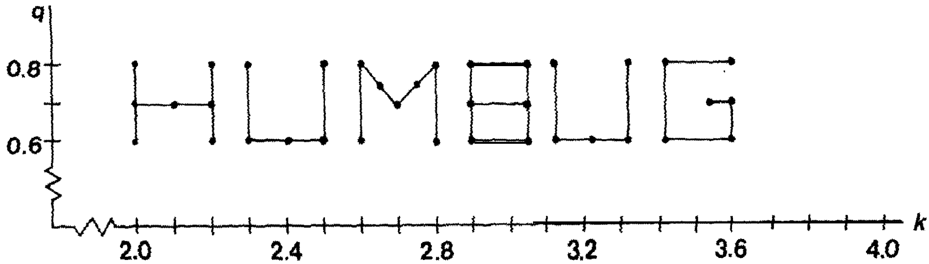

Figure 5.2

with no apparent correlations, so that we may safely assume that B is solely a function of time: $B = B(t)$

This above-mentioned result, combined with the approximate constancy of the profit share s, is sufficient for us to be able to state that even the humbug data can be thought of as having been generated by a Cobb–Douglas production function $q_t = B(t)\, c_0\, k_t^{\beta}$ having constant returns to scale, neutral technical "regress", and marginal products equal to factor rewards!

Let us, however, go on to derive the numerical values involved. To begin with, we follow Solow in setting $B(0) = 1$, and using that to translate the rates of change $\dot{B}/B$ into the series $B(t)$, which is represented in Figure 5.3.

Using the series just mentioned for $B(t)$, one may then derive the underlying production function $f(k) = q/B(t)$, which when plotted versus k in Figure 5.4 gives the same distinct impression of "diminishing returns" that Solow found in *his* data. As we saw in the section on Solow, this pattern is a necessary one, the algebraic consequence of a constant profit share s.

We have already seen that the numerical specification of $f(k)$ can always be anticipated from purely algebraic considerations. For instance, in the Humbug data we use the years 1909–47, and for these years, the average profit share is $\beta = .34$. Moreover, since $q_0 =$ SO, $k_0 = 2.00$, and $B(0) = 1.0$ for Humbug data, we would expect the constant term to be $\ln c_0 = \ln q_0 - \ln B(O) - \beta \ln k_0 = -0.459$.[9] Algebraic considerations therefore tell us that the constant term will be $\ln c_0 =$ -0.459 and the slope $\beta = .34$. The actual regression of $f(k)$ on k, presented below, gives virtually identical results.

$$\hat{f}(k) = -0.453 + .34 \ln k \quad (R^2 = .993) \tag{12}$$

The function $B(t)$ is of course much more troublesome. A simple glance at Figure 5.3 tells us that no linear or log-linear function will suffice for a numerical approximation. Nonetheless, even in this case a fair approximation is possible:[10]

$$\hat{B}(t) = a_0 + a_1 t + \sum_{i=1}^{2} \left[b_i \cos\left(\frac{c_i \pi t}{2}\right) + d_i \sin\left(\frac{e_i \pi t}{2}\right) \right] \tag{13}$$

$a_0 =$	.8565	$a_1 =$	-3.966×10^{-3}	$b_3 =$	.0206
$b_1 =$	$-.0325$	$b_2 =$	.0435	$c_3 =$	.5
$c_1 =$	.4	$c_2 =$	.6	$d_3 =$	$-.0295$
$d_1 =$	.035	$d_2 =$	$-$ .032	$e_3 =$	.4
$e_1 =$	.5	$e_2 =$	.8		

($R^2 = .82$; corrected for degrees of freedom, $\bar{R}^2 = .68$.)[11]

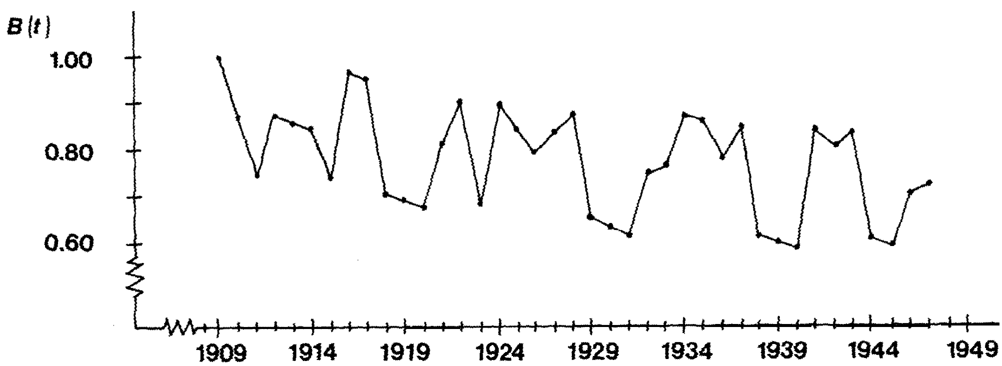

Figure 5.3

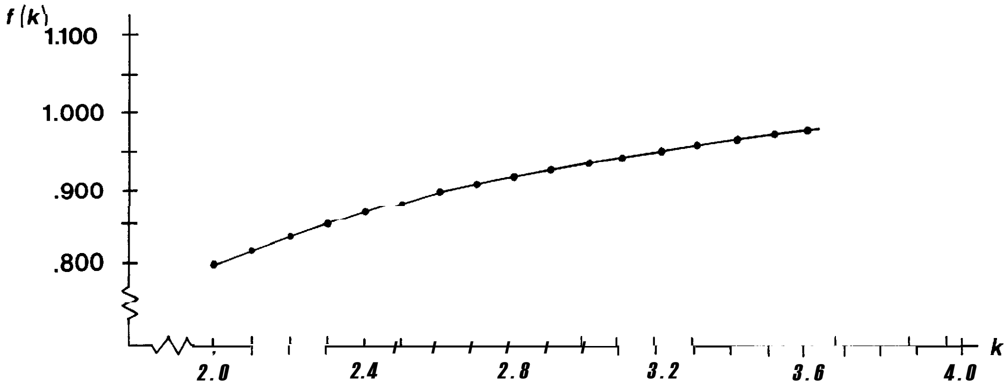

Figure 5.4

Combining these two fitted functions, one arrives at a numerical specification for even the Humbug data (Table 5. I)!

