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Profits equal surplus value on average and the significance of this result for the Marxian theory of accumulation

Being a new contribution to Engels' Prize Essay Competition, based on random matrices and on manuscripts recently published in the MEGA for the first time

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Marx's justification of his theory of surplus value in the face of unequal compositions of capital, by interpreting total profits as a redistribution of surplus value, is not correct in general. However, it is shown here that the equality holds if the input matrices are random and the labour theory of value holds, in a sense to be specified, on average. Manuscripts recently published for the first time confirm that to the end Marx trusted his approach to the theory of value in that he continued to use the identity of the aggregates of capital and surplus in value and in price terms. His insistence was rooted in his philosophy. An attempt is made to clarify his use of a Hegelian methodology by comparing Hegel's and Marx's approaches to the foundation of the infinitesimal calculus. The article concludes with Marx's late reconsiderations of his theory of the falling rate of profit, which also continue to be based on the equality of profit and surplus value.

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1. Introduction

The analytical result of this article consists in the proof that the Marxian transformation of values into prices is correct after all, despite many refutations, if the economic system under consideration is random. Marx wanted to show that the profits reaped

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by capitalists could be understood as redistributed surplus value. The fruit of exploitation, produced in proportion to labour employed, was shared in proportion to capital advanced. The gross product of the economy was of equal magnitude in value and price terms by definition; it followed that the rate of profit measured in price terms was equal to that measured in values. The assumptions needed for this result are, as we shall see in Section 2.3, less restrictive than has been thought so far and similar to those needed for the existence of approximate surrogate production functions (Schefold 2013A). Hence I start from this parellel.

Research in capital theory has taken a new turn in recent years because of new results obtained by means of empirical work with input-output tables. The Cambridge critique of capital theory had been based on the assumption that re-switching and reverse capital deepening occur sufficiently often to undermine the aggregation of capital, the construction of production functions with a diminishing marginal of productivity of capital and, more generally, the neoclassical theory of distribution. As Samuelson (1962) showed, a production function with diminishing returns could be derived from a spectrum of techniques with a constant returns to scale technology, if the wage curves of individual techniques were linear, for the techniques could then be ordered: the techniques chosen at successively higher rates of profit would exhibit successively lower intensities of capital. Reverse capital deepening meant the possibility that, as the rate of profit rose, successive techniques would be chosen that would not necessarily entail a fall of the intensity of capital, but that a rise also could occur. When it was recognised that spectra of technique could easily be constructed for which such cases of reserve capital deepening were possible, advanced economists abandoned the idea of the production function, whilst many empirical economists continued to use it as if nothing had happened. The new empirical results indicate that the question of the existence of the production functions cannot be as neatly solved as had first been thought by either ignoring the paradoxes of capital theory (such as reverse capital deepening) or by regarding them as so important that the notion of marginal productivity of capital should be given up completely.

The early discussions of the paradoxes had been based on relatively simple calculations by means of two- or three-sector models. The development of input-output analysis allowed one to construct wage curves, taking actual input-output tables as data, and the wage curves thus found turned out not to be linear but not to deviate strongly from linearity in the relevant ranges. An investigation by Han and Schefold (2006) considered pairs of empirical input-output tables so that two methods of production were available in each industry, and the possible combinations gave rise to a very large number of techniques and wage curves. One example of re-switching and a number of cases of reverse capital deepening were found, but the number of switch points encountered on all envelopes exhibiting one of various possible paradoxes of capital theory was below 4%, and this seemed to question not the abstract validity but the concrete relevance of the Cambridge critique. Another result, which seemed surprising to us at first, was that the number of wage curves occurring on the envelope was quite low. This result could be interpreted as a new form of the critique of neoclassical capital theory in that it implied that the possibilities of substitution were more limited than the theory assumed.

I have tried to explain these empirical findings in two theoretical papers, introducing the idea of random matrices (Schefold 2013A, 2013B). Essentially, matrices are random, if the elements on each row (which represents the process) are i.i.d. with a distribution around a mean specific for the row. The non-dominant eigenvalue will then tend to be small; this implies that the wage curves will approximate a hyperbola.

The hyperbola will be stretched and a linear wage curve will be approximated, if properties of randomness concerning the labour inputs are added as further assumptions. I believe I have shown that these assumptions are sufficient to explain the empirical findings, and other explanations have, to the best of my knowledge, not been advanced. Moreover, I have isolated special conditions under which approximate surrogate production functions can be rigorously constructed. The idea that surrogate production functions exist only for one-commodity economies turns out to be false. However, the possibilities of substitution are limited, because only a small number of wage curves will appear on the envelope, as shown in Schefold (2013B), so the outcome is ambiguous and calls for more research.

Meanwhile, one may also ask what these results imply for Marxian economics. I intend to show that essentially the same assumptions about technology lead to a new solution of the transformation problem. It turns out that profits are equal to surplus value on average, as had been claimed in the third volume Marx's *Das Kapital*, edited by Friedrich Engels. This is a striking result, which comes as a big surprise, after more than 100 years of discussions of the transformation problem, but the interpretation is delicate, because it involves the question of which analytical reconstruction of the theory of value corresponds to Marx's intentions. I discovered that a somewhat different notion of average was introduced by Engels when he edited *Das Kapital* than Marx had used in the original manuscript, now edited in the Marx-Engels-Gesamtausgabe (*MEGA*^{2}, II, 4.2). My task therefore is not only establishing the analytical result but also interpreting it in the light of the publication of the original Marxian manuscript. As a historian of economic thought, I want to show that the Marxian emphasis on the equality of profits and surplus value is connected with his project of developing the theory of value along the lines of a Hegelian dialectic. To this end we shall look at the Marxian mathematical manuscripts, since they contain an explicit reappraisal of Hegelian logics by the mature Marx. Finally, we shall go through some of the recently published late manuscripts by Marx, in which the equality of profits and surplus value is used in the discussion of tendencies and counter-vailing tendencies for the rate of profit to fall. This law seemed to him to be the most important of political economy. The singular unity of his conception of the theory of value, based on the explanation of profits as redistributed surplus value, will thus come to the fore.

The article is composed of three parts. First is analysis of three ways to transform values into prices (Section 2). Second is interpretation of the solution based on random systems or 'averages', also examining different notions of 'average' in *Das Kapital*. Section 4 covers the importance of the interpretation of profits as redistributed surplus value for the Marxian law of the falling rate of profit. A conclusion follows in Section 5. Readers interested only in the interpretive parts of this article need not read the analysis in detail; those interested in the analytical result but not in my interpretation need not read the later sections and can confine their attention to Section 2.3, which is dedicated to the theory of value in random systems.

2. Analysis

2.1 Sraffa's classical treatment of value and price

Sraffa's theory of prices is well known, and I use it as the frame of reference. Prices for one given technique are defined by

$$(1+r)\mathbf{Ap}+w\mathbf{l}=\mathbf{p}$$

where $\mathbf{A} = (a_{ij})$; $i,j = 1, \ldots, n$ are input-output coefficients, $\mathbf{l} = (l_i)$ is the labour vector, $\mathbf{p}$ are prices of production, w is the wage rate, r the rate of profit. Matrix $\mathbf{A}$ is indecomposable, semi-positive and productive. If prices are expressed in terms of the standard commodity (the standard commodity is the net product of the economy in standard proportions), standard proportions are given by the left-hand side eigenvector of the input matrix, normalised so that total labour in the economy in standard proportions is equal to 1—see Schefold (1989 [1971]), the wage curve is linear and given by

$$w = 1-\frac{r}{R}$$

where R is the maximum rate of profits of the system, and the equations for prices can be given as an infinite series

$$\mathbf{p} = w\left(\mathbf{I}-(1+r)\mathbf{A}\right)^{-1}\mathbf{l} = \left(1-\frac{r}{R}\right)\sum_{t=0}^{\infty}(1+r)^t\mathbf{A}^t\mathbf{l}$$

which converges in the interval $0 \le r < R$ to the prices given by $(1 + r)\mathbf{Ap} + w\mathbf{l} = \mathbf{p}$. In the limit $r \to R$, prices tend to $(1 + R)\mathbf{Ap} = \mathbf{p}$, the prices of the system if wages are zero, also expressed in terms of the standard commodity.

Here we have the perfect representation for single-product systems of how prices depend on past labour inputs and the rate of profit r, which is used as an interest factor. The terms $\mathbf{A}^t\mathbf{l}$ indicate the labour expended, t periods ago, which enters the present product. The cost of this expense for past labour entering present production equals $w\mathbf{A}^t\mathbf{l}$, with an additional interest cost factor given by $(1 + r)^t$. Hence standard prices are equal to past labour expended, therefore equal to labour values, for $r = 0$. As r rises, these labour values are transformed into prices. We may call this Sraffa's transformation.

Prices are shown to be influenced by two opposed tendencies. As the rate of profit approaches its maximum, interest cost rises, and this happens the more, the greater the number of periods that have elapsed since the corresponding labour entering the present product was expended. On the other hand, the wage rate goes linearly to zero as we approach the maximum rate of profit. Now it is clear that labour inputs are the smaller, the more distant they are in time. Formally, $\mathbf{A}^t\mathbf{l}$ is a vector, each component of which tends to zero as t rises. Hence the interest factor, multiplied by the wage rate, has a sharp maximum close to the maximum rate of profit so that labour inputs distant in time can play a significant role in the determination of prices at rates of profits close to the corresponding maximum. If one now thinks that the labour inputs are distributed in an irregular fashion over time, one gets the idea that relative prices may change continuously but dramatically with changes in the rate of profit: they will be quite different from values. In this perspective, it seems totally unlikely that the sum of profits and the sum of surplus value in an economy should be equal except by rare coincidence. Sraffa's analysis seemed to afford a critique of the Marxian theory of value, as far as the equality of profits and surplus value was concerned, which was as drastic as his critique of the neoclassical theory of capital.

2.2 The 'new solution'

A short account of the history of the transformation problem is given by Hunt and Glick (1987); in particular, the critique of the Marxian solution by Bortkiewicz and attempts at solutions by Sweezy, Medio, Shaikh and the 'new solution' are described. Sraffa's theory of prices gave the clearest account of the relation between labour expended and prices of production by means of the reduction to dated quantities of labour, but the central Marxian idea of regarding profits as a redistribution of surplus value amongst the capitalists in the economy was lost. Foley (1982) and Duménil (1983/84) proposed a 'new solution' of the transformation which avoided this drawback by re-interpreting the value of labour power as the labour commanded by the money wage. This solution had also been proposed by Schefold (1973) in an article probably not known to Foley or Duménil. I translate the relevant paragraphs of that article into English to explain the new solution and to make it known that this solution had been proposed earlier and independently, whilst elsewhere I attributed a still earlier version to Robinson (1965).

Schefold (1973, pp. 170–1) formulated as follows:

The relation of surplus labour to necessary labour is not interpreted as the relationship of the labour embodied in the surplus to the labour embodied in the real wage (which is not observable), but as the relationship of the labour commanded by the amount of profits, divided by the labour commanded by the sum of wages (a relationship which is really effective). Since net social product in prices Y corresponds to the total labour expended L, one normalises the prices by setting both equal ($Y = L$). Total living labour L is divided in the relation $P{:}W$ between surplus labour M and necessary labour V, so that $P{:}W = M{:}V$, hence $P = M$, $W = V$, $P + W = Y = L = M + V$.
One obtains by this operation that the sum of values is equal to the sum of prices and surplus value equals profit. The procedure can be formalised in a Sraffa system as follows (for consistency, without danger for the conclusions, I continue to assume that the wage is not advanced). Let the price system be given by

$$(1+r)\mathbf{Ap} + w\mathbf{l} = \mathbf{p}$$

and the system of values by

$$\mathbf{Au} + \mathbf{l} = \mathbf{u}$$

Using the vector $\mathbf{e} = (1,1,\ldots,1)$, we can write net social product as

$$\mathbf{e}(\mathbf{I} - \mathbf{A})\mathbf{p} = Y$$

and we can normalise prices to get $Y = L$:

$$\mathbf{e}(\mathbf{I} - \mathbf{A})\mathbf{u} = \mathbf{e}(\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1}\mathbf{l} = \mathbf{el} = L = Y = \mathbf{e}(\mathbf{I} - \mathbf{A})\mathbf{p}$$

Since the rate of money wages is uniform, necessary labour and surplus labour are divided proportionally within the sectors

$$\mathbf{Au} + \frac{V}{L}\mathbf{l} + \frac{M}{L}\mathbf{l} = \mathbf{u}, \quad M + V = L$$

so that the rate of surplus value is uniform. We repeat: the relationship created between the value system and the price system is based on the assumption that not the labour embodied in the real wage, but the labour commanded by the money expended for wages is interpreted as necessary labour.

