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MATHEMATICAL PROOFS OF THE BREAKDOWN OF CAPITALISM

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THE OLD MARXIST thesis that Capitalism shall break down of its own accord is all too familiar. We know also that among the converging arguments used to support this thesis a prominent place is occupied by the theme of the inadequacy of the accumulation process in the capitalist system. Of late, some Marxists have endeavored to add to this particular argument the prestige of the mathematical demonstration. Apparently, the first attempt in this direction was made by Otto Bauer in 1936, while the last word on the matter seems to be Sweezy's improved version of Bauer's proof.[1] This improved version, however, also starts out with serious mathematical errors which completely invalidate the proof. The presence of these errors has been pointed out by Domar.[2] Yet, even Domar does not seem to have realized precisely where the errors lie. Moreover, in his reworked solution he uses a schema of accumulation entirely different from that assumed by Marxist analysis. We are thus still confronted with the problem of whether or not the Bauer-Sweezy conclusions rigorously follow from the Marxist assumptions about the functioning of the capitalist system.[3] This fact alone would suffice to justify the interest in some probing of that argument, even if the problem of capital accumulation were not in the center of the current preoccupations of theoretical economists and policy advisers as well.

Such probing must ascertain, before anything else, whether the mathematical model used by the argument under scrutiny constitutes a *correct* translation of the Marxist scheme of expanded reproduction. It does not

[1] Paul M. Sweezy, *The Theory of Capitalist Development*, New York: Oxford University Press, 1942, Appendix to Chapter X, pp. 186–189.

[2] Evsey D. Domar, "The Problem of Capital Accumulation," *American Economic Review*, xxxviii (1948), pp. 792f.

[3] In his "A Reply to Critics" (reprinted in Paul M. Sweezy, *The Present as History*, New York: Monthly Review Press, 1953, pp. 352–362) Sweezy rightly points out that in Domar's amended scheme the problem of underconsumption—i.e., the very basis of the Bauer-Sweezy analysis—"simply disappears." In the same article, Sweezy, reflecting upon the mathematical Appendix, states that it was a failure because he attempted to deal with the consumption factor without using Marx's departmental scheme (*ibid.*, pp. 354, 360). Undoubtedly, an aggregative model fails to reflect some problems that only a general equilibrium scheme—be it reduced to two departments—can reveal. But, as I hope to prove, that is not the reason why the argument of the Appendix misses its target. In blaming the aggregative model for this, Sweezy implicitly takes the position that his theory of underconsumption is nevertheless substantially correct.

take long to see that the model used by Sweezy in the mathematical Appendix referred to above, does not correspond at all to the verbal description of that scheme as presented within the Appendix itself as well as elsewhere in the same work. Beyond this point, it is most natural that the analysis should proceed by reworking the argument on the basis of a mathematical model of the Marxist scheme. The real difficulty of the analysis begins precisely here. For as surprising as this may seem, the Marxist scheme of expanded reproduction cannot be cast into a *mathematically* correct model. Indeed, as we shall see in due time, this scheme sins against a most elementary principle, that of dimensional homogeneity.[4] The probing of the Bauer-Sweezy argument could therefore end on this hopeless note. There is, however, a more fruitful procedure: to construct first a scheme free from dimensional contradictions but embodying as many essential points of Marxist rationale as possible, and then to see whether the new dynamic model entails the Bauer-Sweezy conclusions. This is what I propose to do in the present paper. Although I am aware that retouching a Marxist scheme is Marxist anathema, I hope that the results presented here will be useful in two respects: in evaluating the argument of the inadequacy of capitalist accumulation, and in laying bare the purely logical difficulties in the Marxist formulation of expanded reproduction.

1. A DYNAMIC MODEL OF CAPITALISM

A preliminary word about notation. Although this may conflict with many conventional notations of income analysis, we propose to reserve capital letters for denoting *stocks*, and lower case letters for denoting *flows*. By this method we hope to keep the difference between the two concepts present at all times and to allow a quick verification of dimensional homogeneity of all formulae.

Let us refer to the dynamic system we are going to describe as the system (S). We define first the net national income, y, by the standard relation

$$y = c + a \tag{I}$$

where c represents households' consumption, and a, net accumulation. The net national income accrues to the working class as wages, w, and to the capitalist class as surplus value, s, i.e.,

$$y = w + s. \tag{II}$$

[4] Because there seems to be no little disagreement among Marxists themselves on what is the correct interpretation of Marx's analysis of expanded reproduction, it is necessary to add that the above statement refers to Sweezy's presentation. Nothing, however, militates against accepting this presentation as the orthodox one.

Following Marx we shall assume that only the capitalist class accumulates; hence

(III) $$s = l + a$$

where l represents the consumption of capitalists' households. The accumulation is in turn divided into two parts:

(IV) $$a = v + k$$

where v represents the increment of variable capital, V, and k, the increment of constant capital, K. By variable capital we understand the stock of means of subsistence necessary for the maintenance of the working class, and by constant capital the stock of means of production in the strict sense. The terms are Marxist, but the distinction between V and K may be made independently of Marxist economics.

As in any economy, we must have certain technical relations between the factors of production, on the one hand, and the output, on the other. Here too we shall follow the Marxist rationale and assume that the only *purely* technical relation is the proportionality between K and the production of consumers' goods:

(V) $$K = \lambda (w + l + v).$$

As Sweezy reminds us,[5] this relation is equivalent to "the acceleration principle."