Fisher's simulation experiments. Earlier, I mentioned Franklin Fisher's extensive (and expensive) simulation experiments, in which he finds, to his surprise, that aggregate Cobb-Douglas functions seem to "work" for his simulated economy even when the theoretical conditions for such an aggregate function are carefully violated, so long as the particular simulation run happens to have roughly constant wage (and hence profit) shares (Fisher, 1971, p. 306).

It is worth noting at this point that what Fisher means by aggregate production functions working, is not simply that they give a good fit to gross output $Q(t)$ or gross output per worker $q(t)$, but also that the estimated marginal products of labor, and presumably of capital, closely approximate the actual wage and profit rates, respectively (Fisher, 1971).

1 have already demonstrated in section on the laws of algebra why in general an aggregate Cobb-Douglas may be expected to work, in the sense explained earlier, for data which reflect constant wage shares. In this section, however, it will be shown that even Fisher's massive computer simulation is in reality only an application of the laws of algebra.

The structure of the simulation. Fisher's simulated economy consists of N industries, each producing the same type of output Q, using homogeneous labor L, but its own *distinct* type of machine stock K_i. Thus Q_i and Q_j are both quantities of the same good, produced by industries i and j, respectively, whereas K_i and K_j are stocks of different types of machines.

Each industry is assumed to be characterized by a microeconomic Cobb -Douglas production function:

$$Q_i(t) = A_i(t)\ [L_i(t)]^{\alpha}\ [K_i(t)]^{1-\alpha} \quad \text{where } i = 1, \ldots, N \qquad (14)$$

(The α_i are constant over time, but in general $A_i(t)$, $L_i(t)$, and $K_i(t)$ are not.)

At any instant of time, the total stock of labor $L(t)$ in the economy is given. The basic procedure followed in the model is to allocate this given supply among the existing industries so as to equalize the industry marginal products of labor ($MPL_i = MPL_j = MPL$): this of course yields the maximum aggregate output $Q(t) \equiv \sum_{i=1}^{N} Q_i(t)$.

In general, the marginal product of a Cobb–Douglas function is $MPL_i = \alpha_i[Q_i(t)/L_i(t)]$.[12] Since these are all equalized for the various industries to a single level, we can denote this common level by $w(t)$ and write:

$$MPL_i = \alpha_i \frac{Q_i(t)}{L_i(t)} = w(t) \qquad (15)$$

$w(t)$ represents the "imputed rental" (uniform wage rate) of a unit of labor, so that the wage bill in the i^{th} industry is:

$$w(t)L_i(t) = \alpha_i Q_i(t) \qquad (16)$$

Thus, the aggregate wage bill is:

$$w(t)L(t) = \sum_{i=1}^{N} w(t)L_i(t) = \sum_{i=1}^{N} \alpha_i Q_i(t)$$

so that the wage share in total output $Q(t) \equiv \sum_{L=1}^{N} Q_i(t)$ is:

$$\text{wage share} = \frac{w(t)L(t)}{Q(t)} = \sum_{i=1}^{N} \alpha_i \left[\frac{Q_i(t)}{Q(t)}\right] \qquad (17)$$

Finally, since $Q_i(t)$ is the gross output of the i^{th} industry, and $w(t)\ L_i(t) = \alpha_i\ Q_i(t)$ its wage bill, the difference between the two, the gross profit