A note was added:

Only a longer discussion can decide whether this notion of necessary labour is admissible. The value of labour power, numerically equal to necessary labour time, is measured by Marx in the first volume of *Das Kapital* (where prices are equal to values) sometimes in the monetary expression and sometimes in the labour embodied in the real wage. Both definitions of necessary labour therefore coexist at this stage. As soon as prices differ from wages, one has to *choose* between the definitions. If one defines necessary labour time as the labour embodied in the consumption of the workers, one obtains a concept, which can be applied to several different social formations. If one relies on the monetary expression of necessary labour time, one concentrates on a specifically capitalist phenomenon.

Bellofiore (2001, pp. 371–2) attributes such a view to Sraffa himself, based on readings of papers of the 1940s.

It should be noted that the notion of labour commanded has been altered here. The original notion is as follows. Let x_0 be a quantity of bread x, at price p_x, which exchanges at wage rate w for a quantity of labour L_0 so that we have $x_0 p_x = wL_0$. The bread x_0 commands—is able to employ—labour L_0, and the price in terms of labour commanded $p_x / w = L_0 / \mathrm{x}_0$ can also be interpreted as the amount of labour to be delivered to acquire a unit of bread. If this concept is applied unchanged to macroeconomic magnitudes, two curiosities result. First, W / w, the labour commanded by the total wage, is equal to L or total living labour. Second, Y / w, the labour commanded by the net product, is a fictitious amount of labour that exceeds living labour by the labour commanded by profits; this fictitious amount of labour tends to infinity for $w \to 0$ and $r \to R$. Hence the notion of labour commanded was modified. We might speak of labour effectively commanded, if we say that a wage share W / Y allows to buy a share V / L of living labour and not the wage but Y effectively commands L. The normalisation used in Schefold (1973) is to be interpreted in this sense.

I do not claim today that my anticipation of the new solution does full justice to the ideas later exposed by Foley and Duménil. I soon thought—and I now think—that Marx clearly focussed on necessary labour time as the labour embodied in the real wage in the manuscript for the third volume and in the published version as edited by Engels. After the re-interpretation of labour commanded in the new solution, the identification of W and V is little more than a tautology. On the other hand, if necessary labour time is the labour embodied in the real wage, something has to be proved to get $P = M$ and the fact that $P = M$ does not always hold shows that the equality is not trivial. The new solution represents a departure from the Marxian conception; it is not obvious why the link between prices and labour values is maintained. Schefold (1973, p 172) asked whether it was necessary to maintain the link between prices and labour values at all under these circumstances.

2.3 Random systems

We can get an analytical reconstruction of the Marxian derivation of prices of production, which is more sophisticated and closer to his ideas, if we rely on random systems. I introduce them as follows.

Consider the spectrum of eigenvalues of matrix **A**. **A** has a dominant eigenvalue μ_1, which is unique with $0 < \mu_1 < 1$, such that all other eigenvalues $\mu_2, \ldots, \mu_n$ have smaller moduli. We can order them such that $\mu_1 > |\mu_2| \geq |\mu_3| \geq \ldots \geq |\mu_n| \geq 0$ (imprimitive matrices are here excluded—they are not generic—and similarly semi-simple eigenvalues, but not multiple eigenvalues, are disregarded). Mariolis and Tsoulfidis (2014) have shown that the eigenvalues of empirical input-output tables μ_i rapidly fall to zero. More precisely: divide the unit interval [0,1] on the x-axis into n segments $i = 1, \ldots,$ n of equal length, plot μ_i as a point at distance $|\mu_i|$ above segment i and interpolate a curve between these points. According to the published results by Mariolis and Tsoulfidis, this curve will have the shape of a function falling off exponentially and converging rapidly towards zero. In this context, I am indebted to Anwar Shaikh for discussions about his forthcoming book (Shaikh, in press). He has drawn such curves for the same economy, in the same year, at different levels of aggregation; it turns out that the curves are the closer to the x-axis, the lower the level of aggregation and the higher the number of the sectors. The moduli of the non-dominant eigenvalues of empirical input-output tables cluster around zero, as illustrated in Schefold (2013A). These observations may help justify the theoretical postulate employed here: that all eigenvalues, except the dominant, are close to zero.

It is easily seen that an indecomposable semi-positive matrix will have a spectrum with $\mu_1 > 0$, $\mu_2 = \ldots = \mu_n = 0$ if and only if **A** = **cf**, **c** > 0, **f** > 0, where **c** is a column vector, **f** a row vector and **A** has rank 1. If the coefficients of such a matrix are varied by small amounts, $\mu_2, \ldots, \mu_n$ will deviate from zero only by small amounts for reasons of continuity. The theory of random matrices is more precise on this and leads to the special case **f** = **e** = (1, . . . , 1). If the elements of the matrix on each row are distributed independently and identically around a mean specific for the row, with a variance that is so large that many single elements equal to zero are admitted, it is possible to prove, using certain additional assumptions, that the non-dominant eigenvalues will tend to zero as the dimension of the matrix increases. In other words, if the elements of a semi-positive and indecomposable matrix are random, if this matrix approximates the form **A** = **ce** and if the dimension of matrix **A** is sufficiently large, the non-dominant eigenvalues will tend to zero, even if the coefficients of the matrix are perturbed considerably. 'Tend to zero' here means, as usual, that the modulus of any eigenvalue is smaller than any pre-assigned positive number, if n, the dimension of the matrix, is large enough, and if the elements of the matrix are i.i.d. distributed and fulfil a variance and a covariance condition. This has been explained more formally in Schefold (2013A), so it does not seem necessary to build the same mathematical apparatus here. A result of similar precision is not known for the more general case **A** = **cf**. We must be content with the observation that also other distributions of the elements of the matrix on the rows than an identical distribution may cause the non-dominant eigenvalues to disappear.

It can be shown that the wage curves become simpler independent of the numéraire or of the labour vector, and they approximate the form of a hyperbola as the non-dominant eigenvalues tend to vanish. More complex forms of the wage curves than hyperbolas—I regard straight wage curves as stretched hyperbolas—are therefore necessarily due to non-dominant eigenvalues, which do not vanish. If one now assumes that input-output systems have near zero non-dominant eigenvalues, therefore that they are perturbed matrices of matrices of the form **cf**, and if one can also add reasons why the hyperbolas are stretched and tend to be linear, the empirical result is explained that wage curves derived from empirical input-output systems do not deviate much from straight lines. The additional argument, required for the 'stretching', is interesting precisely from the point of view of Marxian theory and relates to the labour vector. It is not necessary that prices are equal to values; it is only to be assumed that this property holds in a certain sense 'on average'. Indeed, as I shall show for the first time, strengthening the results obtained in Schefold (2013A), the wage curves of random systems (perturbations of matrices of the form **ce**) will tend to be linear and prices will be equal to values on average for all labour vectors if the numéraire and the labour vector are independent without additional assumptions. With this, we turn to the application and to Marx.

In his case, wages are advanced, when prices of production are formed. We have a vector of activity levels **y**, equal to the vector of gross output; it serves as numéraire. Hence we have the equations

$$\rho(\mathbf{A}\bar{\mathbf{p}} + \bar{w}\mathbf{l}) = \bar{\mathbf{p}}, \ \ \mathbf{y}\bar{\mathbf{p}} = 1, \ \ \rho = 1 + r$$

We assume that the real wage is given by vector $\mathbf{b} \leq \mathbf{y}$; the surplus product in the hands of the capitals equals $\mathbf{y}(\mathbf{I} - \mathbf{A}) - \mathbf{b} = \mathbf{s}$. Let $\mathbf{A}$ be a semi-positive and diagonalisable, productive and indecomposable input-output matrix, the non-dominant eigenvalues of which are small enough to be neglected; it can be a random matrix. To each of the n eigenvalues (counted with their multiplicities) there corresponds an eigenvector; those on the left are row vectors $\mathbf{q}_i$, those on the right are column vectors $\mathbf{x}_i$; $i = 1, \ldots, n$. We now can interpret the activity vector or vector of gross output $\mathbf{y} = \mathbf{q}_1 + \ldots + \mathbf{q}_n$ as a linear combination of the left-hand eigenvectors, where these eigenvectors are normalised so that the coefficients of the linear combination are all equal to 1. We call this a strong normalisation. The first eigenvector is the one pertaining to the dominant eigenvalue. Its components are in the same proportions as Sraffa's standard commodity. This standard vector may also be interpreted as the average industry, introduced by Marx in the third volume of *Das Kapital*. We introduce the vector **m** of the deviations between the activity levels and the standard vector, defined as

$$\mathbf{m} = \mathbf{y} - \mathbf{q}_1 = \mathbf{q}_2 + \ldots + \mathbf{q}_n$$

We get $\mathbf{m} = 0$ if activity levels correspond to Sraffa's standard proportions. The components of **m** are readily interpreted: if, for example, $\mathbf{m}_1 < 0$, the first industry produces less in the actual system than it would in the standard system. It is clear that profits will equal surplus value, if capital goods, the physical surplus and activity levels (gross product) are all in standard proportions; the rates of profit in value and in price terms will then coincide. Our solution of the transformation problem should not be confused with that relying on standard proportions. It is therefore essential that the deviations **m** are not assumed to vanish.

We then represent the labour vector as a linear combination of the right-hand-side eigenvectors, which are also so normalised (strong normalisation) that the coefficients of the linear combination are all equal to 1; hence we obtain similarly

$$\mathbf{l} = \mathbf{x}_1 + \ldots + \mathbf{x}_n; \ \ \mathbf{v} = \mathbf{l} - \mathbf{x}_1 = \mathbf{x}_2 + \ldots + \mathbf{x}_n$$

These representations of $\mathbf{y}$ and $\mathbf{l}$ as strongly normalised linear combinations of the left-hand-side and right-hand-side eigenvectors of $\mathbf{A}$ are unique, as one proves easily. The vector $\mathbf{x}_1$, pertaining to the dominant eigenvalue, is the vector for which prices would be equal to labour values at all rates of profits, if it were the labour vector, therefore, if $\mathbf{v} = 0$. (It is well known that prices are proportional to labour values at all rates of profit if and only if the labour vector happens to be the right-hand positive eigenvector of the input matrix.) We call $\mathbf{q}_1$ the Sraffa-vector and $\mathbf{x}_1$ the Marx-vector; $\mathbf{m}$ measures the deviation of the activity vector from the Sraffa-vector and $\mathbf{v}$ the deviations of the labour vector from the Marx-vector, for which, if it were the labour vector, prices and values would coincide.

Before extending our considerations of random systems, we derive an equation for the prices, which is based on the eigenvalues and eigenvectors. Using the general formula

$$(\mathbf{I}-\rho\mathbf{A})\mathbf{x}_i = (1-\rho\mu_i)\mathbf{x}_i;\ 0<\rho<\mu_1; 1/\mu_1 = 1+R$$

we obtain an equation homogeneous in non-normalised prices and the wage rate, which simplifies, if $\mu_2 = \ldots = \mu_n = 0$:

$$\begin{aligned}\mathbf{p} &= \rho w(\mathbf{I}-\rho\mathbf{A})^{-1}\mathbf{l}\\ &= \rho w\sum_{i=1}^{n}\frac{1}{1-\rho\mu_i}\mathbf{x}_i = \rho w\left[\frac{\mathbf{x}_1}{1-\rho\mu_1}+\mathbf{v}\right]\end{aligned}$$

In the next step, we replace the row vector of activity levels $\mathbf{y}$, serving as numéraire, by the corresponding linear combination of left-hand-side eigenvectors and use the fact that any two eigenvectors, pertaining to different eigenvalues, are orthogonal:

$$1 = \mathbf{y}\bar{\mathbf{p}} = \rho\bar{w}\left[\frac{\mathbf{q}_1\mathbf{x}_1}{1-\rho\mu_1}+\mathbf{mv}\right].$$

This formula shows that wage curves are hyperbolas in systems in which non-dominant eigenvalues disappear, hence tendentially in systems based on random matrices. On the right-hand side, there appears a constant term, independent of the rate of profit, which is the scalar product of the vectors of deviations, that is, of the deviations of the activity vector (numéraire) from the Sraffa-vector and of the labour vector from the Marx-vector. If these deviations are orthogonal, if this scalar product therefore disappears, the wage curve of the wage paid *ex post* $\rho\bar{w}$ is linear. Now there is in fact no reason why the deviations of activities from the average industry and the deviations of the labour vector from the Marx-vector should be correlated. The activity vector (numéraire) could be chosen randomly, whilst relative prices would be determined independently by the input-output matrix and the labour vector. We therefore assume, according to the well-known formula

$$\mathrm{cov}(\mathbf{m},\mathbf{v}) = 0,\ \text{therefore}\ \mathbf{mv} = n\bar{m}\bar{v}$$

where $\bar{m}$ and $\bar{v}$ are the averages of the components of $\mathbf{m}$ and $\mathbf{v}$ respectively. With this, we obtain a new formula for the wage curve

$$\bar{w} = \frac{1}{\rho}\left[\frac{1}{\frac{\mathbf{q}_1\mathbf{x}_1}{1-\rho\mu_1} + n\bar{m}\bar{v}}\right]$$