Always conforming to the same rationale, we shall assume that a technical relation (in the broad sense) exists between wages and variable capital:

$$V = \mu w.$$

This relation is nothing other than a dimensionally correct formulation of the wage-fund theory which is implied by the Marxist concept of variable capital.[6]

We should emphasize that both λ and μ are *dimensional* constants, of the same dimension as t (time). Consequently, their values depend upon the choice of the time unit. Either λ or μ can then be made equal to unity by a proper choice of the time unit. If this unit is chosen so that $\mu = 1$, we have

(VI) $$V = w.^{7}$

[5] Sweezy, *op. cit.*, p. 187n. See also p. 182.

[6] This statement should be compared with that of J. Steindl, *Maturity and Stagnation in American Capitalism*, Oxford: Blackwell, 1952, p. 243, note 3, who argues that the "weird old monster, the wages fund doctrine, which Marx killed in a brilliant attack, [was nevertheless permitted as a] ghost to muddle up his terminology." I confess I cannot see how we can preserve the notion of variable capital—as conceived and used by Marx—and throw "that fossil" out of Marxist economics.

[7] It is a peculiarity of Marx's system to assume that λ and μ can be made equal to unity simultaneously, a fact which entitles him to write $K + V + s$ for total value. In this formula, however, there is no violation of the principle of dimensional homogeneity, for both K and V are multiplied by $1/\lambda$, $1/\mu$ whose numerical values happen to be unity.

It is extremely important to note also that *no explicit relation is assum to exist between the output and the wage bill (i.e., labor), for according Marxist rationale the latter is determined solely by the behavior of capitalis*

To this behavior we now turn. According to Marxist economics, "it is fundamental feature of capitalism that an increasing proportion of surpl value tends to be accumulated and an increasing proportion of accumulati tends to be invested."[8] Capitalists' behavior, therefore, can be express by the following relations

$$\text{(VII)} \qquad a = a(s), \quad \frac{da}{ds} > 0, \quad \frac{d}{ds}\left(\frac{a}{s}\right) > 0,$$

and

$$\text{(VIII)} \qquad k = k(a), \quad \frac{dk}{da} > 0, \quad \frac{d}{da}\left(\frac{k}{a}\right) > 0.$$

These relations imply that $a(s)$ and $k(a)$ have a first derivative ever where. We need not assume that the same functions have also a seco derivative. But to be realistic we must assume that $a(s)$ and $k(a)$ are *smoo* functions, and this requires that they should have at least a second deri tive to the right and to the left everywhere.[9] We know that in this ca the points where $a(s)$ and $k(a)$ do not have a second derivative are isolat

Finally, we must add the dynamic relations

$$\text{(IX)} \qquad \frac{dV}{dt} = v; \quad \frac{dK}{dt} = k.$$

The system (S) then involves ten unknowns: $V, K, y, c, a, w, s, l, v, k$, a ten relations: (I)—(IX). In this particular case we can do a little bett than comparing the number of unknowns with that of equations. Fr (III), (IV), (VII) and (VIII), we obtain the inverse functions

$$\text{(1)} \qquad l = l(k), \qquad v = v(k),$$

and from (V), (VI), and (IX)

$$\text{(2)} \qquad k = \lambda v + \lambda \left(\frac{dv}{dk} + \frac{dl}{dk}\right)\frac{dk}{dt}.$$

This differential equation determines $k(t)$. The other unknown functic of the dynamic system are then derived from the other equations by straig forward operations.

There are two features of (S) that make it differ from the Marxist sche of expanded reproduction. For the reasons already explained, we sh refer to Sweezy's presentation of the latter scheme as a basis of comparis

[8] Sweezy, *op. cit.*, p. 187; also p. 181.

[9] To illustrate: $y = x^3/|x|$ is a smooth function without a second derivative $x = 0$. However, for $x = 0$, the second derivative to the left is —1, and to the right -

The first difference concerns the composition of surplus value. According to (III) and (IV), for (S) we have

$$s = l + v + k \tag{3}$$

whereas in the Marxist scheme

$$\bar{s} = l + v + k + \frac{dl}{dt}. \tag{3$^{\text{bis}}$}$$

Indeed, Sweezy's explanation of this point carefully aims at leaving no room for misunderstanding. We are told in very explicit terms that the surplus value consists of *four* parts, which in comparative notations are: (1) S_{ac}, corresponding to our k; (2) S_{av}, corresponding to our v; (3) S_c, corresponding to our l; and (4) $S_{\Delta c}$, the increment of l itself, i.e., dl/dt.[10] Moreover, Sweezy sharply criticizes N. Bukharin (the outstanding Marxist theorist liquidated during the Great Purge of the 1930's) for having used (3) instead of (3^{bis}) in presenting the scheme of expanded reproduction. According to the same author, the omission of the term dl/dt from the analysis of surplus value proves that Bukharin was "incapable of imagining an increase in capitalists' consumption."[11] It is elementary, however, that the absence of dl/dt in (3) does not mean at all that l is necessarily a constant, for l like all other variables of the system is determined by all equations together, not by one relation alone.

On the contrary, it is formula (3^{bis}) which is absurd, not *economically*, but in a sense independent of any material interpretation of it. Indeed, (3^{bis}) violates the principle of dimensional homogeneity, which is essentially an arithmetical principle. As long as the letters in that formula stand for measurable material concepts and not for some Hegelian ideals, l and dl/dt cannot be added, any more than can *total* and *average* cost, for instance. I hope to be forgiven for stressing an elementary point that has been, as we know, the source of many economic fallacies. This seemed necessary for reaching the root of the difficulty of translating the Marxist scheme into an arithmetically correct model. For the arithmetical incongruity of (3^{bis}) is not accidental, but reflects a vital aspect of Marxist economics. And that aspect is the notion that a material flow can be the source of its own growth.