Table 5.1. *Humbug data*

Year	Actual share of property income	"Humbug" output per worker	"Humbug" capital per worker					
	s	$q(t)$	$k(t)$	$\dot{q}/q$	$\dot{k}/k$	B/B	$B(t)$	$j(k)$
1909	0.335	0.80	2.00	-0.125	0.000	-0.125	1.000	0.800
1910	0.330	0.70	2.00	-0.143	0.000	-0.143	0.875	0.800
1911	0.335	0.60	2.00	+0.167	0.000	+0.167	0.750	0.800
1912	0.330	0.70	2.00	0.000	+0.050	-0.017	0.875	0.800
1913	0.334	0.70	2.10	0.000	0.048	-0.016	0.860	0.814
1914	0.325	0.70	2.20	-0.143	0.000	-0.143	0.846	0.826
1915	0.344	0.60	2.20	+0.333	0.000	+0.333	0.725	0.828
1916	0.358	0.80	2.20	0.000	0.045	-0.016	0.965	0.830
1917	0.370	0.80	2.30	-0.250	0.000	-0.250	0.948	0.843
1918	0.342	0.60	2.30	0.000	0.044	-0.015	0.710	0.845
1919	0.354	0.60	2.40	0.000	0.042	-0.015	0.700	0.857
1920	0.319	0.60	2.50	+0.167	0.000	+0.167	0.690	0.870
1921	0.369	0.70	2.50	+0.143	0.000	+0.143	0.805	0.870
1922	0.339	0.80	2.50	-0.250	0.040	-0.264	0.921	0.869
1923	0.337	0.60	2.60	+0.333	0.000	+0.333	0.678	0.885
1924	0.330	0.80	2.60	-0.063	0.019	-0.069	0.902	0.887
1925	0.336	0.75	2.65	0.067	0.019	-0.073	0.810	0.893
1926	0.327	0.70	2.70	+0.071	0.019	+0.065	0.780	0.897
1927	0.323	0.75	2.75	+0.067	0.018	+0.061	0.830	0.903
1928	0.338	0.80	2.80	-0.250	0.000	-0.250	0.880	0.908
1929	0.332	0.60	2.80	0.000	0.036	-0.012	0.660	0.908
1930	0.347	0.60	2.90	0.000	0.052	-0.018	0.652	0.920
1931	0.325	0.60	3.05	+0.167	0.000	+0.167	0.641	0.935
1932	0.397	0.70	3.05	0.000	-0.049	+ 0.019	0.74%	0.335
1933	0.362	0.70	2.90	+0.143	0.000	+0.143	0.764	0.916
1934	0.355	0.80	2.90	0.000	0.052	-0.018	0.874	0.916
1935	0.351	0.80	3.05	-0.125	0.000	-0.125	0.860	0.930
1936	0.357	0.70	3.05	0.143	0.033	+0.132	0.752	0.930
1937	0.340	0.80	3.15	0.250	0.000	-0.250	0.852	0.940
1938	0.331	0.60	3.15	0.000	0.032	-0.011	0.638	0.940
1939	0.347	0.60	3.25	0.000	0.031	-0.011	0.633	0.948
1940	0.357	0.60	3.35	+0.333	0.000	+0.333	0.626	0.960
1941	0.377	0.80	3.35	0.000	0.070	-0.026	0.843	0.950
1942	0.356	0.80	3.60	0.000	-0.042	+0.015	0.820	0.975
1943	0.342	0.80	3.45	-0.250	0.000	-0.250	0.832	0.964
1944	0.332	0.60	3.45	0.000	0.044	-0.015	0.624	0.964
1945	0.314	0.60	3.60	+0.167	0.000	+0.167	0.614	0.978
1946	0.312	0.70	3.60	0.000	-0.014	+0.004	0.717	0.975
1947	0.327	0.70	3.55	—	—	—	0.721	0.970

in the i^{th} industry, is treated as the "imputed rental" of its unique machine stock $K_i(t)$. Defining this gross profit (imputed machine rental) as $\pi_i(t)$, we have:

$$\pi_i(t) = (1 - \alpha_i)Q_i(t) = \text{gross profits in } i\text{th industry} \quad (18)$$

Since output $Q_i(t)$ and labor $L_i(t)$ are homogeneous across industries, their respective aggregates are derived by simple addition. But since each industry has a *unique* type of machine, an aggregate capital stock cannot be derived by adding machines together, each machine being a different type. An index of aggregate capital has therefore to be constructed, and it is known that in general any such index will violate the strict conditions under which the microeconomic Cobb-Douglas production functions can be theoretically aggregated into a macroeconomic Cobb-Douglas production function (Fisher, 1971, pp. 307-08). On the basis of aggregation theory, therefore, one would not expect the macroeconomic variables in this simulated economy to behave as if they were generated by a Cobb-Douglas function, even if aggregate shares happen to remain roughly constant over time. That, of course, is the reason for Fisher's surprise at his results.

Fisher chooses to construct an aggregate index in two steps. First, he runs the model economy over its 20-year period, from which he gets the gross profits $\pi_i(t)$ of any given industry, for each of 20 years. Similarly, over each of the 20 years he knows the machine stock $K_i(t)$ in the same industry: the ratio of the 20-year sums of these two is the average rate of return in the i^{th} industry:

$$r_i = \left(\sum_{t=0}^{20} \pi_i(t) \Big/ \sum_{t=0}^{20} K_i(t)\right)$$

$= 20$ year average rate of return in i^{th} industry (19)

The units of each average return r_i are output per machine type i. Thus Fisher can use these r_i in any one period t to aggregate the individual industry machine stocks into an aggregate index of capital $J(t)$:

$$J(t) = \sum_{i=1}^{N} J_i(t) = \sum_{i=1}^{N} r_i K_i(t) \tag{20}$$

It is useful to note that in the above expression the r_i are **not** functions of time, since they represent average rates of return over the whole 20-year period.

The constancy of wage shares. From Equation (19), the wage share is

$$\text{wage share} = \sum_{i=1}^{N} \alpha_i \left[\frac{Q_i(t)}{Q(t)}\right]$$

Now, as Fisher notes, since the parameters α_i are independent of time, the wage share will be roughly constant over time only if the relative outputs $Q_i(t)/Q(t)$ are roughly constant over time (Fisher, 1971, p. 321, footnote 21). Let us denote these roughly constant relative outputs by p_i, and the constant wage share by $(1 - s)$, the lack of time subscript denoting their constancy:

$$\frac{Q_i(t)}{Q(t)} \approx p_i \tag{21}$$

$$(1 - s) = \frac{w(t)L(t)}{Q(t)} \cong \sum_{i=1}^{N} \alpha_i p_i \tag{22}$$

In each industry, the wage bill, as derived in Equation (17), is $w(t)L_i(t) = \alpha_i Q_i(t)$. From (22), the aggregate wage bill is $w(t)L(t) = (1 - s)Q(t)$, and dividing one by the other, we get:

$$\frac{L_i(t)}{L(t)} = \frac{\alpha_i}{1 - s}\frac{Q_i(t)}{Q(t)} \simeq \frac{\alpha_i p_i}{1 - s} \tag{23}$$

Finally, to prepare us for the last step, we need to note that the rough constancy of relative outputs $Q_i(t)/Q(t)$ and relative employment $L_i(t)/L(t)$ implies that each firm's output and employment grow at roughly the same rate. That is, dropping time subscripts and denoting time derivatives by dots: [13]

$$\frac{\dot{Q}_i}{Q_i} = \frac{\dot{Q}}{Q} \quad \text{and} \quad \frac{\dot{L}_i}{L_i} = \frac{\dot{L}}{L} = .03 \tag{24}$$

Algebraic considerations. It is the central result of this paper that *given constant shares,* any aggregate data Q, K, L whatsoever can be described by a function of the form $Q(t) = B(t)c_0K^{\beta}L^{1-\beta}$, providing the residual $\dot{B}/B$ is solely a function of time. What we must therefore do for Fisher's experiments. in order to see why aggregate Cobb-Douglas functions work for them, is to examine this residual $\dot{B}/B$.