Now we make a third assumption, which fits well with Marx, about the random character of the system: we assume that $\bar{v} = 0$, hence that on average the deviations of the labour vector from the Marx-vector disappear. This is a new assumption, as far as I can see, in the literature on Sraffa and Marx. It means that, because the individual deviations of the labour vector from the Marx-vector do not disappear but its average disappears, the labour theory of value does not hold for the single prices but on average, as it were. A more precise formulation of this observation follows shortly; we shall see that $\bar{v} = 0$ does not have to be assumed but is tendentially implied, if **A** is random. If, however, we simply make the assumption, to begin with, we obtain a linear wage curve for the wage paid *ex post* $\rho\bar{w}$, as in Sraffa, and we obtain here, because the wage is advanced, for $\bar{w}$ the hyperbola

$$\bar{w} = \frac{1}{\rho}\frac{1-\rho\mu_1}{\mathbf{q}_1\mathbf{x}_1}$$

Now we may pass from prices not normalised to normalised prices:

$$\bar{\mathbf{p}} = \frac{\bar{\mathbf{p}}}{\bar{w}}\bar{w} = \frac{\mathbf{p}}{w}\bar{w} = \bar{w}\rho\left[\frac{\mathbf{x}_1}{1-\rho\mu_1} + \mathbf{v}\right] = \frac{\mathbf{x}_1}{\mathbf{q}_1\mathbf{x}_1} + (1-\rho\mu_1)\frac{\mathbf{v}}{\mathbf{q}_1\mathbf{x}_1}$$

It turns out that the three assumptions about the randomness of random systems lead to prices of production, which are linear functions of the rate of profit. Prices actually are constant and equal to labour values, if the deviations of the labour vector from the Marx vector disappear not only on average but individually; we then have $\mathbf{v} = 0$. Moreover, we get, as already indicated: the deviations of the labour vector from the Marx-vector disappear on average if and only if prices of production are equal to values on average.

The theorem follows from pre-multiplying the price formula from the left with **e** and forming the difference $\mathbf{e}\{\bar{\mathbf{p}}(r) - \bar{\mathbf{p}}(0)\} = -\rho\mu_1\frac{\mathbf{ev}}{\mathbf{q}_1\mathbf{x}_1} = 0$, for $\mathbf{ev} = n\bar{v}$; hence $\mathbf{e}\bar{\mathbf{p}} = \mathbf{e}\bar{\mathbf{p}}(0)$, if and only if $\bar{v} = 0$.

Now we get a stronger result for random systems (perturbations of matrices of type **ce**): In random systems, prices tend to be equal to labour values on average, whatever the labour vector, for $\bar{v}$ (and $n\bar{v}$) tends to zero.

This theorem is important, because one gets the linear wage curve for the wage paid *ex post*, if $\mathbf{mv} = n\bar{m}\bar{v} = 0$. Because our statements about random systems hold only tendentially for large n with any desired accuracy, it is not enough to know that $\bar{v}$ is small or in the limit equal to zero, since n increases and $n\bar{v}$ might not be small. But we can prove

$$n\bar{v} = \mathbf{ev} = \mathbf{ex}_2 + \ldots + \mathbf{ex}_n \to 0$$

since, according to the assumptions about random matrices, the elements of each row of $\mathbf{A}$ are identically and independently distributed, so that the sums of all columns must tend to be equal and $\mathbf{eA} \to \mu_1\mathbf{e}$ for large n. Hence $\mathbf{e}$ is proportional to $\mathbf{q}_1$ and is a left-hand-side eigenvector of $\mathbf{A}$ and orthogonal to $\mathbf{x}_2, \ldots, \mathbf{x}_n$.

If $\mathbf{A}$ is not the perturbation of a matrix of type $\mathbf{ce}$, but more generally of $\mathbf{cf}$, $\bar{v} = 0$ is an assumption in what follows. Whether it is fulfilled in reality is, as with all the other assumptions, an empirical question.

Whether or not $\bar{v} = 0$, prices are linear functions of the rate of profit in systems for which all non-dominant eigenvalues vanish. I showed more than 40 years ago that in general if the assumptions about averaging do not hold, the prices as functions of the rate of profit exhibit complex movements. The price vector assumes n linearly independent values for n different levels of the rate of profit, so that prices as functions of the rate of profit never stay within any hyperplane, let alone on a line (Schefold 1989 [1971]). That this holds on average after all has surprised me very much. It implied that the critique of neoclassical theory had to be modified. What are the implications for Marx?

None of the manuscripts by Marx, which have recently been published in the MEGA edition (Marx, 1975sq), not even the manuscript of the third volume of *Das Kapital* in MEGA^2 (II, 4.2) leads to a modification of the requirement for the transformation of values into prices, which is formulated in the third volume in the 10th chapter as edited by Engels as follows: 'Consequently, the sum of the profits in all spheres of production must equal the sum of the surplus values, and the sum of the prices of production of the total social product equal the sum of its value' (Marx, 1975sq, *MECW* 37, p. 172).

The fulfilment of the second requirement, that the sum of the prices of the commodities in the gross product shall be equal to the sum of their values, has been secured by normalisation of the prices. We therefore have, in conventional notation

$$C + V + M = K + W + P$$

where C, V, M are constant capital, variable capital and surplus value in values and K, W, P the capitals and the profits in prices. We have the corresponding identity in real terms

$$\mathbf{y} = \mathbf{yA} + \mathbf{b} + \mathbf{s}$$

and the distribution in real terms and in price terms are connected and define the rate of profit:

$$P = \mathbf{s}\bar{\mathbf{p}}(r) = r(\mathbf{yA}\bar{\mathbf{p}} + \mathbf{b}\bar{\mathbf{p}}) = r(\mathbf{yA}\bar{\mathbf{p}} + \bar{w}\mathbf{yl}) = r(K + W)$$

As a matter of fact, one can prove that there is one rate and only one rate of profit r; $0 < r < R$; such that the wage $\bar{w}\mathbf{yl} = W$ at $\bar{w}(r)$ buys exactly $\mathbf{b}\bar{\mathbf{p}}(r)$ and the profit $r\left(\mathbf{yA}\bar{\mathbf{p}}(r) + \bar{w}(r)\mathbf{yl}\right)$ buys exactly $\mathbf{s}\bar{\mathbf{p}}(r)$. We call this the Marxian equilibrium condition and show it as follows. Multiplying the quantity equations from the right by $\bar{\mathbf{p}}(r)$, considering r as the unknown, and the price equations from the left by $\mathbf{y}$, one gets the identity, with $\mathbf{b}\bar{\mathbf{p}} > 0$, $\mathbf{s}\bar{\mathbf{p}} > 0$:

$$\mathbf{y}\bar{\mathbf{p}} - \mathbf{yA}\bar{\mathbf{p}} = \mathbf{b}\bar{\mathbf{p}} + \mathbf{s}\bar{\mathbf{p}} = \bar{w}\mathbf{yl} + r(\mathbf{yA}\bar{\mathbf{p}} + \bar{w}\mathbf{yl}) = W + P$$

The equation, read from the right, shows that the net income buys the net product. It is Say's law which will be violated in a crisis of realisation. To show that the components of income buy the corresponding components of output at the equilibrium rate of profit, observe that there is exactly one r; $0 < r < R$; such that $\mathbf{b}\bar{\mathbf{p}} = \bar{w}\mathbf{yl}$, for $(\mathbf{b} / \mathbf{yl})(\bar{\mathbf{p}} / \bar{w})$ is continuous in $0 < r < R$, strictly monotonically rising and tends to infinity for $r \to R$. At $r = 0$ we have $(\mathbf{b} / \mathbf{yl})(\mathbf{I} - \mathbf{A})^{-1}\mathbf{1} < (\mathbf{b} + \mathbf{s})(\mathbf{I} - \mathbf{A})^{-1}\mathbf{1} / \mathbf{yl} = 1$. There is therefore exactly one r with $\mathbf{b}\bar{\mathbf{p}} = \bar{w}\mathbf{yl} = W$ (Weierstrass's theorem) and it follows that $\mathbf{s}\bar{\mathbf{p}} = P$.

On the other hand, the surplus value consists of vector $\mathbf{s}$, the commodities which remain of the gross product after the deduction of the commodities forming constant and variable capital, valued in terms of labour values; we therefore have $\mathbf{s}\bar{\mathbf{p}}(0) = M$. If the equality $M = P$ holds, one obtains the well-known equality

$$\frac{M}{C+V} = \frac{P}{K+W}$$

the rate of profit in value terms then is equal to the rate of profit in price terms. In a way, this equality is the real test of the Marxian proposition of $P = M$, for it is independent of the numéraire chosen, whilst $P = M$ could result from the choice of the numéraire. Moreover, by virtue of the identity of the rate of profit in values and prices, the laws about the rate of profit, derived by Marx in his theory of accumulation by means of the labour theory of the value, are transported into the sphere of prices.

Now we get

$$P = \mathbf{s}\bar{\mathbf{p}} = \frac{1}{\mathbf{q}_1\mathbf{x}_1}\left(\mathbf{s}\mathbf{x}_1 + (1 - \rho\mu_1)\mathbf{s}\mathbf{v}\right)$$

There is no reason for a correlation of the components of $\mathbf{s}$ with those of $\mathbf{v}$, hence we may assume here as well that

$$\mathrm{cov}(\mathbf{s}, \mathbf{v}) = 0, \text{ therefore } \mathbf{sv} = n\bar{s}\bar{v}$$

Now we already have assumed $\bar{v} = 0$. Moreover, if $\mathbf{A}$ is random in the sense of being a perturbation of matrix of the type $\mathbf{ce}$, we have the stronger *result* $n\bar{v} = 0$. In either case, we obtain that, independently of the rate of profit,

$$P = \frac{\mathbf{s}\mathbf{x}_1}{\mathbf{q}_1\mathbf{x}_1}$$

therefore,

$$P = M$$

holds: the sum of profits may be interpreted under average condition as the sum of surplus values, redistributed amongst the capitalists—a most surprising result, obtained after 120 years of discussions of the transformation problem.

Profit can be explained completely as redistributed surplus value under the assumptions we have made about the formation of averages, because the deviations of prices from values in the valuation of the commodities representing profits cancel—deviations that may exist, which may even be large, but which correspond to the deviations of the labour vector from the Marx-vector, and these cancel in total because of our new and decisive assumption $\overline{v} = 0$, or, in the stronger case, because random systems of type **ce** *imply* $n\overline{v} = 0$.

What we have derived is, first, a result of pure theory, valid just as much under the stated assumptions as other theories under their assumptions, like the theory of the standard commodity with the derivation of the linear wage curve. Sraffa's theory, however, is based on linear algebra, whilst the present result is based on stochastics. To present it with full mathematical rigour would require building a formidable mathematical apparatus. I have been content with a more elementary approach and hope that the reader will have understood that the theorems are algebraically correct if $\mathbf{A}$ is of rank 1, that they are true as limit theorems, if $\mathbf{A}$ is random (perturbation of matrix of type **ce**) and n tends to infinity and that they are true as approximations for finite n. The quality of the approximation follows in this case by means of established methods (see Schefold 2013A), whilst an analogous result is not yet available for matrices of type **cf** (hence the importance of assuming that $\overline{v}$ is strictly zero in the latter case; we could show that $n\overline{v}$ tends to zero in the former).

The theorems may also be interpreted in the perspective of applied economics. This would involve a double approximation. Input-output matrices are more highly aggregated than would be desirable from the theoretical point of view. There are the various difficulties to measure the coefficients, and prices, wage rates, profit rates are not uniform and so on. Then there is the question, more relevant for the present enquiry: in what sense and to what extent are input-output tables random? The very interpretation of the coefficients of an input-out table or of the labour vector as random variables, the character of their distribution and the possibility of applying the law of large numbers will be treated in another publication.