This position makes also for the second point of difference between (S) and the scheme of expanded reproduction. For (S), from (II)—(IV) and (VI) we derive

$$y = V + s = V + l + v + k \tag{4}$$

[10] *Ibid.*, p. 163, reemphasized on p. 181.

[11] *Ibid.*, p. 164n.

whereas in Marxist dynamics

$$(4^{\text{bis}}) \qquad \bar{y} = V + v + \bar{s} = V + l + 2v + k + \frac{dl}{dt}.$$[12]

This time, too, we are strongly warned against an "unguarded haste" that may lead to confusing the concept of national income with formula (4).[13]

The line of reasoning behind (4^{bis}), however, is not as simple and obvious as in the case of (3^{bis}). First, the *flow* increment v is added to the *stock* V during the same period in which the flow is produced. Then the stock $V + v$ is used as a basis for equating the wage bill to $V + v$. This is how it comes about that (4^{bis}) contains one more v than (4). At bottom, this means that the flow v can be both *consumed* and passed over into the next period as an *increment* of a stock, or, in other words, that "the growth of variable capital constitutes an outlet for accumulation and *at the same time* signifies a growth in consumption."[14] In explicit terms, this means that

$$(5) \qquad a = v + k \quad \text{and} \quad \bar{c} = V + v + l + \frac{dl}{dt}.$$[15]

Since Marxist practice sees nothing wrong in adding to a flow its own dynamic increment during the very period in which this increment is produced, it is difficult to see why the surplus value should not be given by

$$(3^{\text{ter.}}) \qquad \bar{\bar{s}} = l + \frac{dl}{dt} + v + \frac{dv}{dt} + k + \frac{dk}{dt}.$$

For one may explain that in each period the surplus value is divided into six parts: the previous value of l plus its increment, and so forth.[16] Actually, once the principle of dimensional homogeneity is rejected, there is no reason for not continuing to add the increments of increments of increments....

But the difficulty, nay, the impossibility, of casting the Marxist scheme of expanded reproduction into a mathematical model can be illustrated in a more concrete fashion. Let us suppose that such a model has been constructed and that the corresponding system has been solved for its ten unknown functions. Let us also assume that the solution gives $l = A + Bt$, for instance. If now to the question "what is the value of the flow of capi-

[12] *Ibid.*, p. 63. In explicit mathematical terms the formula is given in Appendix A (p. 373 combined with p. 368n). That Appendix is written by Shigeto Tsuru, but Sweezy refers to it for every question pertaining to the composition of national income.

[13] *Ibid.*, pp. 248n, 371, *passim*.

[14] *Ibid.*, p. 222.

[15] Cf. *ibid.*, p. 372. Tsuru's justification of the double counting of v in (4^{bis}) is highly instructive: the double counting is the natural result of three "metamorphoses" of money, whatever this may mean.

[16] To be sure, we have

$$s(t + \Delta t) = s(t) + \Delta l + \Delta v + \Delta k$$

but, clearly, this is not what is meant by (3^{bis}) and (4^{bis}).

talists' consumption at $t = t_0$?" one answers $A + Bt_0$, then, according to Marxist rationale, the answer is wrong: for this answer does not include the "increment" dl/dt, which is B. If the answer is that the value of the flow is $A + B + Bt_0$, then by all logic it follows that $l = A + Bt$ does not represent the consumption of capitalists. What does it represent then? And if we say that capitalists' consumption is given by $A + B + Bt$, what shall we do with the term dl/dt in (3^{bis})? If we drop it, we depart from strict Marxist rationale; if we retain it, we shall never know the amount of capitalists' consumption.[17]

Finally, let us observe that the position that a *material* flow can be the source of its own growth is tantamount to the belief in the existence not only of perpetual motion but of perpetual *accelerated* motion as well. But if a flow cannot be the source of its own growth, one may ask, what is the source of economic growth? The answer to the apparent puzzle is not difficult. Since human economy is not an isolated system, economic growth is the result of a continuous tapping of other stocks: the stocks of natural deposits, of various forms of free energy, and above all of that peculiar energy which is accumulated in the body of living organisms. The economic *process* consists precisely in this tapping. To be sure, this *process* grows without any counterbalancing decrease in something else, just as physical entropy grows without any decrease in the total energy of the universe. Only in this sense can we speak of the economic *process* being Hegelian, i.e., containing the source of its own development. But the *material* elements involved in the process must obey the universal laws of matter and energy.

2. THE ARGUMENT OF THE INADEQUACY OF CAPITALIST ACCUMULATION

We have already mentioned that the Bauer-Sweezy mathematical proof of the inadequacy of capitalist accumulation uses a model different from the Marxist scheme of expanded reproduction. But the model has a more generally applicable shortcoming. Indeed, the proof proceeds from the definition of national income by the formula

$$y^* = w^* + l^* + k. \tag{6}$$

In the absence of an explicit statement by Sweezy, it is rational to assume that he adheres to the practice of adding the corresponding incremental flows to both wages and capitalists' consumption, i.e.,

$$w^* = V + v, \quad l^* = l + \frac{dl}{dt}. \tag{7}$$

With this, (6) becomes

$$y^* = \bar{y} - v. \tag{8}$$

[17] The same remarks apply to the solution $V = V(t)$.