By definition, from Equation (3)

$$\frac{\dot{B}}{B} = \frac{\dot{q}}{q} - s\frac{\dot{k}}{k} \tag{3}$$

Here, $q = Q/L$, and $kK/L = J/L$ since Fisher's index of capital is denoted by J. Thus $\dot{q}/q = \dot{Q}/Q - \dot{L}/L$, and $\dot{k}/k = \dot{J}/J - \dot{L}/L$ and:

$$\frac{\dot{B}}{B} = \frac{\dot{Q}}{Q} - \frac{\dot{L}}{L} - s\frac{\dot{J}}{J} + s\frac{\dot{L}}{L}$$

$$\frac{\dot{B}}{B} = \frac{\dot{Q}}{Q} - s\frac{\dot{J}}{J} - (1 - s)\frac{\dot{L}}{L} \tag{25}$$

Since s and $1 - s$ are (roughly) *constant* profit and wage shares, respectively, we need only examine the rates of change of $L(t)$, $J(t)$, and $Q(t)$.

The first is easy. In all of his simulations, Fisher specifies that "labor grows at an average rate of 3% trend" with small random deviations from the trend (Fisher, 1971, p. 309). Ignoring the small random deviations then,

$$\frac{\dot{L}}{L} \approx .03 \tag{26}$$

The growth rate of the aggregate capital index $J(t)$ is a bit more complicated. In Equation (20) we defined

$$J(t) = \sum_{i=1}^{N} J_i(t) = \sum_{i=1}^{N} r_i K_i(t)$$

where the r_i are constant over time. Differentiating this with respect to time,

$$\frac{dJ(t)}{dt} = \sum_{i=1}^{N} r_i \frac{dK_i(t)}{dt}$$

$$= \sum_{i=1}^{N} [r_i K_i(t)] \frac{dK_i(t)}{dt}\frac{1}{K_i(t)}$$

$$= \sum_{i=1}^{N} J_i(t)\frac{\dot{K}_i}{K_i}$$

Dividing through $J(t)$, we get:

$$\frac{\dot{J}}{J} \equiv \frac{dJ(t)}{dt}\frac{1}{J(t)} = \sum_{i=1}^{N}\left(\frac{\dot{K}_i}{K_i}\right)\left[\frac{J_i(t)}{J(t)}\right]$$

During all his simulations, Fisher assumes that each capital stock $K_i(t)$ grows at an essentially *constant* rate one which in general differs from industry to industry.[14] Thus,

$$\frac{\dot{K}_i}{K_i} \cong \beta_{i1} \tag{27}$$

and this in turn implies

$$\frac{\dot{J}}{J} \cong \sum_{i=1}^{N} \beta_{i1}\left[\frac{J_i(t)}{J(t)}\right] \tag{28}$$

Therefore $\dot{J}/J$ is a weighted average of the β_{i1}, with weights which sum to one, since $J(t) \equiv \sum_{i=1}^{N} J_i(t)$. (This type of weighted average is known as a convex combination, and implies that $\dot{J}/J$ will always be between the largest and smallest β_{i1}.)

Finally, we come to the growth rate of aggregate output $Q(t) \equiv \sum_{i=1}^{N} Q_i(t)$. From Equation (14), we know $Q_i(t)$, so

$$Q(t) = \sum_{i=1}^{N} Q_i(t) = \sum_{i=1}^{N} A_i(t)[L_i(t)]^{\alpha}[K_i(t)]^{1-\alpha}$$

From this, we can derive $\dot{Q}/Q$:[15]

$$\frac{\dot{Q}}{Q} = \left[\sum_{i=1}^{N}\frac{\dot{A}_i}{A_i} + \alpha_i\left(\frac{\dot{L}_i}{L_i}\right) + (1-\alpha_i)\left(\frac{\dot{K}_i}{K_i}\right)\right]\frac{Q_i}{Q} \tag{29}$$

Of the terms in expression (29), we already know that $Q_i/Q \cong p_i$ from (21), $\dot{L}_i/L_i \cong \dot{L}/L \cong .03$ from (24), and $\dot{K}_i/K_i \cong \beta_{i1}$ from (27). To this, we need only add the fact that in general, ignoring small **random** deviations, Fisher assumes that the shift parameter A_i grows at an essentially constant rate, which differs from industry to industry.[16]

$$\frac{\dot{A}_i}{A_i} \cong \gamma_{i1} \tag{30}$$

All of this gives

$$\frac{\dot{Q}}{Q} \cong \sum_{i=1}^{N} \gamma_{i1}p_i + \sum_{i=1}^{N} .03\alpha_i p_i + \sum_{i=1}^{N}(1-\alpha_i)\,\beta_{i1}p_i$$

But $\sum_{i=1}^{N} \alpha_i p_i = 1 - s$ = constant wage share, from (22). So

$$\frac{\dot{Q}}{Q} \cong \sum_{i=1}^{N} \gamma_i p_i + \sum_{i=1}^{N}(1 - \alpha_i)p_i\beta_{i1} + .03(1-s) \tag{31}$$