What separates us from Marx are not so much the modern methodological questions but the foundation of the theory. Prices are here not derived from values, but without having recourse to values from the structure of production or of the values in use, represented by $\mathbf{A}$ and $\mathbf{l}$, and from the distribution, represented by r. The formal redundancy of the theory of surplus value remains. It is one thing to say that using prices based on input-output relations, the quantity of total profits happens to be nearly equal to the quantity of surplus value (both expressed in terms of the total product), and another to affirm that profits are nothing but redistributed surplus value created by labour. If we are interested in why Marx insisted so much on the theory of value, consistently used for his analysis, we must turn to the history of economic thought and to interpretation.

We shall have to discuss how the result obtained here and that intended by Marx differ, concerning what is meant by 'average'. Provisionally, we are content with stating that Marx thought of an average to be formed across all sectors for a general system, not necessarily random as in our case, and that he thought that even then the differences of values and prices would compensate. By contrast, I use random systems ($\mathbf{A},\mathbf{l}$), which have specific average properties. It will have to be admitted that the notion of average cannot easily be transferred from single-product systems with circulating capital to fixed capital, to land and to joint production.

3. Interpretation

3.1 The historicity of Marxian economics

The recent completion of the second division of the MEGA, in particular the publication of the manuscripts Marx wrote to prepare the final version of volumes II and III of

Das Kapital, the manuscripts Engels used for the publication of volumes II and III after the death of Marx without reproducing all of them completely, lead to an inevitable reappraisal of the analytical construction of the theory of value, price and accumulation in Marx; we shall have to modify earlier positions and must check how much closer we can get to Marx's intentions as revealed in these manuscripts. The Cambridge school treated Marx as if he had been one of theirs, and my own earlier interpretations of Marx followed their line (Schefold, 1973). In particular, I attempted to summarise the results regarding the transformation problem obtained along the lines of the Cambridge school in my introduction written for the new edition of the third volume of *Das Kapital* as volume II, 15 of the new MEGA edition (Schefold, 2004). A well-known difficulty of this approach is the fact that it ignores Marx's Hegelian roots.

His theory is historical, like Hegel's philosophy and the approach of the Historical school which was, however, less successful in formulating theories adapted to different historical circumstances than Marx; they in effect worked more often with historical analogies. I use one of the frequent discussions of Aristotle in Marx to illustrate the historical nature of his departure in the theory of value, using a passage that appeared in the first edition of *Das Kapital* only in an appendix, which was later transferred to the main text with only formal changes:

> But the fact that in the form of commodity-*values* all labours are expressed as *equal human labour* and hence *as counting equally* (*als gleichgeltend*) could not be *read out of* the value-form of commodities by Aristotle, because *Greek* society rested on *slave labour* and hence had *the inequality of people and their labours as a natural basis*. The secret of the expression of value, *the equality of all labours* and the fact that *all labours count equally* because and insofar as they are *human labour as such* can only be deciphered when *the concept of human equality* already possesses the fixity of a popular prejudice. But that is only possible in a society in which *the commodity-form* is the general form of the product of labour and thus also the relation of people to one another *as possessors of commodities* is the ruling social relation. The genius of Aristotle shines precisely in the fact that he *discovers in the expression of value* of commodities a *relation of equality*. Only the historical limit of the society in which he lived prevents him from finding out what, 'in truth', this *relation of equality* consists in. (Marx, 1978,2(1), pp. 141–42)

According to this, the theory of value, as historically specific, can be recognised only in a society in which commodity owners preponderate, a condition that is thought to lead to the fixation of the idea of human equality. This explanation seems at first sight to contrast with Marx's intention to elaborate on the relationship of exploitation as the fundamental characteristic of capitalism. But as owners of commodities, the worker owning his labour power and the capitalist owning the means of production are in fact equal. Aristotle asked how the commodities could be equal in value, the persons entering in the exchange being unequal. They will become equal in the exchange, adds Aristotle (Aristoteles, *Eth. Nic* 1133, 17–20, 25–26). Aristotle's view appears in this perspective to stand in between that of a premonetary and a monetary economy. He regarded the polis-state as essentially a community of land-owning citizens with largely self-sufficient households; money historians have challenged this view and emphasized the importance of credit and banking in the Athenian economy (Schefold, 2013C). Marx seems to have favoured a more primitive interpretation (Engels was perhaps more modernist). According to this quote, Marx must have thought that commodity exchange was not yet the dominating form of economic intercourse, but he recognised the existence of commodity production by independent artisans in ancient Athens and he (not just Engels) treated the labour theory of value as the explanation of exchange value prior to prices of production (Schefold, 1995). He mentions this problem of historical transition when dealing with the transformation of values into prices and in his discussion of absolute rent.

The importance of the passage, however, is in its methodological implication: the more complex and more concrete forms are to be derived from the more simple and more general. Marx denotes the simple form of relative value in the first edition of *Das Kapital* also as 'the "cellular form" or, as Hegel would have said, the *An sich* [in itself] of money' (MEGA² II. 5, p. 28) The way from the abstract to the concrete, from the double character of the commodity to value, from the form of value to money or from abstract labour to average labour is also a way from the core to the surface. In research, it is necessary to discover the core, starting from the surface. The representation, by contrast, is effected by moving from the inside towards the outside. The 'in truth' at the end of the citation refers to the Aristotelian ἐν ἀληθείᾳ, to the 'what is unconcealed'—when truth has been discovered.

The new manuscripts recently discovered confirm that Marx stuck unwaveringly to his approach of explaining the phenomena of the capitalist mode of production and of its genesis by means of the intellectual development which starts from the theory of value. His late manuscripts, written in preparation of the publication of volumes II and III of *Das Kapital* and now published in MEGA², in large parts seem to consist only of inept calculations. Marx seems to be playing with numerical examples, in which the components of values and prices and the aggregates of constant and variable capital and surplus value are connected by means of ratios, such as the rate of surplus value and the rate of profit in varying combinations. It can be shown, however (see Section 4.2), that these bewildering efforts still aimed at finding 'laws of development' according to Marx's method. Well known are those of the production of absolute and relative surplus value and the tendency of the rate of profit to fall in consequence of the rise of the organic composition of capital. Clearly, he tried to find variations of these laws and perhaps others.

The main objections, raised against him from the point of view of modern theory, are dual. First, it is asserted that he was wrong when he affirmed that profit could be interpreted as redistributed surplus value. Second, he is said to have landed in contradictions in his analysis of the tendency of the rate of profit to fall; in particular, he is said to have overlooked that a rising organic composition of capital can be introduced under competitive conditions only if it is accompanied by some gain, a rise of profits or wages. Reading the manuscripts, one can observe that he checked his 'laws' (or, to use a more modest expression, his formulas) time and again, trying to find plausible examples for concrete applications. The modern theory of price was not at his disposal, however, so one might say he failed to take the decisive step in his movement from the abstract to the concrete because he failed to show how prices of production are formed simultaneously, affecting the values of products and of capital goods.

Marx wanted to get concrete step by step. Therefore, *Das Kapital* says in the third volume that the equality of the mass of surplus value and of the mass of profit was to hold 'on average' (MECW 37, p. 160), and he recognised several opposing tendencies for the general tendency of the rate of profit to fall. To this extent, his position was flexible. But he stuck rigidly to the fundamental position of starting from the labour theory of value and continued to attribute a central role to his 'law' of the tendency of the rate of profit to fall. He would have regarded it as petit-bourgeois to waver with regard to his fundamental principles. He had postulated the law of value and maintained it, and he continued to regard the tendency of the rate of profit to fall as the primary factor and the opposing tendencies as secondary. From a merely formal point of view, he could just as well have argued the reverse.[1]

There is a reason not to expect too much from the manuscripts. Engels knew them very well and was, on the whole, able to choose the best passages. In so doing, he was inclined to render the Marxian theses more dogmatic, and our problem is no exception.

3.2 The problem of the transformation of values into prices in the manuscripts

Marx came to discuss the problem of transformation several times in the manuscripts of 1861–63 in MECW 31 (MEGA² II.3), under the title 'Zur Kritik der politischen Ökonomie'. It is the manuscript from which Kautsky first chose passages concerned with the history of the theory, which he then published as *Theorien über den Mehrwert*.

We find a very simple example (MECW 31, p. 263). There are three capitals of equal magnitudes in three sectors α, β, γ; total capital in each of the three cases is denoted by C and the surplus value by M.[2]

> Since the rate of profit is determined by the ratio of surplus value to capital advanced, and as on our assumption this is the same in α, β, γ, etc., then if C is the capital advanced, the various rates of profit will be $= \frac{3M}{C}, \frac{2M}{C}, \frac{M}{C}$. Competition of capitals can therefore only equalise the rates of profit, for instance in our example, by making the rates of profit equal to $= \frac{2M}{C}, \frac{2M}{C}, \frac{2M}{C}$ in the spheres α, β, γ. α would sell his commodity at $1M$ less and C at $1M$ more than its value. The average price in sphere α would be below, and in sphere γ would be above, the value of the commodities α and γ.
> As the example of β shows, it *can* in fact happen that the average price and the value of a commodity coincide. This occurs when the surplus value created in sphere β itself equals the average profit; in other words when the relationship of the various components of the capital in sphere β is the same as that which exists when the sum total of capitals, the capital of the capitalist class, is regarded as a single *magnitude* on which the whole of surplus value [is] calculated, irrespective of the particular sphere of the total capital within which it has been created.

In this manner Marx summarised his idea of the transformation of values into prices clearly and visually, in the context of his critique of Rodbertus. A little later he added: 'The capitalists, like hostile brothers, divide among themselves the loot of other people's labour which they have appropriated so that on an average one receives [*aneignet*] the same amount of unpaid labour as another' (MECW 31, p. 264).
Instead of asking how the capital goods advanced, be they constant or variable capital, are valued, Marx moves on to the question of how rents also arise from the division of total surplus value. He concludes that the property of natural objects is 'not a source from which flows value', but it is a source of income insofar as receivers of rent appropriate a 'part of the unpaid labour squeezed out by the capitalist' (p. 276). The problem of transformation occurs again in the same manuscript. The concept of the average rate of profit is introduced as an average of quite different individual rates of

[1] According to Marx, it is characteristic for the petit bourgeois to waver between one side and the other; the petit bourgeois is a living contradiction (On P. J. Proudhon, MECW 20, p. 33).

[2] I have changed the notation here. Marx denotes the sectors by A, B, C, inviting a confusion of sector C with capital C. M is chosen for surplus value to be consistent with quotes from the German original and with our notation in Section 2.3.

profit (MECW 33, p. 95). He then asserts that the transformation of values into prices is to be determined in detail in a theory of competition, which is to be worked out in a subsequent step, but the step is not taken in this manuscript (p. 101). The famous letter Marx sent on 2 August 1862 to Engels to inform him about his advances in his analysis does not lead on in this regard either (MECW 41, pp. 394–8).

The step was taken only in the later manuscript for the third volume of *Das Kapital* in the chapter 'Bildung einer allgemeinen Profitrate (Durchschnittsprofit) und Verwandlung der Waarenwerthe in Productionspreise' (MEGA^2 II, 4.2, p. 230; 'formation of a general rate of profit and transformation of values of commodities into prices of production', my translation). Marx examines the organic compositions of different capitals, where the organic composition of capital is said to depend on the technological relationship between the composing parts of capital and the prices of the means of production. Instead of proceeding to solve this difficulty, he adds another: that of determining how much of the value of fixed capital is transferred to the product. This amortisation is called by the French expression *Déchet*. It is a matter of course that the amortisation would have to be calculated simultaneously with the prices of production, but Marx drops this consideration and assumes that all capital can be treated as circulating, therefore as capital, the whole value of which is transferred to the product in the period under consideration. He then calculates, using different numerical examples, how five commodities, produced in five different sectors by means of capitals of different composition, produce a certain surplus value, assuming equal rates of surplus value in each sector, and how this surplus value is redistributed in proportion to the value of the capital advanced, so that the sum of prices of production equals the sum of values. This whole exposition then entered the manuscript of *Das Kapital*, as edited by Engels (compare MEGA^2 II, 4.2, p. 235 and MECW 37, pp. 158–9).

Here follows a crucial passage, the beginning of which I paraphrase: Marx observes that the deviations of the prices from values also hold for the commodities that are used as capital goods, and this concerns the capital goods entering constant capital as well as indirectly those that constitute the means of subsistence of workers, the values of which constitute the variable part of capital. Marx continues, as edited by Engels:

> However, this always resolves itself to one commodity receiving too little of the surplus value while another receives too much, so that the deviations from the value which are embodied in the prices of production compensate one another. (MECW 37, p. 160)

Here follows the sentence that is central for all the quarrels about the transformation problem:

> Under capitalist production, the general law acts as the prevailing tendency only in a very complicated and approximate manner, as a never ascertainable average of ceaseless fluctuations. (MECW 37, p. 160)

The original formulation in the manuscript by Marx had been (which Engels has changed):

> Es ist überhaupt bei dieser ganzen bürgerlichen Scheisse immer nur in a very complicated, and very rough way, daß sich das allgemeine Gesetz als die beherrschende Tendenz durchsetzt. (MEGA^2 II, 4.2, p. 237)
> [My translation] It is generally with this whole bourgeois shit always only in a very complicated, and very rough, way that the general law prevails as the dominating tendency.