If, however, w and l have their normal meaning,

$$(8^{\text{bis}}) \qquad y^* = y - v.$$

It is thus seen that according to any interpretation, whether Marxist or not, (6) fails to include the accumulation of variable capital in the national income.[18]

The second observation about the Bauer-Sweezy argument concerns the mathematical inaccuracies to which the introduction of this paper alluded. Sweezy states[19] that from "the fundamental feature of capitalism," i.e., from (VII) and (VIII), it follows that

$$(9a) \qquad 0 < \frac{dw^*}{dk} < 1, \quad \frac{d^2w^*}{dk^2} < 0,$$

$$(9b) \qquad 0 < \frac{dl^*}{dk} < 1, \quad \frac{d^2l^*}{dk^2} < 0.$$

But (VII) and (VIII) together yield

$$(\text{VII}^{\text{bis}}) \qquad -1 - \frac{dv}{dk} < \frac{dl^*}{dk} < \frac{l^*}{v+k}\left(1 + \frac{dv}{dk}\right),$$

$$(\text{VIII}^{\text{bis}}) \qquad -1 < \frac{dv}{dk} < \frac{v}{k}.$$

Rigorously speaking, according to (VII$^{\text{bis}}$) dl^*/dk may even be negative, while d^2l^*/dk^2 may have either sign. The supposition that (VIII$^{\text{bis}}$) entails (9a), however, reveals a more essential error of the argument.

According to either of the two possible interpretations of w^*, dw^*/dk depends on dV/dk. But the latter's value cannot be determined or restricted

[18] This seems the proper place to mention a position taken by Steindl, *op. cit.*, p. 243, note 3, which, if valid, would upset the entire argument developed so far. Steindl objects to Sweezy's analysis of expanding surplus value on the ground that s cannot even include such a term as v. To assume that it does "implies that some part of [the] national income flow is *wages*, and at the same time is also *surplus value* (profits) *in the same period*; that some part of the value created in *a given year* is unpaid labor and at the same time also paid labor!" On the surface, this argument sounds identical to that advanced against (4$^{\text{bis}}$) above. But on closer examination it shows itself to be based upon how stocks grow from flows. Steindl's position implies that V either grows *by itself* or remains constant. At bottom, all this shows that it is impossible to get rid of the "fossil" and still have a consistent Marxist scheme of expanded reproduction. (cf. *supra*, fn. 6). A two-department scheme offers no escape from the dilemma (if there is one). So that one can heartily agree with Sweezy that his problem is "a standing challenge to Marxian economists" ("A Reply," p. 360).

[19] *Ibid.*, p. 187.

by the behavior of capitalists alone, i.e., by (VII) and (VIII). This value is determined only by the system as a whole, *including the technical relation* (V).[20]

The Bauer-Sweezy thesis, however, may be perfectly valid in spite of the false start in the mathematical proof. But this question cannot be elucidated without examining the same thesis in the light of a consistent model. And since (S) comes as close as possible to the Marxist rationale, we shall proceed on this basis.[21]

The Bauer-Sweezy argument can be summarized as follows:

(1) Capitalists' behavior being that described by (VII) and (VIII), if $dy/dt > 0$, then $dk/dt > 0$;

(2) On the other hand, the technical condition (V) together with capitalists' behavior require $dk/dt < 0$ if $d^2y/dt^2 < 0$;

(3) Therefore, in the case where the national income grows at a decreasing rate the *behavior* value of dk/dt is greater than the *equilibrium* value. Hence, "the output of consumption goods will display a continuous tendency to outrun demand."[22]

It can be proved, however, that both premises (1) and (2) on which the conclusion (3) rests are false. The following theorem shows this for (1).

THEOREM 1. *There are functions of t satisfying* (VII) *and* (VIII) *and such that* $\dot{y} > 0$ *and* $\dot{k} < 0$.[23]

PROOF: Let us consider the following functions of t for $t \geqslant 0$:

$$(10) \qquad k = Ae^{-(\alpha+\beta)t}, \quad v = e^{-\alpha t}(1 - Ae^{-\beta t}), \quad l = B - \gamma e^{-\alpha t}$$

where

$$(11) \qquad 0 < \alpha < 1, \quad 1 < \beta, \quad 0 < A < 1 + \alpha(\gamma - 1), \quad 0 < \gamma < 1, \quad \gamma < B.$$

Since $\dot{k} < 0$, from (VII) and (VIII) we first obtain

$$(12) \qquad \dot{k}\frac{da}{dk} = \dot{a} < 0, \quad \dot{a}\frac{ds}{da} = \dot{s} < 0.$$

[20] Domar's criticism failed to realize this aspect of the problem. After remarking that the fundamental feature of capitalism entails neither (9a) nor (9b), Domar asserts that by this feature it is given that $d(k/y)/dt > 0$ (*op. cit.*, p. 793). But this expression involves dV/dk and V/k. Domar's conclusions regarding Sweezy's argument need, therefore, to be reexamined.

[21] Another alternative would be to proceed *per absurdum* by accepting (3^{bis}) and (4^{bis}) as relations in *pure numbers*. I want to stress the fact that even in this alternative the mathematical truth of the subsequent theorems is not in the least invalidated. The choice to proceed otherwise aims only at avoiding the incongruities described in Section 2.

[22] Sweezy, *op. cit.*, p. 189.

[23] For the sake of compactness, we shall hereafter use the dot notation for the derivatives with respect to time: $\dot{x}$ for dx/dt, $\ddot{x}$ for d^2x/dt^2, etc.