Combining the expressions for $\dot{L}/L$, $\dot{J}/J$, and $\dot{Q}/Q$, we return to the all important residual $\dot{B}/B$ of equation (25);

$$\frac{\dot{B}}{B} \cong \frac{\dot{Q}}{Q} - s\frac{\dot{J}}{J} - (1-s)\frac{\dot{L}}{L}$$

$$\frac{\dot{B}}{B} \cong \sum_{i=1}^{N} \gamma_{i1}p_i + \sum_{i=1}^{N}(1-\alpha_i)p_i\beta_{i1} + .03(1-s)$$

$$- s\sum_{i=1}^{N}\beta_{i1}\left[\frac{J_i(t)}{J(t)}\right] - .03(1-s)$$

$$\frac{\dot{B}}{B} = \sum_{i=1}^{N} \gamma_{i1}p_i + \sum_{i=1}^{N}\beta_{i1}\left[(1-\alpha_i)p_i - s\frac{J_i(t)}{J(t)}\right]$$

$$\frac{\dot{B}}{B} = \sum_{i=1}^{N} \gamma_{i1}p_i + s\sum_{i=1}^{N}\beta_{i1}\left[\frac{(1-\alpha_i)p_i}{s} - \frac{J_i(t)}{J(t)}\right]$$

Given that the constant wage share $1 - s = \sum_{i=1}^{N} \alpha_i p_i$ we can write the profit share $s = 1 - \sum_{i=1}^{N} \alpha_i p_i$. But by definition $p_i = Q_i/Q$, so that

$$\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} Q_i/Q = 1$$

Thus,

$$s = \sum_{i=1}^{M} p_i \quad \sum_{i=1}^{N}\alpha_i p_i = \sum_{i=1}^{N}(1-\alpha_i)p_i = \sum_{i=1}^{N} s_i,$$

where $s_i = (1-\alpha_i)p_i$. From this, we at long last get

$$\frac{\dot{B}}{B} = \sum_{i=1}^{N} \gamma_{i1}p_i + s\sum_{i=1}^{N}\beta_{i1}\left[\frac{s_i}{s} - \frac{J_i(t)}{J(t)}\right] \tag{32}$$

in which it is important to note that the terms s_i/s and $J_i(t)/J(t)$, when summed over i, each sum to 1.

Laws of algebra and laws of simulation. In the expression (32) for $\dot{B}/B$ the basic structural parameters are β_{i_1} and γ_{i_1}. Of these, β_{i_1} represents the rate of growth of the i^{th} machine stock over any given simulation run, whereas γ_{i_1} represents the rate of technical change in the i^{th} industry. (Since the α_i are constant over any given run, changes in the shift parameter $A_i(t)$ represent the only possible technical change in any industry.)

Fisher partitions his simulations into two basic groups. In the first of these, which he calls "Hicks experiments," he sets all $\beta_{i_1} = 0$. Thus, in each of these experiments, there is technical change $(\gamma_{i_1} \neq 0)$ but no growth in the size of the machine stock $(\beta_{i_1} = 0)$. Under these conditions, $\dot{B}/B$ reduces to a constant over time.

$$\frac{\dot{B}}{B} \cong \sum_{i=1}^{N} \gamma_{i_1}p_i \cong b_1 \quad \text{(a constant over time)} \tag{33}$$

Thus, for Hicks experiments, one can expect from purely algebraic considerations that

$$\ln B(t) \cong a_0 + b_1 t \tag{34}$$

where a_0 is a constant of integration.

From the laws of algebra (Equation 5), we know that in general if $\dot{B}/B$ is solely a function of time, any data associated with constant shares $s \cong \beta$ can be represented by the functional form below (since Fisher uses J as an index of **capital,** what we previously called $k = K/L$ is now

$$j = (J/L): q = B(t)c_0 j^{\beta} \tag{5}$$

Taking natural logs,

$$\ln q = \ln B(t) + \ln c_0 + \beta \ln j$$
$$= (a_0 + \ln c_0) + b_1 t + \beta \ln j$$

and combining the constants into. a single constant b_0, we get

$$\ln q = b_0 + b_1 t + \beta \ln j \tag{35}$$

What we have shown therefore is that for *Hicks experiments, purely algebraic* (as opposed to econometric) *considerations* lead us to the conclusion that whenever shares are (roughly) constant Fisher's aggregate data can be generated by what *appears* to be a Cobb-Douglas "production" function with a constant rate of technical change and a marginal product of labor equal to the actual wage.

This is precisely the result Fisher gets for his Hicks experiments: for this set of experiments, the functional form which repeatedly works the best (in the sense that the estimated marginal product of labor most closely approximates the actual wage) is one which assumes constant returns to scale and a constant rate of technical change.[17]

We now turn to the second set of experiments, what Fisher calls his "Capital experiments," in which all $\gamma_{i_1} = 0$. In this set of experiments, therefore, there is positive or negative growth of the i^{th} machine stock ($\beta_{i_1} \neq 0$) but NO technical change ($\alpha_{i_1} = 0$). Equation (32), the general expression for the residual, now becomes:

$$\frac{\dot{B}}{B} \approx s \left[\sum_{i=1}^{N} \beta_{i1} \frac{s_i}{s} - \sum_{i=1}^{N} \beta_{i1} \frac{J_i(t)}{J(t)} \right] \tag{36}$$