The vulgar outburst detracts attention from the fact that Engels changed the Marxian phrase not only to eliminate what in English is a four-letter word and to provide

German counterparts for English expressions, mixed in with the German phrase, but that he modified (he certainly thought he clarified) the meaning. Marx, too, had spoken of averages, but in these chapters only in the context of an average rate of profit where the term 'average' is according to the elementary definition. Sums are formed by adding capitals or surplus values which are then divided by the number of the terms added, so that averages result that refer to a given period. Engels, by contrast, spoke of the 'average of ceaseless fluctuations', and he thus referred to other forms of averaging in economics over time: natural prices resulted as averages over the oscillations of market prices, and in this manner an average of business cycles results in modern analysis in growth along a trend. The deviations of market prices from natural prices are random. Values and prices of production, by contrast, are formed by deterministic calculation. One acute reader, Böhm-Bawerk, had observed the problem in Engels's reformulation, but, not knowing the manuscript, attributed it to Marx: 'Here Marx confounds two very different things: an *average of fluctutations*, and an *average between permanently and fundamentally unequal quantities*' (Böhm-Bawerk 1949, p. 37).

Is it possible to relate the difficulty, as it was recognised by Marx, to calculate in detail how deviations of prices from values cancelled, to a random process? The vulgar expression was introduced because Marx realised that his analysis was not precise, but he did not know how to render it rigorous and he seems not to have been eager to try a more complex analysis and be it only by means of calculating an example of a transformation involving both inputs and outputs. He did not base himself on a random process. He reconsidered how deviations of prices from values might compensate each other only once more, when he considered the deviation of prices from values for the cost price, therefore for some of the capital goods entering the production of a commodity, evaluated in prices, but no new insight of importance for our question was found (MEGA² II, 4.2, p. 241 and MECW 37, p. 164). The formation of averages is rendered a little more precise later. Marx observes that the general rate of profit is determined by the organic compositions in the several spheres of production and by the distribution of the social capital in total amongst these (MEGA² II, 4.2, p. 239). The cost price is smaller than the price of production as a matter of course, but Marx does not seem to be completely sure whether it is also always smaller than the value of a commodity (MEGA² II, 4.2, p. 241 and MECW 37, p. 164).

3.3 Marx as a critical Hegelian and Engels's prize competition

Engels's attitude as the editor of the Marxian manuscripts was that Marx had provided the definitive treatment of the problem of transformation. In the preface to the first edition of the second volume of *Das Kapital* he challenged the academic community to solve the riddle of how the theory of value and exploitation could remain true, once one had recognised that the products of capitals of unequal organic composition but of equal value could not be sold at equal prices. This has been called Engels's prize competion. When Engels published the third volume, he reviewed the solutions found up to then, and a modern history of this competition has been written by Howard and King (1987). Howard and King listed the passages where Marx had addressed the problem in his published writings prior to his death. None of them provide the essential hint; the most important and the most rich in implications is the following citation from the first volume of *Das Kapital* (the passage underwent no essential change between the first and the last editions):

Every one knows that a cotton spinner, who, reckoning the percentage on the whole of his applied capital, employs much constant and little variable capital, does not, on account of this, pocket less profit or surplus value than a baker, who relatively sets in motion much variable and little constant capital. For the solution of this apparent contradiction, many intermediate terms are as yet wanted, as from the standpoint of elementary algebra many intermediate terms are wanted to understand that 0/0 may represent an actual magnitude. Classical economy, although not formulating the law, holds instinctively to it, because it is a necessary consequence of the general law of value. It tries to rescue the law from collision with contradictory phenomena by a violent abstraction. It will be seen later how the school of Ricardo has come to grief over this stumbling-block. (MEGA^2 II, 9, pp. 264–5; first English edition)

Marx starts from the assumption that the workers get a uniform wage and have the same working day, so that the value of labour power also is uniform and with it the rate of surplus value. Hence, surplus value is proportional to labour expended and independent of the mass of constant capital used. The long way followed by Marx to get to his theory of prices of production is curiously compared with the way leading from elementary algebra to the idea that 0 / 0 could represent a 'real magnitude'. What Marx means has become comprehensible, since we have been introduced to his mathematical writings at least in provisional editions and since we have been able to compare them with modern mathematics on the one hand and with Hegel's philosophy of mathematics in his *Logics* on the other hand. Hence, if we want to comprehend the one and only significant hint Marx gave to explain the evident incompatibility between the uniform rate of surplus value and the uniform rate of profit in a theory of labour values, we must look at his understanding of mathematics, and this requires a comparison with Hegel. It turns out that both the mature Hegel and the mature Marx regarded the foundation of the infinitesimal calculus as a testing ground for their theories of dialectical logics.

We begin with the historical point of departure. The renowned historian of mathematics Dirk J. Struik describes the difficulties left behind by Newton's version of the infinitesimal calculus in 'A Concise History of Mathematics' as follows:

Newton had tried to make his position clear by the theory of 'prime and ultimate ratios', which he introduced in the *Principia* and which involved the conception of limits, but in such a way that it was very hard to understand it. . . . The misunderstandings were not removed until the modern limit concept was established. (Struik, 1967, p. 111)

The discovery by Leibniz is characterised similarly:

Leibniz' explanation of the foundation of the calculus suffered from the same vagueness as Newton's. Sometimes his *dx, dy* were finite quantities, sometimes less than any assignable quantity and yet not zero. In the absence of rigorous definitions he presented analogies. (p. 114)

The literature on this historical problem is voluminous. Struik comments on it briefly. He observes that a thorough change was introduced by Cauchy and quotes Hilbert (p. 159), who ascribes the clarification of the notion of limit to Weierstrass.

The Marxian mathematical manuscripts are edited in English by Yanovskaya (1983). They were earlier accessible in Germany in the form of the book edited by Wolfgang Endemann (1974); moreover, there is the book with more extensive comments by Alain Alcouffe (1985) in French. Alcouffe demonstrates that Marx's mathematical manuscripts were composed partly to follow, partly to overcome Hegel's *Logics* (the following is, apart from quotes from the English edition, based on my reading of Hegel's *Logik* in Hegel, 1986).

Hegel writes (2010, p. 205):

> The *mathematical infinite* is of interest because of the expansion and the significant new results which its introduction into mathematics has produced in it, but also because of the oddity that this science has to date still been unable to justify its use conceptually (. . .) Ultimately, the justifications are made to rest on the *correctness* of *results . . . as demonstrated on other grounds*, not on the clarity of the subject matter.

The 'peculiar interest' (p. 215) of the infinite shows for Hegel in particular insofar as in the 'so-called *infinitesimal differences*' the quantitative significance is 'entirely lost'. '*dx*, *dy*, are no longer quanta . . . but have . . . *the meaning of mere moments*'. They are neither finite, nor '*nothing*, not a null void of determination' (p. 215). In the ratio of differentials, the 'quantum is truely made complete as a qualitative existence; it is posited as actually infinite' (p. 215). Hegel here sees an 'intermediate state', which the mathematicians do not perceive, between 'being and nothing'. For him, the unity of being and nothingness is not a state, but it is 'the becoming, is alone the truth' (p. 216). The infinitesimals therefore are for him only in a transition. 'Becoming' was between 'Being' and 'Nothingness', as in Greek philosophy. He believes that the idea of differential calculus cannot be determined more correctly, than 'as Newton stated it' (p. 217). 'It may be objected that vanishing magnitudes do not have a *final ratio*, because any ratio before the magnitudes vanish cannot be the last, and once vanished, there is no ratio any more.' But the relationship of vanishing magnitudes is to be understood as occuring, 'not *before* or *after* they vanish, but the ratio *with which* they vanish (*quacum evanescunt*)' (p. 217). The 'final ratios' in differentiating are limits, to which the magnitudes diminishing without limit are closer than any given finite difference. Small magnitudes (we should say, of second or higher order of smallness) were often left aside in Newtonian and Leibnizian calculus, and this was justified because 'correct' derivatives could thus be obtained. But Hegel criticises the comparison of the omission of small magnitudes with empirical approximations, of which, unfortunately, Wolff was guilty. He has objections to Euler and also Laplace, because as soon as 'the terms of a ratio are quantitatively a zero' (p. 221), there is in Laplace (according to Hegel) no conceptual understanding of this ratio. Hence again the suggestion to interpret the differentials as qualities. In consequence, the intrusion of the qualitative into mathematics is affirmed with other examples, beginning with the natural numbers; they show in the beginning only an external quantitative development, but qualitative moments result, for instance, from the musical harmonies.

Marx begins his explanation of the operation of differentiation with a linear function, where the finite differences

$$\frac{\Delta y}{\Delta x} = a$$

in

$$\frac{dy}{dx} = a$$

seem to be cancelled (Yanovskaya, 1983, p. 5). The difficulty in differentiating is for him in the fact that in general, if the ratio of the differences and the ratio of the differentials do not coincide as in the linear case, the transition from the former to the latter, which he calls a '*negation of the negation*' (p. 3; emphasis in original) is not so simple. Obviously, the mathematical development conceals a silent struggle with Hegel.

Marx then distinguishes the 'developed algebraic expression' of the function $y = f(x)$, which he calls the original function, from the modification obtained by 'differentiaton', which he calls the 'preliminary derived function' (p. 6). This denomination confuses the modern reader; the provisionally derived function is the function formed by means of finite differences prior to the transition to the limit. We now have

$$\frac{\Delta y}{\Delta x} \neq \frac{dy}{dx}$$

and as long as the differences in the ratio on the left remain finite, the inequality remains. How can difference be so transformed as to determine $\frac{dy}{dx}$? Only via the transition to the limit—he here speaks of the 'process of differentiation'—there arises the 'derived function'.

Next it is proved: if the function is not linear,

$$\frac{\Delta y}{\Delta x} \neq \frac{dy}{dx}$$

and (now) there must, 'with no subterfuge about merely approaching infinitely [closely]' (p. 7), Δy and Δx become zero, so that the provisional derivation leads to $0 / 0$. But the 'transcendental . . . mistake which appears only on the left-hand side has perhaps already lost its terror' (p. 8). Marx can say this because the derivative is obtained on the right-hand side. Similarly, it is stated on p. 77, that $f'(x)$ is derived 'by means of actual differentiation [i.e., formation of differences] and subsequent cancellation alone'. The formation of the limit therefore is at least represented by a 'process of differentiation', even if that is not defined in the precise modern fashion, and this representation is interpreted by Marx as cancellation (p. 37). He fails to observe that the limit does not always exist and that not each function is differentiable everywhere.

The problem obviously is to interpret $\frac{dy}{dx}$, the expression on the right. For that one obtains reasonable derivatives is shown by means of differentiating polynomials and an exponental function (no more general functions are considered). But what is the apparent ratio of vanishing terms? Let us give a known example corresponding to the calculations executed by Marx: let $y = f(x) = x^2$ be given; the function is to be derived in x'. One forms the provisional derivative and transforms it as Marx does it:

$$\frac{x^2 - (x')^2}{x - x'} = \frac{(x + x')(x - x')}{x - x'} = x + x'$$

If one now allows x to become x', one obtains the harmless $2x'$ on the right-hand side, the derivative (in the modern sense) of $y = x^2$, which every student knows; on the left-hand side one has $0 / 0$, however.

Is it now possible to satisfy oneself by considering $\frac{dy}{dx}$ only as a symbol on one side for the derivation to be performed on the other?

Marx suggested the idea, but matters are generally not quite as simple as in this special case. Marx treats differentiation in the simple cases like the prescription of a series of algebraic transformations designed to eliminate the vanishing nominator. But this cannot always be done explicitly. Marx asks in his drafts, as Hegel had done earlier, how a product of two functions is to be derived, where the rule for differentiation of two functions $y(x)$ and $z(x)$ yields

$$\frac{d(yz)}{dx} = y\frac{dz}{dx} + z\frac{dy}{dx}$$

Since the right-hand side now does not define the left-hand side, it becomes necessary to clarify what dy, dz, dx mean, and even more so, if the differential $d(yz) = ydz + zdy$ is formed. It seems to me that Marx takes his distance from Hegel by treating these expressions as symbols or operators and not as magnitudes that are infinitely small or qualities. He writes: ‘The equation is thus only a symbolic indication of the operations to be performed’ (p. 49). They are ‘operational symbols’, and only by executing the derivations does the differential become what he now calls ‘real value’:

$$\frac{d(yz)}{dx} = yz' + zy'$$

Hence he renounces to the infinitely small, as Hegel had done, but in a different way. Hegel would treat the differentials as qualities. In this, Marx does not follow him; he treats the differentials as operators. There is manifestly a selectively critical relationship with Hegel. The important notion of ‘negation of the negation’ is kept, whilst other expressions of dialectical logics are not used. Marx thus felt challenged by Hegel’s mathematics, as Alcouffe (1985) has shown. He wanted to demonstrate the superiority of the materialist version of the dialectical method. In the area of mathematics, materialist could only mean ‘rational’. He did not accept Hegel’s proposal to interpret the vanishing differentials as qualities; this must have seemed insufficient to him as a rational explanation.