These conditions are clearly fulfilled by (10), for

$$\dot{a} = \dot{k} + \dot{v} = -\alpha e^{-\alpha t}, \quad \dot{s} = \dot{a} + \dot{l} = -\alpha\,(1-\gamma)e^{-\alpha t}. \tag{13}$$

The other conditions of capitalists' behavior are also satisfied:

$$\begin{aligned} a\dot{k} - \dot{a}k &= -\beta A e^{-(2\alpha+\beta)t} < 0 \\ \dot{a}l - a\dot{l} &= -\alpha B e^{-\alpha t} < 0. \end{aligned} \tag{14}$$

Yet, we have

$$\dot{y} = v + \dot{s} = e^{-\alpha t}\,(1 - \alpha + \alpha\gamma - Ae^{-\beta t}) > 0. \tag{15}$$

Q.E.D.

The falsity of the second premise of the argument is shown by Theorem II. Before enunciating this theorem, however, we need to introduce some definitions and prove some useful lemmas.

DEFINITION I. *If all functions v, k, l, y as well as their first derivatives with respect to t are positive, then we call* (S) *a growing system.*

From (V), (VI) and (IX) we obtain

$$k = \lambda\,(v + \dot{v} + \dot{l}), \quad \dot{k} = \lambda\,(\dot{v} + \ddot{v} + \ddot{l}) \tag{16}$$

and

$$\dot{y} = \frac{k}{\lambda} + \dot{k}, \quad \ddot{y} = \frac{\dot{k}}{\lambda} + \ddot{k}. \tag{17}$$

For a growing system, relations (VII) and (VIII) become

$$\frac{\dot{v} + \lambda\,(\dot{v} + \ddot{v} + \ddot{l})}{v + \lambda\,(v + \dot{v} + \dot{l})} > \frac{\dot{l}}{l}, \quad \frac{\ddot{v} + \ddot{l}}{\dot{v} + \dot{l}} > \frac{\dot{v}}{v}. \tag{18}$$

REMARK I. To have a growing system (S), we need only to determine v and l as positive and increasing functions of t and such that: (1) both expressions (16) be positive, and (2) inequalities (18) be satisfied. We should observe, however, that according to our strictly necessary assumptions, the functions $v(t)$ and $l(t)$ do not necessarily possess a second derivative everywhere.[24] That raises the problem of the meaning of the second formula (16) and of (18). According to (2), however, $\dot{k}(t)$ has a derivative everywhere (except where the sum $dv/dk + dl/dk$ would be zero). It follows that even if $\ddot{v}, \ddot{l}$ may not exist everywhere, the integral of (2) is such that it makes $\ddot{v} + \ddot{l}$ a continuous function of t wherever $\dot{k}$ is continuous. Since only the sum $\ddot{v} + \ddot{l}$ enters in the formulae (16) and (18), we can use the

[24] *Supra*, p. 228.

notation $\ddot{v}$, $\ddot{l}$ even though these derivatives may not exist separately. We need, therefore, to choose v and l such that $\ddot{v} + \ddot{l}$ be continuous (in addition to the conditions mentioned previously in this Remark).

DEFINITION II. *If a growing system* (S) *satisfies the inequalities*

$$\frac{\ddot{v} + \ddot{l}}{\dot{v} + \dot{l}} > \frac{\dot{v}}{v} > \frac{\dot{l}}{l} \tag{18bis}$$

we shall call it a strongly growing system.

LEMMA 1. *The system*

$$v = A + Be^{\alpha t}, \quad l = A' + B'e^{\alpha t} \tag{19}$$

where

$$A, B, A', B', \alpha > 0, \quad A'B - B'A > 0 \tag{20}$$

is a strongly growing system.

The proof is immediate.

LEMMA 2. *Given a strongly growing system* (v, l) *that is not of the form* (19), *we can derive another strongly growing system* (v^*, l^*) *different from* (v, l).

PROOF. We first choose arbitrarily the origin of t. Then we determine A, B, A', B', α by the system:

$$\begin{aligned} A + B &= v(0), & A' + B' &= l(0), \\ \alpha B &= \dot{v}(0), & \alpha B' &= \dot{l}(0), \\ \alpha^2(B + B') &= \ddot{v}(0) + \ddot{l}(0). \end{aligned} \tag{21}$$

Because (l, v) is a strongly growing system, the unknowns of the system are easily seen to satisfy (20). Hence the system

$$v^*(t) = \begin{cases} v(t) \text{ for } t \leqslant 0, \\ A + Be^{\alpha t} \text{ for } t \geqslant 0, \end{cases} \quad l^*(t) = \begin{cases} l(t) \text{ for } t \leqslant 0, \\ A' + B'e^{\alpha t} \text{ for } t \geqslant 0, \end{cases} \tag{22}$$

is a strongly growing system different from (l, v).

Q.E.D.

THEOREM 2. *There exist strongly growing systems* (S) *such that* $\ddot{y}$ *be negative for some values of* t.

PROOF. Let (v, l) be a strongly growing system such that $\ddot{y}$ be positive for all values of t where $\ddot{y}$ exists, and let us assume that $t = 0$ is such a value. Let us put

$$v_1(t) = v(0) + \dot{v}(0)t + v_2 \frac{t^2}{2!} + v_3 \frac{t^3}{3!},$$

(23)

$$l_1(t) = l(0) + \dot{l}(0)t + l_2 \frac{t^2}{2!} + l_3 \frac{t^3}{3!},$$

where

$$v_2 + l_2 = \ddot{v}(0) + \ddot{l}(0) \tag{24}$$

and v_3, l_3 are some parameters to be determined later. Because $v_1(0) = v(0)$, $\dot{v}_1(0) = \dot{v}(0)$, $l_1(0) = l(0) = \dot{l}_1(0) = \dot{l}(0)$, $\ddot{v}_1(0) + \ddot{l}_1(0) = \ddot{v}(0) + \ddot{l}(0)$, and because (v, l) is a strongly growing system, the continuity of the functions (23) warrants the existence of a non-null interval $0 \leqslant t \leqslant t'$ in which (v_1, l_1) satisfy the conditions of a strongly growing system. If v_1, l_1 are introduced in (16) and (17), we obtain:

$$\ddot{y}_1(t) = \dot{v}(0) + \ddot{v}(0) + \ddot{l}(0) + \lambda(v_2 + v_3 + l_3) + (v_2 + v_3 + \lambda v_3 + l_3)t + v_3 \frac{t^2}{2}. \tag{25}$$