In Equation (36) each term in the brackets is a convex combination (a weighted average whose weights sum to one) of the β_{i1}, so that each term lies between the largest and the smallest β_{i_1}. One would therefore expect the *difference* of these terms to be close to zero; in addition, since the constant wage share $1 - s = \sum_{i=1}^{N} \alpha_i \beta_i$ is itself a convex combination of the parameters α_i, it itself will be within the range of *these* parameters;[18] since the unweighted average of the α_i is 0.75, the profit share s will be roughly around *0.25.* Given that the term in the brackets is likely to be small, multiplying it by $s \cong .025$ will yield a number even closer to zero. In capital experiments algebraic considerations would therefore lead us to expect:

$$\frac{\dot{B}}{B} \simeq 0 \tag{37}$$

so that

$$\ln B \cong a_0, \quad \text{where } a_0 \text{ is a constant} \tag{38}$$

In setting this result into the general functional form of Equation (5) $q = B(t) c_0 j^{\beta}$ and taking natural logs of both sides, $\ln q \cong \ln B(t) + \ln c_0 + \beta \ln j = (a_0 + \ln c_0) + \beta \ln j$ and combining the constant terms into a single constant b_0

$$\ln q \cong b_0 + \beta \ln j \tag{39}$$

For the capital experiments, therefore, purely algebraic considerations lead us to expect that Fisher's data can be represented by what *appears* to be a Cobb-Douglas production function with a constant level of technology and a marginal product of labor equal to the actual wage. *Once again this is precisely the result Fisher gets for his capital experiments.*[19]

It is important to note that Fisher himself never presents the exact regression results involved (an understandable omission considering that there were a total of 1010 runs of this simulated economy, each run covering a 20-year period). Instead, he tells us only that the best fits to the aggregate data were derived from an equation of the form $\ln q = b_0 + b_1 t + \beta \ln j$ for *Hicks experiments,* and one of the form $\ln q = b_0 + \beta \ln j$ for *capital experiments.* To Fisher this result comes as a surprise. But it should not, for as we have just seen, Fisher's complicated and expensive experiments have merely rediscovered the laws of algebra.

Cross-section aggregate production functions. The direct analogy to constant shares in time series is the case of uniform **profit margins** (profits per dollar sales) in cross-section data. Using the subscript i for the i^{th} industry (or firm), and defining $\beta = s_i = r_i k_i / q_i$ as the uniform profit margin, we can rewrite Equation (3) as

$$\frac{dq_i}{q_i} = \left[(1 - \beta) \frac{dw_i}{w_i} + \beta \frac{dr_i}{r_i} \right] + \beta \frac{dk_i}{k_i} \tag{40}$$

Then, so long as the term in brackets is uncorrelated with dk_i/k_i, the above equation is algebraically similar to a simple linear regression model $y_i = bx_i + u_i$, with the term in brackets playing the part of the disturbance term u_i. Obviously, for any data in which the bracketed term is small and uncorrelated with the dependent variable dk_i/k_i, the "best" fit will be a cross-section Cobb-Douglas production function with constant returns and factors paid their marginal products.

There are still other ways in which one may explain the apparent success of a Cobb-Douglas in cross-section studies, the best single reference being Phelps Brown's (1957) critique. In a subsequent note, Simon and Levy (1963) show that any data having uniform wage and profit rates across the cross section can be closely approximated by the ubiquitous Cobb-Douglas function having "correct" coefficients, *even* though the data reflect only mobility of labor and capital, not any specific production conditions.

Once again, it would seem that the apparent empirical success of the Cobb-Douglas function having "correct" coefficients is perfectly consistent with wide varieties of data, and cannot be interpreted as supporting aggregate neoclassical production and distribution theory.

Summary and conclusions

It is characteristic of theoretical parables that they illustrate the *fundamental truth* of a paradigm, truths which more developed theoretical structures may modify and elaborate, but cannot undermine. In the neoclassical progression of parables from simple exchange to capitalism as the final solution to Man's "natural" greed, one central theme which emerges right in the beginning is the conception of equilibrium prices as "scarcity prices:" relative prices which reflect the relative scarcity of commodities.

In their most developed form, neoclassical parables have sought to present the notion of scarcity pricing as an explanation of the distribution of income between workers and capitalists. Here, the task is to portray a capitalist economy in such a way that the wage and profit rates may be seen to be the scarcity prices of labor and capital, respectively. But for this to be even a logical *possibility,* it is at the very least necessary that the wage and profit rates behave as *if* they were scarcity prices – i.e., that the profit rate fall as the capital-labor ratio rises, and the wage rate fall as the labor-capital ratio rises. This correlation is minimally necessary for the internal consistency of the parable (though of course its existence would hardly justify the implied causation).

Alas, the grand neoclassical parables have fallen on hard times, and after repeated demonstrations of their logical inconsistencies, they have been abandoned by the high-brows of the theory; not without regret, though, for as Samuelson so insightfully notes, within the parable "the apologist for capital and for thrift has a less difficult case to argue" (Samuelson, 1966).

"If all this causes headaches for those nostalgic for the old time parables of neoclassical writing, we must remind ourselves that scholars are not born to live an easy existence We must respect, and appraise, the facts of life" (Samuelson, 1966).

Not everyone was ready to give up the old time parables though, and those who chose to ignore the previously mentioned facts of life sought succor – where else? – in the "facts." The "real world," whose vulgar intrusions neoclassical theory had in the past so carefully avoided, became its last refuge. Facts, after all, are always better than facts-of-life.

And what are these facts? Simply, that again and again, aggregate Cobb-Douglas production functions work – that is, they not only give a good fit to aggregate output, but they also generally yield marginal products which closely approximate factor rewards. Since the aggregate production function is the simplest form of the grand neoclassical parable, its apparently strong empirical basis has often been taken as providing a good measure of support for the old time religion, regardless of what the theory says,

The main purpose of this chapter has been to show that these empirical results do not, in fact, have much to do with production conditions at all. Instead, it is demonstrated that when the distribution data (wages and profits) exhibit constant shares, there exist broad classes of *production* data (output, capital, and labor) that can *always be related to each other through a functional form which is mathematically identical to a Cobb-Douglas "production function" with constant "returns to scale," "neutral technical change," and "marginal products equal to factor rewards."*

Since this result is a mathematical consequence of any (unexplained) constancy of shares, it is true even for very implausible data. For instance, data points that spell out the word "HUMBUG" were used as an illustration, and it was shown that even the humbug economy can be represented by Cobb-Douglas production function having all the previously mentioned properties.