As long as the Marxian mathematical manuscripts are not edited completely, our judgements about his approach to mathematics must remain speculative to some extent. My impression is that he seems to have kept his distance from the axiomatic method. Mathematics in Marx do not appear like a construct, resulting from different conceptions defined by axioms, but it is a coherent realm of objects, to be analysed by experience and research—hence the ‘intermediate terms’, which are there to connect algebra and the infinitesimal calculus. Ways for their representation are discovered and operations introduced, by which new mathematical objects are engendered, like the derivatives from functions, as if the functions were to be worked on. The lack of an axiomatic build-up does not mean that Marx did not formulate his hypotheses also in the mathematical realm to analyse their consequences. But as his realism always made him look for plausible assumptions, close to reality, in economics, he examined in his mathematics usable functions, capable of orderly differentiation. He was not after ‘pathological’ constructs such as a non-Euclidean geometry or a function, which would be continuous, but, in the origin, not differentiable, such as $y = x\sin(1 / x)$.

His limited ‘rational’ use of dialectics in mathematics is remarkable in view of the fact that the mature—indeed, the old—Marx proceeded in a different way in *Das Kapital*. Here, dialectics is used to comprehend the material relations in economics and to represent what Marx saw as ‘irrational’ in capitalism. He laid the basis for his theory of the forms of value in the first volume, which starts with the double character of the commodity as a unity of value in exchange and of value in use, and it leads to the discovery of the double character of labour. Down to the fourth editon of the first volume of *Das Kapital* we read that the 'whole mystery' is in the 'elementary form of value' 'x commodity $A = y$ commodity B'. 'The linen expresses its value in the coat; the coat serves as the material in which the value is expressed.' (*MEGA* II, 9, p. 41) As Endemann (1974) has pointed out in his discussion of the Marxian mathematical manuscripts, this equation, from the point of view of physics, is simply wrong concerning the dimensions; it would have to be replaced by two equations—more exactly, by an equation and an inequality. A body, for instance, has extension and mass. A physicist can say of water in a container that, when it is transformed into ice, it fills a greater volume than it does in the form of a fluid but has the same mass. To proceed in the same way with commodities, let $v(x)$ denote the value of x; then one could write:

$$v\left(x\,\text{commodity}\,A\right) = v(y\,\text{commodity}\,B)$$

and, meaning the values in use,

$$x\,\text{commodity}\,A \neq y\,\text{commodity}\,B$$

For Marx, however, 'use-value becomes the form of manifestation, the phenomenal form of its opposite, value' (MEGA II.9, p. 48); in this manner, the dimensions are connected, but not confused. As we can see, Marx was fairly quick to eliminate the 'terror' in the 'transcendental mistake' of 0 / 0 by his representation of his derivative. Here, however, the horror is increased until the value of the commodity appears as *Verdopplung* (redoubling) in the form of money alongside the commodity itself.

Translators have not always had the courage to render this provocative Marxian formula literally. Marx says of the process of exchange in chapter 3 of *Das Kapital* (we here use the third edition of volume I and the English translation in the MEGA):

> Er producirt die Verdopplung der Waare in Waare und Geld, einem äußeren Gegensatz, worin sie ihren immanenten Gegensatz von Gebrauchswerth und Werth darstellen. (MEGA[2] II.8, p. 128)

The translation renders this as

> The process then differentiates them into commodities and money, and thus produces an external opposition corresponding to the internal oposition inherent in them, as being at once use-values and values. (MEGA[2] II.9, p. 90)

But a more literal translation would read:

> It [the exchange process] produces the redoubling of the commodity into commodity and money, an external opposition, within which they represent their immanent opposition of value in use and value.

The redoubling of the commoditiy into commodity and money is also mentioned in chapter 2 (MEGA[2] II.8, p. 114), but the same emasculated translation is used ('differentiation of commodities', p. 75). The translation suggests that some commodities remain commodities, others become money. The original text affirms that a new object is created, alongside the one that is already there. This seems paradoxical as long as money is commodity money, but commodity money will entail the creation of notes. There will be capital, and this, too, will redouble: there is capital in real form, in monetary form, eventually capital as shares. Once there are banks, there will be funds and so on.

Marx develops a kind of modal logic according to which these forms of value arise of necessity (see MEGA[2] II.9, p. 75: 'money is a crystal formed of necessity in the course of exchanges'). As money seemed to him latent in the relative form of value (see Section 3.1), so paper money was latent in commodity money, and dialectics would reproduce the real redoubling in a theoretical language.

A similar logic operates in the realm of surplus value creation. Our concern was with profits, which usually are understood as a sum of money, of which it was to be shown, however, that it derived from a redistribution of surplus value. In the equation $P = M$, there is value on the left-hand side as money and on the right-hand side as labour. The quantitative equality is interesting because of the difference of forms: The money earned by capitalists is really exploited labour. Total profits are then divided into rent, profit of the entrepreneur and interest. Surplus value, as it were, multiplies: not in quantity, but in its phenomenal forms (*Erscheinungsformen*)—a central category of Marxian logics. There are qualitative consequences: it becomes a value in use of capital to generate interest; capital is traded and the rate of interest becomes its price. Aristotle had criticised that money could, as a medium of exchange, not bear money. The scholastics followed him and called usury unnatural because of this conundrum, which they interpreted as a logical problem. Marx similarly formulated that 'interest, signifying the price of capital' was 'from the outset quite an irrational expression' (MECW 37, p. 352). How should 'a sum of value' (the capital) 'have a price besides its own price, besides the price expressed in its own money form?' (p. 353). Similarly, Thomas of Aquinas thought that the price of a loan of money consisted in the re-payment of the money, since money was a medium of exchange and could as such only be given or taken for an equivalent in value, whilst later scholastics softened the critique of usury and interpreted a contract involving credit as an inter-temporal exchange. Böhm-Bawerk followed them, eliminating the irrational aspect of the problem of dimensions: the money I shall receive tomorrow is, by the temporal dimension, different from the money I lend today in exchange for the promise of interest.

Marx uses a different dialectical description of the connection, in which the dimensions are mixed up, to capture what seemed irrational in the appearances of capitalism. Mathematics by contrast is essentially rational, so he was concerned in his mathematical writings to take away the 'horror' from the 'transcendental accidents'. We have gone into this difficult territory of Marxian dialectics to explain these distinct and contrasting uses Marx makes of the dialectical method: it is used in an attempt to make calculus more rational and operational and to reproduce what Marx sees as the irrational features of capitalism. To attempt to construct a logic of what is regarded as only partially rational is an extraordinary undertaking; many of his followers, including some of his translators, did not dare represent this enigmatic aspect of his work in their interpretations.[3] But the issue must be faced, if one wants to grasp the meaning of $P = M$ for what Böhm-Bawerk aptly called the 'closure' of the Marxian system.

[3] Kliman (2014), who believes that all 'Sraffian' interpretations of Marx are whiggish and distortions, does not face this issue either.

As Hegel derived the totality of the world from its logical and mathematical foundations down to its historical evolution, Marx tried to derive the whole of capitalism from his fundamental hypothesis, sticking to his main principle, the theory of value. The representation was to follow not the idea but the real givens, and it was to make transparent the laws operating in the core of the real world. The Marxian mathematical manuscripts confirm that this inversion of Hegelian thought was also to lead down to the depths of the founding logical and mathematical principles, and that operations, acting on well-defined objects, were to replace a speculative development. Where Hegel wanted to explain the difficulties of differential calculus by the denoting the differentials as qualities (which might have been justified according to Alcouffe, 1985, by means of modern so-called non-standard analysis), Marx understood differentiation as an operation. The distinction between differentiable and non-differentiable functions is lacking, but he came closer to the modern standard interpretation of the infinitesimal calculus.

How many 'intermediate terms' Marx introduced in *Das Kapital* to get from his theory of value to the formation of prices and to the conditions of competition becomes clear if we consider that the whole second volume with the process of circulation of capital was inserted between the derivations of the theory of value and the theory of prices. The break in the transformation of values into prices, which occurred because capital goods had to be evaluated in prices as well, was bridged by means of a four-letter word: an expression of the disappointment that his conception of the representation of average profit as a redistributed surplus value could not be presented with full rigour.

Engels went one step further. He rendered the formation of averages according to algebraic rules as a formation of averages in random processes. Our solution is not rigorous in the sense of a complete solution of the Marxian thesis but a proof that the transformation is possible in a smaller domain and as Engels presented it: with single production, omitting joint production and rents, and assuming a random character of input coefficients and the labour vectors. The equality of the sum of profits and the sum of surplus values will result in a process, in which input coefficients of single-product systems are random and in which the deviations of labour vectors from Marx vectors will vanish on average.

Hence our contribution does not represent a complete solution of the transformation problem—which is not possible—but it is, I hope, a respectable contribution to Engels's prize competition.

4. $P = M$ and the falling rate of profit

4.1 Limits for the rates of exploitation and of profit

We now get to the applications of $P = M$ in Marx. We first have to clarify that the rate of surplus value, not the rate of profit, is the primary relationship, as Marx insisted and as Robinson (1965) saw. A comparison with Sraffa illustrates this point. He referred to Marx in his book of 1960, when he introduced the maximum rate of profit:

> The notion of a maximum rate of profits corresponding to a zero wage has been suggested by Marx, directly through an incidental allusion to the posssibility of a fall in the rate of profits 'even if the workers could live on air. (Sraffa, 1960, p. 94)

Sraffa further refers to the Marxian critique of the Smithian conception of representing the price of each commodity as a sum of wages, profits and rents, which is possible in a finite number of steps only, if there is a commodity, which can be produced without other commodities as means of production. If there is no basic commodity in the economy, there is no fixed limit for the rise of the rate of profit.

'If the workers could live on air'—Sraffa obviously thinks of the maximum rate of profit arising in his system, if the wage of the workers is set equal to zero. The rate of profit then is equal to the rate of net product, divided by the cost of capital, or equal to what in neoclassical theory is called the productivity of capital. Marx, however, had a somewhat different maximum rate in mind in the passage, to which Sraffa refers:

> The surplus value, however, as a total, is determined first by its rate, and second by the mass of labour simultaneously employed at this rate, or, what amounts to the same, by the magnitude of the variable capital. One of these factors, the rate of surplus value, rises, and the other, the number of labourers, falls (relatively or absolutely). Inasmuch as the development of the productive power reduces the paid portion of employed labour, it raises the surplus value, because it raises its rate; but inasmuch as it reduces the total mass of labour employed by a given capital, it reduces the factor of the number by which the rate of surplus value is multiplied to obtain its mass. Two labourers, each working 12 hours daily, cannot produce the same mass of surplus value as 24 who work only 2 hours, even if they could live on air and hence did not have to work for themselves at all. (MECW 37, p. 246)

Marx observes that progress reduces the number of workers, so that surplus value is reduced, given the rate of exploitation. But surplus value can be increased by raising the rate of exploitation as in this example, where the number of workers is reduced, but they work more individually (in total less than in the original situation). Marx's concern is to determine the limit for the increase of the rate of surplus value or of exploitation. The idea that workers might live on air, at a limit unattainable in reality, is not concerned directly with the rate of profit in Marx but immediately with surplus value. Workers who are able to live on air imply for him only indirectly a finite maximum rate of profit; directly they mean an infinite rate of surplus value!