We can choose v_2, v_3, l_3 such that $\ddot{y}_1(0) < 0$. There is then an interval $0 < t \leq t''$ where $\ddot{y}_1(t) < 0$. Let T be the smaller of t', t''. But at T the values of v_1 and l_1 satisfy the conditions of a strongly growing system. According to Lemma 2, at T we can splice the functions $v^*(t)$, $l^*(t)$ of the type (19).

Thus the system corresponding to

$$\bar{v}(t) = \begin{cases} v(t) & \text{for } t \leqslant 0, \\ v_1(t) & \text{for } 0 \leqslant t \leqslant T, \\ v^*(t) & \text{for } T \leqslant t, \end{cases} \quad \bar{l}(t) = \begin{cases} l(t) & \text{for } t \leqslant 0, \\ l_1(t) & \text{for } 0 \leqslant t \leqslant T, \\ l^*(t) & \text{for } T \leqslant t \end{cases} \tag{26}$$

is a strongly growing (S), for which $d^2\bar{y}/dt^2 < 0$ for $0 < t \leq T$.

Q.E.D.

The inadequacy of capitalist accumulation is derived by Sweezy from another argument as well: that capitalists' behavior can lead only to a decreasing ratio between the rate of consumption and the rate of capital investment, while the technical conditions require this ratio to be a constant.[25] Clearly this argument, if correct, would be far stronger than that summarized on page 233, for according to it capitalism could never be in equilibrium, whether $\ddot{y} > 0$ or $\ddot{y} \leqslant 0$. It can be shown, however, that this argument, too, is fallacious.

[25] *Ibid.*, p. 182. The technical ratio is $1/\lambda$, from (V).

THEOREM 3. *There are functions of t satisfying the behavior conditions* (VII) *and* (VIII), *such that the ratio between the rate of growth of output and the rate of growth of constant capital be increasing.*[26]

PROOF: Using the functions of (10), we have

$$\frac{v + \dot{v} + \dot{l}}{k} = \frac{1}{A}\left[(1 - a + a\gamma)e^{\beta t} + A(a + \beta - 1)\right] > 0 \,. \tag{27}$$

Q.E.D.

3. A FUNDAMENTAL PROPERTY OF THE SYSTEM (S)

Lemma 1 proves that a "capitalist" system (S) may be growing in such a way that the net national income be continuously increasing at an *increasing* rate. On the other hand, Theorem 2 proves that there are growing systems (S) for which the net national income increases at a *decreasing* rate in some intervals. The question now is whether the symmetrical situation of Lemma 1 exists, namely, whether a growing system exists for which $\ddot{y}$ may remain negative after a certain value of t. The answer to this question is negative, as shown by the following:

THEOREM 4. *No growing system* (S) *exists such that* $\ddot{y} \leqslant 0$ *for all* $t \geqslant t_0$.

PROOF: Because the details of the proof vary according to whether $\lambda \gtrless 1$, (without however requiring a different method of demonstration) we are going to give the proof only for the case $\lambda > 1$.[27] The proof will show that the assumption $\ddot{y} \leqslant 0$ for all $t \geqslant 0$ is in contradiction with the properties of (S).

Let

$$\ddot{y} = -\, e^{t/\lambda}\, p(t) \tag{28}$$

where $p(t) \geqslant 0$. To avoid unnecessary complications we shall assume that $p(t)$ exists everywhere except for a number of isolated values.

From (17) we obtain

$$e^{t/\lambda}\left(\frac{k}{\lambda} + \ddot{k}\right) = -\, p(t) \tag{29}$$

[26] Since Sweezy (*op. cit.*, p. 187) defines consumption as $w^* + l^*$, and since $w^* = w + v$, his "consumption" is in fact "output" in our sense (save for the term dl/dt). Strictly speaking, it is much more realistic to relate constant capital to output than to consumption.

[27] In view of the fact that the value of λ is determined by the condition $\mu = 1$ (*supra*, p. 227), I doubt that on a priori grounds we can hold that $\lambda > 1$ is the only realistic case.

which yields

(30) $$e^{t/\lambda}\,\dot{k} = C' - \int_0^t p(t)dt.$$

But since $\dot{k} > 0$, we must have

(31) $$C' \geqslant \lim_{t=+\infty} \int_0^t p(t)dt, \quad \lim_{t=+\infty} p(t) = 0.$$

Let us put

(32) $$\chi(t) = \int_t^\infty p(t)dt.$$

Because of (31),

(33) $$\lim_{t=+\infty} \chi(t) = 0.$$

From (30), we obtain

(34) $$\dot{k} = [C + \chi(t)]\, e^{-t/\lambda}, C \geqslant 0$$

and further

(35) $$k = \lambda\,[C_0 - Ce^{-t/\lambda} - \gamma(t)]$$

where

(36) $$\gamma(t) = \frac{1}{\lambda} \int_t^\infty e^{-t/\lambda}\,\chi(t)dt.$$

Because of (33), the latter integral exists. And since k is a positive and increasing function of t, we must have

(37) $$k(0) = \lambda\,[C_0 - C - \gamma(0)] > 0, \quad k\,(+\infty) = \lambda C_0 > 0.$$