Similarly, we have examined Solow's famous paper on measuring technical change; and here too it is shown that the underlying production function which he isolates, by removing the effects of technical change, can be *algebraically* anticipated, even down to the fitted coefficients of his regression.

Next, Franklin Fisher's mammoth simulation experiments are examined and once again it becomes clear that the laws of algebra can anticipate the laws of simulation from the structure of the experiments alone.

Lastly, in the final part of this chapter, the analysis is extended to provide a simple explanation for cross-section aggregate production functions. The overall impact of these discussions, it is hoped, will be to demonstrate that the *reality* to which the neoclassical hangers-on clutch so desperately is as empty as their own abstractions.

Postscript

The point of this chapter is to demonstrate that as long as distributive shares are constant, it is an algebraic law that the Cobb-Douglas function "fits" almost any data. Hence, Solow's paper and the Humbug data stand on the same footing.

Solow has recently claimed that all along the intention of his 1957 paper was to "yield an exact Cobb-Douglas and tuck everything else into the shift factor" (Solow, 1974, p. 121). But his own printed words give quite a different impression: in the original paper, after he has derived the so-called shift factor $A(t)$, Solow expressly states his intention to "discuss the shape of $f(k, 1)$ and reconstruct the (underlying) aggregate production function" (Solow, 1957, p. 317). To this end, he constructs a graph of f(k) versus k, noting with obvious satisfaction that in spite of "the amount of a priori doctoring which the raw figures have undergone, the fit is remarkably tight" (Solow, 1957, p. 317), giving rise to "an inescapable impression of curvature, of persistent but not violent diminishing returns" (Solow, 1957, p. 318).

If, as Solow now claims, he knew all along that the underlying production function would be a Cobb-Douglas, then why bother "reconstructing" it? Why the surprise at the tightness of fit and the "inescapable impression of curvature"? Why does Solow need regression analysis to "confirm the visual impression of diminishing returns . . ." (Solow, 1957, p. 319). If Solow had indeed understood his own method, he should have known that regardless of the amount of a priori doctoring of the data, the laws of algebra dictate that the fit of f(k) versus k would be very tight as well as being inescapably curved. But it is hardly necessary to rediscover these algebraic artifacts by means of graphs and regressions.[20]

Having just said that his method and his education lead him to conclude that even the Humbug economy is neoclassical, Solow next asserts the very opposite. With the help of Samuel L. Myers, he runs a regression of the form $\ln q = a_0 + a_1 t + b \ln k$ on the Humbug data, and finds to his obvious delight that this leads not only to a very poor fit but also gives rise to a negative coefficient for $\ln k$. The moral seems clear: production functions do not "work" for the Humbug data, whereas they do for real data (Solow, 1974, p. 121).

But once again, his method and education betray him, The laws of algebra show that almost any production data associated with a constant profit share β could be cast in the form $Q = B(t)k^{\beta}$. The Humbug data was an illustration of this, and it was sufficient for my purpose in the original paper to show that even in this case the "underlying" function $f(k)$ was extremely well fitted by the Cobb-Douglas form $f(k) = k^{\beta}$ (R^2 = .993) and that the so-called shift factor B was solely a function of time. Hence, even Humbug data would be consistent with a neoclassical production function having "neutral technical change" and "marginal products equal to factor rewards".

Obviously, given that the underlying function f(k) was numerically specified by the laws of algebra (Equation (12) and note 9, in this chapter), all that would have been necessary for a complete *numerical* specification was a fitted function for B(t). However, since such a fitted function was not necessary to the logic of my argument, I was content with merely graphing B(t) versus time, as in Figure 5.3.

A glance at Figure 5.3 is sufficient to indicate that no simple linear or log-linear function will fit *B(t). And yet this is precisely the form that Solow* uses *in his regression,* [21] He naturally gets a very poor fit. How clever.

In this version of the paper, for the sake of completeness, I do actually specify a fitted function for *B(t),* with an R^2 = .82 (Equation 13). But the logic of the argument does not require this step; it only requires that the so-called shift factor be a function solely of time: *there is nothing in neoclassical theory, no law of production or of nature, which requires B(t) to be linear or log-linear.* Struggling under the weight of their bag of tools, Solow and Myers seem to have forgotten that linearity is merely a convenient assumption whose applicability must at all times be *justified,* not merely assumed.

Notes

1 ". . . the core of the theory of a private ownership economy is provided by the theory of exchange" (Walsh, 1970, p. 159).
2 Garegnani in fact does not state it this way. He shows that the necessary and sufficient condition is that the wage-curves all be straight lines, and shows that this in turn is true when all industries have the same capital-labor ratios, i.e., when prices are proportional to labor values (Garegnani, 1970, p 421)
3 Q(t) ≡ value of output; K(t) ≡ value of the utilized stock of capital; L(t) ≡ employed stock of labor; t ≡ time.
4 I thank Professor Luigi Pasinetti for having pointed this out in his comments on an earlier version of this paper.
5 R. R. Nelson gives a summary of subsequent refinements (Nelson, 1964).
6 "In order to isolate shifts of the aggregate production function from movements along it" (Solow, 1957, p. 314).
7 The discrete equivalent for $\dot{A}/A$ is ΔA/A, where $\Delta A = A(t+1) - A(t)$. Thus $A(t+1) = A(t)[1 + \Delta A/A]$; in 1909, $t = 0$, and by setting $A(0) = 1$, Solow derives a series for $A(1), A(2) \ldots$, from the data on $\dot{A}/A$.
8 Since Solow's calculations contained an arithmetical error, the points representing the years

1943-1949 clearly lay outside the range of any hypothesized curve. After expressing some hesitance, Solow leaves them out of his regressions (Solow, 1957, p. 318).