The difference of the approaches is visible even better in *Value, Price and Profit*, a posthumously published essay read by Marx on 20 and 27 June 1865 in the provisional Central Committee of the International Workingmen's Association, polemicising with John Weston. Weston, an Owenist, was for co-operatives and against trade unions, which were quite active organising strikes in 1865 because of the boom in Europe. Weston thought this activism would only raise money wages and that the consequent increase of demand would raise prices. Marx opposed this by expounding the fundamentals of the theory of value for the first time in public; he denied the possibility of shifting the increases of wages onto prices. In the course of this argument, he also discussed a 'maximum rate of profit':

> But as to *profits*, there exists no law which determines their *minimum*. We cannot say which is the ultimate limit of their decrease. And why can we not fix that limit? Because, although we can fix the *minimum* of wages, we cannot fix their *maximum*. We can only say that, the limits of the working day being given, the *maximum of profit* corresponds to the *physical minimum of wages*; and that wages being given, *the maximum of profit* corresponds to such a prolongation of the working day, as is compatible with the physical forces of the labourer. The maximum of profit is therefore limited by the physical minimum of wages and the physical maximum of the working day. It is evident that between the two limits of this *maximum rate of profit*, an immense scale of variations is possible. The fixation of its actual degree is only settled by the continuous struggle between capital and labour; the capitalist constantly tending to reduce wages to their physical minimum, and to extend the working day to its physical maximum, while the workingman constantly presses in the opposite direction. (MEGA^2 I. 20, p. 183)

This consideration is hardly ever reproduced in analytical reconstructions of the Marxian theory, although it is quite simple to provide one. Let input matrix **A** be divided into a matrix **C** of the constant capitals used in each industry; a matrix $\mathbf{X} = \mathbf{lx}$, which represents the necessary wage (**x** is the row vector of the wage goods physically required per unit of labour); and in a matrix $\mathbf{Y} = \mathbf{ly}$ of a surplus wage, which results from trade union activity and is subject to oscillations (**y** is the row vector of the commodities contained in the surplus wage). The maximum rate of profit corresponding to the minimum wage according to Marx, given the working day, then is the maximum rate of profit in the sense of Sraffa, if in the expression

$$\mathbf{A} = \mathbf{C} + \mathbf{X} + \mathbf{Y}$$

one puts $\mathbf{Y} = 0$. But how is the prolongation of the working day to be represented? This is the lacuna of the literature. We assume that **C** represents the means of production (constant capital represented by use values), which are used during a working day, to produce one unit of gross output in each industry. The labour vector **l** indicates the number of workers, which are needed, given a certain length of the labour day of, say, eight hours. If the labour time is increased from 8 hours to 16 hours each day, without a change of the daily wage of the individual worker and without a change of the intensity of his labour, only half as many workers will be needed and each component of the labour vector will be halved. Given these assumptions, we see that a change of the daily labouring time by the factor α ($\alpha > 1$, if labour time is increased, $\alpha < 1$ if it is diminished) leads to a change of the input matrix

$$\mathbf{A} = \mathbf{C} + (1/\alpha)\mathbf{lx} + (1/\alpha)\mathbf{ly} = \mathbf{C} + (1/\alpha)(\mathbf{X} + \mathbf{Y})$$

The maximum rate of profit in the sense of Marx then is the one in the sense of Sraffa, which results from that of **A**, where α is the maximum amount of labour time to be performed, given the wage.

Marx had put it more simply in the *Grundrisse*: 'In the 'EXISTING RELATIONS BETWEEN WAGES and PROFITS the rate of profit is at its maximum and that of wages at its minimum' (MECW 28, 'Grundrisse', p. 453). In *Das Kapital* volume I, Marx adds the complication that labour may be intensified to the extent that necessary wages cannot be kept constant, but must be increased. He repeatedly observes that the prolongation of labour time leads to a better use of fixed capital. This could be represented in a similar manner as here the constant capital. It would only be necessary to decide between different possible representations of fixed capital: as a joint product as in Sraffa (which is only alluded to in Marx, see Sraffa, 1960, p. 94) or by means of a linear rule of depreciation, which Marx uses frequently, because it is consistent with the labour theory of value approach and is often used in practice. Yet other theories of depreciation could be used.

The point of the exercise is to remind us of the fact that in Marx, because of his departure from the labour theory of value with its ramifications concerning the working day, the rate of surplus value is the primary macroeconomic phenomenon and the rate of profit is derived. The rate of surplus value looks like a simple concept, but for Marx it is subject to complicated and contrasting influences because it is influenced by changes in the number of workers (which in turn depends on different forms of technical progress), the length of the working day, the intensity of work and the composition of the wage (which is paid in different forms as daily wage, hourly wage etc. and consists of necessary wage and possibly a surplus wage—immiseration really means that the latter is negative). We have to keep this in mind when we consider how Marx uses his aggregates, in particular the mass of surplus value, in the consideration of the theory of accumulation with its central law of the tendency of the rate of profit to fall.

4.2 From the rate of surplus value to the fall of the rate of profit: evidence of the manuscripts

The theory of value was to serve the theory of accumulation. There are no fewer than four manuscripts, written by Marx after the publication of volume I, on the theme of the transformation of surplus value into profit and of the rate of surplus value into the rate of profit. The editors of the MEGA surmise that they were mostly written in 1868. They were used by Engels in part for the third volume of *Das Kapital* and complement the main manuscript of the third volume, which had been written three years earlier. They confirm the importance of $P = M$, since Marx measures the rate of surplus value and the organic composition in value terms in his analysis of the production of relative surplus value; the results would not carry over to the analysis of of the rate of profit, if $P = M$ did not hold, as we saw in our critique of the new solution.

Marx quoted a phrase by Malthus in the *Grundrisse*, which he used later repeatedly and also in these manuscripts: ' The capitalist expects an equal profit on all parts of the capital which he advances' (quoted in MECW 29, 'Grundrisse', p. 203). In the manuscript of the third volume in MEGA^2 II 4.2, p. 52, we read:

> 'Aus der Verwandlung von . . . , der Rate des Mehrwerths in Profitrate, ist die Verwandlung des Mehrwerths in Profit abzuleiten, nicht umgekehrt. Und in der That ist die Profitrate das, wovon historisch ausgegangen wird. Mehrwerth und Rate des Mehrwerths sind, relativ, das Unsichtbare und das zu erforschende Wesentliche, während Profitrate und daher die Form des Mehrwerths als Profit sich auf der Oberfläche der Erscheinung zeigen.'[4]

In the *Grundrisse,* Marx had formulated:

> Capital setting out from itself as the active subject . . . does in fact appear to be determined by the movement of capital as capital independently of its relation to labour . . . it behaves towards surplus value as posited by and based upon capital; it relates itself as the source of production to itself as the product; as the producing value to itself as the value produced (*MECW* 29, 'Grundrisse', p. 130.)

In all these texts Marx battles with the paradox that constant capital does not create surplus value, although this appears to be the case. Indeed, it must appear to us to be the case because capital 'expects' the same advantage on all its parts.

Marx examines in this perspective times and again in the manuscripts, from the first publication of the first volume of *Das Kapital* to his death, how the rate of surplus value and the rate of profit are related. If one looks only at the arithmetic, one cannot understand how this intelligent mind could return repeatedly to such elementary calculations. The editors of MEGA^2 II, 4.3 observe succinctly on p 600:[5]

[4] My translation: 'The transformation of surplus value into profit is to be derived out of the transformation of the rate of surplus value into the rate of profit, not vice versa. And in fact, it is the rate of profit where one starts from historically. Surplus value and rate of surplus value are relative, invisible and the essential to be investigated, while the rate of profit and hence the surplus value in the form of profit show on the surface of what appears.'

[5] This volume contains the economic manuscripts 1863–68. It has been edited by Carl-Erich Vollgraf aided by Larisa Mis'kevic.

Er lotete zunächst arithmetisch, 'ganz formell' . . . aus, welche Bewegung eines Faktors welche Auswirkung auf die Mehrwert- oder Profitrate hat. Sodann prüfte er, welche ökonomische Relevanz die Entwicklungsrichtung eines Faktors hat—hinter dem Anwachsen des variablen Kapitals könnte beispielsweise die Erhöhung der Beschäftigtenzahl stehen, ebenso könnte eine Erhöhung des Arbeitslohns stattgefunden haben—ob dieser Fall überhaupt möglich ist, oder ob er zu den 'abgeschmackten', auszusortierenden Fällen . . . gehört.[6]

This is in fact the key to these dark, dusty rooms in which Marx moves about, looking for a treasure such as a further law of the kind of the fall of the rate of profit. If the analytical reconstruction is primarily directed at a mathematically consistent representation, it may miss Marx's secret intentions; the point is to guess them and develop the theory.

His manner of working shows clearly in the essay 'Laws of the Rate of Profit', written (according to the editors) between October and December 1867; Engels does not seem to have used it. The laws here are simple, of the kind 'die Profitrate stets kleiner als die Rate des Mehrwerts' ('the rate of profit always smaller than the rate of surplus value'; p 57.) Marx writes:

$$\frac{m}{c+v} < \frac{m}{v}$$

and notes that equality occurs with $c = 0$. He adds: 'Aber dieß letzte, nie erreichbare Grenze' ('but this ultimate limit, never to be reached'). As modern economists, used to constructing models, we would say that the limit is theoretically attainable and would ignore the practice. Marx, contemporary of the historical school, was a realist, methodologically speaking, and thus he excluded $c = 0$.

A second law states: 'Bleibt C unverändert [$C = c + v$], so steigt u. fällt die Profitrate, wie m wächst od. fällt' ('If C remains unchanged, the rate of profit rises and falls as m increases or falls'). Here he is concerned with variants of the movement of the rate of profit, such as an increase of surplus value, the rate of surplus value being unchanged, which presupposes an increase of variable capital and, since C has been assumed to be constant, a diminution of c. But that is only formal. What he is really interested in is the converse case: if c rises and v falls; then would 'eine kleinre Anzahl Arbeiter grössres constantes Kapital, d.h. eine Productionsmittelmasse von wachsendem Werth in Bewegung setzen' ('a smaller number of workers move larger constant capital, that is a mass of means of production of growing value'). So he gets from an almost playful use of simple expressions to the 'Steigen der Productivität der Arbeit' ('rising of the productivity of labour'; p. 58).

The same rate of surplus value can be expressed in different rates of profit in a further 'law'; he again arrives at a simple formula, which he interprets, going from the abstract to the concrete:

Verwohlfeilert sich der Arbeiter, so können also mit demselben v mehr Arbeiter geworben werden u. m wächst doppelt, 1) weil der einzelne mehr Mehrwerth liefert u. 2) weil die Anzahl der exploitirten Arbeiter, die mehr Mehrwerth liefern, gewachsen ist.[7] (p. 61)

6 My translation: 'He fathomed first arithmetically, 'quite formally', which movement of a factor would have which effect on the rate of surplus value or on the rate of profit. Then he checked which economic relevance has the direction of development of a factor—behind the growth of variable capital there could operate the increase of the number of the employed; similarly, an increase of the wage could have taken place—whether the case is possible at all or whether it belongs to the 'insipid', to the cases to be excluded'.

7 My translation 'If the worker gets cheaper, more workers can be employed by means of the same v and m grows doubly, 1) because each furnishes more surplus value and 2) because the number of workers exploited, who deliver more surplus value, has grown.'

Here we find considerations, which play a role in the context of the counter-vailing causes of the tendential fall of the rate of profit: if the rate of surplus value increases, the fall of the rate of profit becomes smaller or is inverted.

At some length he is concerned with another law, which reads as a formula (p. 69):

$$\frac{m}{C}\,(\text{the rate of profit}) : \frac{m}{v}\,(\text{the rate of surplus value}) = \frac{v}{C}$$

he also writes

$$p' = m\,/\,C, \quad m' = m\,/\,v, \text{therefore}\; p' : m' = v : C$$

Elsewhere in the manuscript (p. 70), this is rewritten as

$$p' = m'\frac{v}{C}$$

Nothing but tautologies, the reader will say, but Marx questions the equations in the manuscript for the third volume (MEGA^2 II, 4.2, pp. 73, 24–8). What happens if the real wage changes, or the length of the working day or the intensity of work? Are perhaps the magnitudes on both sides of the equation to be interpreted in different ways according to the application, so that they become inequalities?