Now, because v must grow less rapidly than k, $v(+\infty)$ must be finite. Therefore, $a = k + v$ also is bounded. But l must grow less rapidly than a; hence l, too, must be bounded. Since $\dot{l} > 0$, it follows that

(38) $$\lim_{t=+\infty} \int_0^t \dot{l}dt < +\infty, \quad \lim_{t=+\infty} \dot{l} = 0.$$

From (V) we get

(39) $$v + \dot{v} = C_0 - Ce^{-t/\lambda} - \gamma(t) - \dot{l}(t)$$

which, by integration, yields

(40) $$v = C_0 - \frac{C\lambda}{\lambda - 1}\, e^{-t/\lambda} + [v_0 - \Gamma(t) - L(t)]\, e^{-t}$$

where

(41) $$\Gamma(t) = \int_0^t e^t \gamma(t)dt, \quad L(t) = \int_0^t e^t\, \dot{l}(t)dt.$$

To see whether (35) and (40) are compatible with a growing (S), we shall examine this problem for t sufficiently large. In this case, the corresponding formulae of k and v can be replaced by some asymptotic expressions, $\tilde{k}$ and $\tilde{v}$.

To determine these asymptotic expressions, let us observe that since $\lambda > 1$, $e^{-t/\lambda}$ tends more slowly towards zero than e^{-t}, and more slowly than both $\gamma(t)$ and $\Gamma(t)$. Indeed, from (36) and (41) we have

$$\lim_{t=+\infty} e^{t/\lambda}\, \Gamma(t) = \lim_{t=+\infty} e^{t/\lambda}\, \gamma(t) = 0\,. \tag{42}$$

There are several alternatives to be considered.

(A) $C \neq 0$. In this case $e^{-t/\lambda}$ represents the first order infinitesimal, and we have

$$\begin{aligned} \tilde{k} &= \lambda\,(C_0 - Ce^{-t/\lambda}),\\ \tilde{v} &= C_0 - \frac{\lambda C}{\lambda - 1}\, e^{-t/\lambda} - L(t)\, e^{-t}\,. \end{aligned} \tag{43}$$

Now, if

$$\lim_{t=+\infty} e^{t/\lambda}\, \dot{l} = M < +\infty\,, \tag{44}$$

we have

$$e^{-t}\, L(t) \cong Me^{-t/\lambda} \tag{45}$$

which, introduced in (43), yields

$$\tilde{v} = C_0 - \left(\frac{\lambda C}{\lambda - 1} + M\right) e^{-t/\lambda}. \tag{46}$$

It is easy to see that $\tilde{v}$ grows faster than $\tilde{k}$, and hence the assumption (44) is incompatible with the structure of (S).

If in (44) $M = +\infty$, then $e^{-t/\lambda}$ tends faster toward zero than Le^{-t} and we have

$$\tilde{k} = \lambda C_0, \quad \tilde{v} = C_0 - L(t)e^{-t}. \tag{47}$$

This case must be rejected for the same reason as above.

(B) $C = 0$. In this case $\gamma(t) \neq 0$. There are several alternatives.

($B1$) *Both integrals* (41) *converge for* $t = +\infty$. It is not possible to have

$$v_0 - \Gamma(+\infty) - L(+\infty) \geqslant 0\,, \tag{48}$$

for then

$$v_0 - \Gamma(t) - L(t) > 0 \tag{49}$$

and $\dot{v}$ would be negative. And if

$$v_0 - \Gamma(+\infty) - L(+\infty) < 0 \tag{50}$$

we have

$$\tilde{k} = \lambda C_0 \, , \tag{51}$$
$$\tilde{v} = C_0 + [v_0 - \Gamma(+\infty) - L(+\infty)]\, e^{-t} \, .$$

This must be rejected because v would grow faster than k for t sufficient large.

($B2$) *Only Γ converges.* In this case,

$$\tilde{k} = \lambda C_0, \qquad \tilde{v} = C_0 - e^{-t} L(t) \, , \tag{52}$$

which gives a decreasing ratio $\tilde{k}/\tilde{v}$.

($B3$) *Only L converges.* If Γ diverges while

$$\lim_{t=+\infty} e^t \Gamma(t) < +\infty \, , \tag{53}$$

we have

$$\tilde{k} = \lambda C_0, \qquad \tilde{v} = C_0 - e^{-t} \Gamma(t) \, , \tag{54}$$

which must be rejected for the same reason as in ($B2$).
We should then examine the alternative

$$\lim_{t=+\infty} e^t \Gamma(t) = +\infty \, . \tag{55}$$

If this alternative is true, $\dot{\gamma}(t)e^t$ cannot tend toward zero as t tends towa $+\infty$. On the other hand, we have

$$\Gamma(t) = \gamma(t)e^t - \gamma(0) - \int_0^t \dot{\gamma}(t)\, e^t dt \, . \tag{56}$$

And, since by (36) $\dot{\gamma}(t)$ is negative, it follows that

$$\tilde{\Gamma}(t) = \varepsilon(t)\gamma(t)e^t, \quad \varepsilon(t) > 1 \, . \tag{57}$$

Therefore,

$$\tilde{k} = \lambda(C_0 - \gamma), \quad \tilde{v} = C_0 - \varepsilon\gamma \, , \tag{58}$$

which again gives a decreasing ratio $\tilde{k}/\tilde{v}$.