9 The deviation of the numerical value of the constant term is explained on pp. 20-21 of Solow's 1957 paper.

10 I wish to thank Larry Heinruth and especially Peter Brooks, for the time and effort expended in deriving this fitted function. Two steps were involved in the fitting. First, a two-year moving average $\bar{B}(t)$ was constructed from the data for B(t), by means of the formula $\bar{B}(t) = [B(t) + B(t+1)]/2$ in which the year 1909 represents $t = 1$, 1910 by $t = 2$, etc. Second, the function $\hat{B}(t)$ of Equation (13) was fitted to this moving average B(t), with a $R^2 = .82$.

11 Since the fitted function has $K = 16$ parameters to it, and since there are $T = 38$ data points in the moving average $B(t)$, the R^2 corrected for degrees of freedom is (Goldberger, 1964):

$$\bar{R}^2 = R^2 \frac{K}{T \quad K-1}(1 - R^2)$$
$$= .82 - \frac{16}{21}(.18) = .68$$

12 By the definition $MPL_i = \partial Q_i(t)/\partial L_i(t)$. Applying this to the expression for $Q_i(t)$ in Equation (14) yields $MPL_i = \alpha_i[Q_i(t)/L_i(t)]$.

13 From Equation (21), $Q_i(t) = p_i Q(t)$, where p_i is constant over time. Thus

$$\frac{dQ_i(t)}{dt} = p_i \frac{dQ(t)}{dt}$$

and

$$\frac{dQ_i(t)}{dt}\frac{1}{Q_i(t)} = p_i \frac{dQ_i(t)}{dt}\frac{1}{p_i Q_i(t)} = \frac{dQ(t)}{dt}\frac{1}{Q(t)}$$

Similarly for employment from (23).

14 Fisher assumes ln $K_i(t) = \beta_{i0} + \beta_{i1} t$ + (small random deviations). Ignoring the small deviations, and differentiating gives (Fisher, 1971, p. 309)

$$\frac{dK_i(t)}{dt}\frac{1}{K_i(t)} \cong \beta_{i1}$$

15 $Q(t) = \sum_{i=1} A_i(t)[L_i(t)]^{\alpha}[K_i(t)]^{1-\alpha}$

Dropping the time subscript, and differentiating,

$$\dot{Q} = \sum_{i=1}^{N} [\dot{A}_i L_i^{\alpha} K_i^{1-\alpha} + A_i(\alpha_i \dot{L}_i)L_i^{\alpha-1}K_i^{1-\alpha} + A_i L_i^{\alpha}(1-\alpha_i)\dot{K}_i K_i^{1-\alpha-1}]$$

$$\dot{Q} = \sum_{i=1}^{N}\left[\left(\frac{\dot{A}_i}{A_i}\right)A_i L_i^{\alpha} K_i^{1-\alpha} + \left(\frac{\alpha_i \dot{L}_i}{L_i}\right)A_i L_i^{\alpha} K_i^{1-\alpha} + (1-\alpha_i)\left(\frac{\dot{K}_i}{K_i}\right)A_i L_i^{\alpha} K_i^{1-\alpha}\right]$$

$$\dot{Q} = \sum_{i=1}^{N}\left[\left(\frac{\dot{A}_i}{A_i}\right)Q_i + \left(\alpha_i \frac{\dot{L}_i}{L_i}\right)Q_i + (1 \quad \alpha_i)\left(\frac{\dot{K}_i}{K_i}\right)Q_i\right]$$

so that

$$\frac{\dot{Q}}{Q} - \sum_{i=1}^{N}\left[\left(\frac{\dot{A}_i}{A_i}\right) + \left(\alpha_i \frac{\dot{L}_i}{L_i}\right) + (1-\alpha_i)\left(\frac{\dot{K}_i}{K_i}\right)\right]\frac{Q_i}{Q}$$

16 Fisher (1971, p. 309) assumes $\ln A_i = \gamma_{i0} + \gamma_{i1} t$, so that $(dA_i/dt)(1/A_i) = \gamma_{i1}$.

17 The function form in Fisher's equation, (17), the best form for Hick's experiments, is log $(Y^*/L) = a + b \log(J/L) + dt$ where his Y^*/L corresponds to our q and his J/L to our j. Fisher uses "log" for natural logarithms (Fisher, 1971, p. 313).

18 Fisher has two ranges of α_i: $.7 \leq \alpha_i \leq .8$, and $.6 \leq \alpha_i \leq .9$, in both the unweighted average = 0.75 (Fisher, 1971, p. 309).

19 The functional form Fisher finds best for Capital experiments is $\log(Y^*/L) = a + b\log(J/L)$ which, allowing for notation differences, is identical to equation 5.38 (Fisher, 1971, p. 313).

20 Yet confronted with the humbug data, Solow says: "If you ask any systematic method or any educated mind to interpret those data *using a production function and the marginal productivity relations*, the answer will be that they are exactly what would be produced by technical regress with a production function that must be very close to Cobb-Douglas" (Solow, 1957, p. 121). What kind of "systematic method" or "educated mind" is it that can interpret almost any data, even the humbug data, as arising from a neoclassical production function?

21 Solow uses the form $\ln q = a_0 + a_1 t + b \ln k$; since the general form under consideration is $q = A(t) f(k)$, so that $\ln q = \ln A(t) + \ln f(k)$, Solow has obviously specified $A(t)$ as log-linear: $\ln A(t) = a_0 + a_1 t$.

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