Then, after all these deliberations, we find something radically new concerning the foundation of the tendency of the rate of profit to fall (MEGA^2 II, 4.3, p. 109):

> *Es geht eine technologische Verändrung im Arbeitsprozeß vor* [Hervorh. v. KM] ... Die grössre Masse Productionsmittel, die von derselben Arbeitsmasse in Bewegung gesezt wird, drückt *stets* [emphasis added] Steigrung in der Productivkraft der Arbeit aus. Aber in der Agrikultur u. extraktiven Industrie ist es möglich, daß diese künstliche Steigrung in der Productivkraft der Arbeit nur stattfindet, um grössren natürlichen Widerstand zu brechen, also z.B. um dieselbe Productmasse zu erhalten. Die Abnahme im natürlichen Factor der Production soll hier compensirt werden durch Steigrung in der Anwendung von *c*. Die Productivkraft der Arbeit wird nach der einen Seite gesteigert, weil sie nach der andren abnimmt. Von diesem Fall wird hier abstrahirt, weil er später (Grundrente) näher betrachtet wird. Bei der folgenden Untersuchung wird also vorausgesetzt, daß *dieser Fall nicht stattfindet* u. das Wachsen von *c* daher stets Index des Wachsens der absoluten Productivkraft der Arbeit ist.[8]

Note that if it is not a matter of diminishing returns, by means of the word 'stets' ('always'), which I have emphasized in the quote, Marx carefully excludes a growth of c, given v, which does not lead to increased productivity, hence he excludes mis-investment that raises the organic composition of capital merely by adding capacity at a constant level of employment of labourers. The rise of the organic composition of

[8] My translation: '*There occurs a technological change in the work process* [emphasis in original] . . . The larger mass of means of production, which is set in motion by the same amount of labour, *always* [emphasis added] expresses an increase in the productivity of labour. But it is possible in agriculture and in extractive industry that this artificial increase in the productivity of labour occurs only in order to break increased natural resistance, therefore e.g. in order to obtain the same mass of product. The diminution in the natural factor of production shall here be compensated by means of increasing the application of *c*. The productivity of labour is increased in one direction, because it diminishes in the other. We abstract from this case here, because it will be considered more closely later (ground rent). Therefore, in the following enquiry, it is assumed that *this case does not take place* and that hence growth of *c* is always an indicator of the absolute force of production of labour.'

capital is not the investment in an eleventh spade, when there are 10 workers and 10 spades, so that the eleventh spade is superfluous, but the replacement of the spades by equipping the men with plows and horses. But if there are diminishing returns: is this not a return to a Ricardian foundation for the limitation of growth? Is this not the Ricardian argument for the rate of profit to fall, because of the transition to inferior lands or because of some other lack of natural resources? I found a similar argument in Marx in the *Theorien über den Mehrwert*, which was written earlier than the manuscript in question (MECW 26.3 pp. 359–60, MECW 33, p. 290f.); Marx had considered there that perhaps a tendency for the increase of the organic composition of capital might be countered by the use of cheaper raw materials. He distinguished, to find a counter-tendency to the counter-tendency, between organic and inorganic matter. He expected that humans would never have the same control over the former as over the latter, and organic materials would become more expensive because of the 'insipid law of rent'. Even the inorganic materials would eventually become more expensive because of the depletion of the mines. I called this 'Marx's temporary retreat to Ricardo', because here, after all the Marxian polemics against Malthus's and Ricardo's reasoning regarding the falling rate of profit, Marx seems to follow their arguments (Schefold, 1997 [1976], pp. 270–3). The manuscript quoted here (1867–69, see MEGA^2 II, 4.3, p. 610sq., see also the comments on this text by the editors) demonstrates that Marx did not drop these arguments but wanted to extend them in the third volume of *Das Kapital*, undoubtedly in the context of his discussion of differential rent.

Incidentally, it is interesting that Marx establishes a link between the theory of accumulation and changes in the value of money in the same manuscript, which I cannot recall having seen in *Das Kapital*:

> Daß aber speziell in der eigentlichen Industrie, die Productivität der Arbeit einen Anstoß erhält durch den sinkenden Geldwerth, das bloße Anschwellen der Geldpreise, u. die allgemeine internationale Hetzjagd auf die vermehrte Geldmasse, ist ein historisches fact u. speziell nachzuweisen von 1850 bis 1860. Umgekehrt, wenn der Geldwerth steigt u. daher die Preise fallen.[9] (p. 119)

Marx therefore observes that an inflation increases and a deflation dampens demand.

Having considered so many different possibilities, Marx finally asks himself whether their foundation might be questionable after all and reaffirms his conviction (p. 139):

> Wie immer—in Folge der Ausgleichung der Profitrate zur *allgemeinen od. average Profitrate*, durch die Verwandlung der Preise in Produktionspreise etc die Preise der Waaren von ihren Werthen abweichen mögen u. sich daher die Masse des Mehrwerths od. des Profits unter die verschiedenen Geschäftszweige vertheilen mag, dieß kann nur die *Erscheinungsform* ändern, worin jene Einflüsse sich geltend machen, aber die Gesetze, denen sie folgen, in keiner Weise alteriren od. gar aufheben.[10]

[9] My translation: 'But that, especially in industry proper, the productivity of labour gets a stimulus by the sinking of the value of money, the mere blowing-up of money prices, and by th general international hunt for the increased mass of money is a historical fact and specifically to be verified from 1850 to 1860. Vice versa, if the value of money increases and hence prices fall.'

[10] My translation: 'However—in consequence of the equalisation of the rate of profit to the general or average rate of profit—the prices of commodities may deviate from their values via the transformation of prices into prices of production etc. and however hence the mass of surplus value or of profit may be distributed amongst several branches of business: this can only change the *phenomenal form* in which those influences make themselves felt, but it can alter the laws, which the influences follow, in no way, let alone suspend them.'

5. Conclusion

There have been endless discussions regarding whether Marxian 'laws' were right. Marxians and their critics will continue to disagree, whilst the historical situation will change and suggest different judgements. But the mistake, made by Marx, in transforming values into prices, is probably really not big enough to be decisive for this discussion. On average—if the appropriate definition is chosen—he was right to put $P = M$. Surprisingly, Sraffa himself once considered such a conclusion. He wrote around 1940:[11]

The propositions of M. are based on the assumption that the comp. of *any large aggr.* of commodities (wages, profits, const cap.) *consists of a random selection* [emphasis added], so that the ratio between their aggr. (rate of s.v., rate of p.) is approx. the same whether measured at 'values' or at the p. of prod. corresp. to any rate of s.v.
This is obviously true, and one would leave it at that, if it were not for the tiresome objector [Sraffa?!], who relies on hypothetical deviations: suppose, he says, that . . . the caps switched part of their consumption from comms of lower to higher org. comp., while the workers switched to the same extent theirs from higher to lower, the aggr. price of each remaining unchanged. . . . It is clear that M's pros are not intended to deal with such deviations.
They are based on the assumption (justified in general) that the aggregates *are* of some average composition. This is in general justified in fact, and since it is not intended to be applied to detailed minute differences it is all right.
This should be good enough till the tiresome objector arises. If then one must define which is the average to which the comp. should conform for the result to be exact and not only approximate, it is the St. Comm. . . ..
But what does this average 'approximate' to? i.e. what would it have to be composed of (what weights shd the average have) to be exactly the St. Com.?
i.e. Marx *assumes* that wages and profits consist *approximately* of quantities of st. com.

A comparison of Sraffa's reasoning with our own will help us bring our conclusion into focus. He says that wages, profits and constant capital in commodity form are of random composition so that changes of relative prices, induced by changes in distribution, have no relevant effect on the valuation of the aggregates. In particular, the valuation is not significantly different, whether it is a measurement by means of labour values or prices at the actual rate of profit. He clearly thinks of prices expressed in terms of some numéraire, composed of commodities (the proposition would not be true, if prices were expressed in terms of labour commanded), and the proposition seems plausible ('obviously true') to him independently of any other consideration about the nature of price movements and the character of the underlying technology. Using the same methods of elementary stochastics as in the main part of this article and reverting to the notation introduced in Section 2.3, we may formalise Sraffa's intuition by saying that the vectors of wages $\mathbf{b}$, of profits $\mathbf{s}$ and of constant capital (with activity levels $\mathbf{y}$) $\mathbf{yA}$ are each a perturbation of a vector that is proportional to a given common vector $\mathbf{z}$. Denoting the unperturbed vectors by an asterisk, we have $\mathbf{b}^* = \beta\mathbf{z}$, $\mathbf{s}^* = \sigma\mathbf{z}$ and $\mathbf{y}^*\mathbf{A} = \gamma\mathbf{z}$. The rate of surplus value P/W and the rate of profit r then are respectively (we need not fix a numéraire:

$$P/W = \mathbf{s}^*\mathbf{p}/\mathbf{b}^*\mathbf{p} = \sigma\mathbf{zp}/\beta\mathbf{zp} = \sigma/\beta;\ r = P/(K+W) = \sigma\mathbf{z}/(\gamma\mathbf{z}+\beta\mathbf{z}) = \sigma/(\gamma+\beta).$$

[11] Sraffa Papers D 3/12/111:140; original in English, taken from Bellofiore (2001, p. 369), suggested by Franklin Serrano. On Sraffa's temporary flirt with a 'statistical' approach see Gehrke and Kurz (2006); they suggest the approximate dating of the paper and describe how Sraffa came ultimately to abandon the 'statistical hypothesis' (p. 143).

The valuations of the perturbations are thus assumed to be equal to zero: $(\mathbf{s}-\mathbf{s}^*)\mathbf{p}=0$, $(\mathbf{b}-\mathbf{b}^*)\mathbf{p}=0$, $(\mathbf{y}-\mathbf{y}^*)\mathbf{Ap}=0$, but Sraffa has no argument to prove that, apart from his intuition, and he does not see how to make a theory based on stated assumptions out of his (or Marx's) proposition about a stochastic property of the capitalist system.

As a matter of fact, he directly abandons the stochastic approach in the next step by asking what happens to the valuations in values and prices, if the capitalists decide to consume a capital-intensive **s** and the workers a labour-intensive **b**. There may be socioeconomic reasons for class-specific consumption patterns, but to assume that the corresponding habits change suddenly in a stochastic setting is a bit like assuming that all oxygen molecules (O_2) in the air surrounding me in my room suddenly fly to the left and the nitrogen molecules (N_2) to the right, so that I get high when I turn to the left and I asphyxiate when I turn to the right.

Sraffa soon abandoned the stochastic perspective altogether. Gehrke and Kurz (2006) have shown that Sraffa investigated a statistical approach for a short time, without getting significant results. He asks, in our quote, for an 'exact' result, which would be not only 'approximate'. This is an ambiguous formulation, since probability theory is as rigorous as algebra; we need not be and are not content with a casual approximation. Sraffa did not know (or did not try) to render the Marxian idea rigorous in a stochastic setting. To get exact results, he refers to the standard commodity. If the economy is in standard proportions **q**, normalized so that employment $\mathbf{ql}=1$, and prices normalized so that $\mathbf{q}(\mathbf{I}-\mathbf{A})\mathbf{p}=1$, the rate of surplus value in price terms is $P/W=(1-w)/w$, or, with the wage paid *ex post*,

$$P/W=r/R-r.$$

With this result, Sraffa could still see himself in the Marxian tradition, but the Marxian vision of the process of exploitation, culminating in $P=M$, was not reproduced. Our solution is different from Sraffa's. It is not necessary to assume, as Sraffa says at the end of the quote, 'that wages and profits consist *approximately* of quantities of st[andard] com[modity]'. Why should the economy be close to standard proportions? It essentially suffices to assume that the randomness is in the structure of the economy, then $P=M$ will result with any activity levels. To illustrate the point, we look at a relationship that is independent of the numéraire and show that the rate of profit is the same in value and in price terms, whatever the activity levels. Using the formula for prices **p** of Section 2.3 and simplifying, one gets

$$r=\frac{\mathbf{sx}_1+(1-\rho\mu_1)\mathbf{sv}}{(\mathbf{yA}+\mathbf{b})\mathbf{x}_1+(1-\rho\mu_1)(\mathbf{yA}+\mathbf{b})\mathbf{v}}$$

We showed that $sv=n\overline{sv}$ tends to zero. Going immediately to the limit, we have

$$\begin{aligned}\mathbf{yAv}&=(\mathbf{q}_1+\ldots+\mathbf{q}_n)\mathbf{Av}=\mu_1\mathbf{q}_1\mathbf{v}+\ldots+\mu_n\mathbf{q}_n\mathbf{v}=\mu_1\mathbf{q}_1\mathbf{v}\\&=\mu_1\mathbf{q}_1(\mathbf{1}-\mathbf{x}_1)=\mu_1\mathbf{q}_1(\mathbf{x}_2+\ldots+\mathbf{x}_n)=0\end{aligned}$$

and, by the same reasoning, $\mathbf{yv}=0$, hence $\mathbf{bv}=(\mathbf{y}-\mathbf{yA}-\mathbf{s})\mathbf{v}=0$.

We therefore obtain

$$r=\frac{\mathbf{s}_{\mathbf{x}}}{(\mathbf{yA}+\mathbf{b})\mathbf{x}_1}$$

independently of the rate of profit, at which prices are calculated, and independently of the composition of **y**. The vectors **s**, **b** and **yA** are all in the same hyperplane, being orthogonal to **v**, but not necessarily proportional; the solution therefore is quite different from Sraffa's.

My way up to this result has been full of unexpected turns. That an approach similar to the analysis here proposed had also been considered by Sraffa was not the least of the surprises.

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