($B4$) *Both Γ and L diverge.* The only case which cannot be immediat reduced to one of the types examined under ($B2$) and ($B3$), is that wh

$$\lim_{t=+\infty} L(t)/\Gamma(t) = \delta, \quad 0 < \delta < +\infty \, . \tag{59}$$

The case where (53) would be true is disposed of by the same argum as in ($B3$). If, however, (55) prevails, then

$$\tilde{k} = \lambda(C_0 - \gamma), \quad \tilde{v} = C_0 - (\varepsilon + \delta)\gamma \, , \tag{60}$$

which also gives a decreasing $\tilde{k}/\tilde{v}$.

With this, all alternatives have been examined and proved to contradict some property of (S).

Q.E.D.

REMARK 2. It is important to note that $\ddot{y}$ may end by remaining negative in a system (S) where v and k grow but l decreases. This is shown by the following example:

$$k = \lambda(C_0 - Ce^{-t/\mu}), \quad v = C_0 - Be^{-t/\mu}, \quad l = l_0 + (\mu C - \mu B + B)e^{-t/\mu} \tag{61}$$

where

$$\mu < \lambda, \quad B > C > C_0, \quad 0 \leqslant t. \tag{62}$$

4. CONCLUDING REMARKS

We shall now summarize the salient points of the preceding analysis and tie them together with a few additional remarks.

(A) As we have seen (Theorem 3), from the mere knowledge of the behavior of capitalists as described by Marxist economics, it is not possible to deduce that the ratio between the means of production, K, and the output of consumers' goods, $w + l + v$, is decreasing.

(B) We have also seen that an economic system corresponding to the Marxist description of capitalism (corrected for dimensional absurdities), can be *growing* although its net national income increases only at a *decreasing* rate (Theorem 2).

Assuming therefore that this Marxist description of capitalism is epistemologically valid, a period in which the net national income of a capitalist economy increases at a decreasing rate does not justify any prediction of the breaking down of capitalism. Moreover, the number of periods in which the net national income of a growing capitalist economy increases at a decreasing rate may be unlimited (Theorem 2). Therefore, even the repetition of such phases does not justify that prediction.

(C) It is true, however, that a phase in which the national income of a growing capitalist economy increases at a decreasing rate cannot last indefinitely (Theorem 4). Thus, it seems that we could proclaim the end of the capitalist system provided we knew that we have actually entered into such an everlasting phase. But how can we know in practice that the phase in point is everlasting and not a temporary one?

(D) On the other hand, it is not too clear why in a growing capitalist economy, capitalists' consumption should be increasing. Actually, according to the very Marxist theory of concentration the number of capitalists should continually decrease. The speedier the concentration, the more likely it is that the consumption of the capitalist class should decrease. If this happens, then neither is a lasting phase in which the national income increases at a decreasing rate incompatible with a growing economy in all other respects (Remark II).

(E) At the appropriate time we emphasized the fact that, according to Marxist rationale, no technical relation exists between output and variable capital, i.e., between output and employment.[28] The aggregate production function of the capitalist system is thus reduced to a relation between output and constant capital alone. This is entirely in agreement with the position that the magnitude of employment depends solely upon the interest of capitalists in having a greater or smaller recourse to the reserve army. In other words, the amount of employment is determined by the amount of variable capital capitalists are willing to set up, and not by a technical condition.[29] Therefore, given that the behavior of capitalists is that of (VII) and (VIII), we could not possibly assume also an *independent* relation between output and employment. For if we added such a relation, the number of equations of (S) would increase by one unit and would thus become greater than that of the variables of the system. The capitalist system would then be an impossible system *ab initio*, as impossible as a square with five sides.

It is hard to see then how one can reconcile Marxist economics with the assertion that capitalism produces more consumers' goods than the demand for them. For if there is no technical relation between employment and output, there also is no demand equation in the system. The *employed* workers have no demand: they always receive and consume exactly what results from capitalists' behavior.[30] This peculiarity of the Marxist position has puzzled many followers of Marx who—like Rosa Luxemburg, especially—kept on asking: "Where does the demand come from [in a capitalist system]?"[31]

Of course, capitalists may not hit at once upon their *true* preferences when they are confronted with entirely new personal situations. All economic decisions in an evolutionary system are subject to this type of error. Clearly, such errors of appreciation introduce some *shocks* into the functioning of the system, but it is highly improbable for these shocks to accumulate in the same direction so as to produce a lasting deviation from the trend determined by the system of equations itself.

[28] *Supra*, p. 228.

[29] K. Marx, *Capital*, Chicago, 1932, Vol. I, Ch. xxv, Sec. iii.

[30] And to recall, one of the highlights of Marx's teachings is that capitalism produces always less than the demand, if "demand" is to be interpreted as "need." See the categorical statement on this point in K. Marx and F. Engels, *Correspondence*, 1846–1895, New York, 1935, p. 199.

[31] Rosa Luxemburg, *The Accumulation of Capital*, New Haven: Yale University Press, 1951, p. 19.

(F) In the preface of an admirable little book, Erwin Schrödinger expressed the thought that the difficulty of analysing the process of life does not reside in the complication of mathematics, but in the fact that the process is too complicated for mathematics.[32] This remark applies admirably also to the problem of the future of capitalism. For capitalism, like all other economic systems that preceded it and that will be produced by the continuous evolution of human society, is a form of life. Some aspects of its functioning lend themselves perfectly to mathematical analysis. Yet, when we come to the problem of its *evolution*, of its mutation into another form, mathematics proves to be too rigid and hence too simple a tool for handling it. Mathematical proofs of future evolutionary changes in any domain should, therefore, be viewed with skepticism, even if, unlike those analysed in this paper, they are logically irreproachable.

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[32] Erwin Schrödinger, *What is Life?*, Cambridge, England: The University Press, 1955, p. 1.