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Potron and the Perron–Frobenius Theorem

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ABSTRACT *Maurice Potron is an important precursor of the study of linear models of production and, in particular, of input–output analysis. We show that, contrary to Abraham-Frois and Lendjel's interpretation which we consider as unfaithful to Potron's model, there is a clear connection between his theory and the Perron-Frobenius theorem.*

KEY WORDS: Frobenius, Leontief, Perron, Potron, Sraffa

1. Reading Potron

When Emeric Lendjel (2000, 2002) discovered the forgotten work of Maurice Potron, he must have felt like a diamond hunter who finds a huge gem: Potron's (1911a) very first contribution to economic theory is based on an early application of the Perron–Frobenius theorem, one generation before it was recognized as a useful tool in economics. From 1911 onwards, Potron studied a linear model of production and had clearly in mind the structure of an input–output table. His theoretical calculations on the adjustment of production to global consumption led him to introduce a 'Leontief inverse matrix'. For these contributions Potron fully deserves to be ranked as a major precursor of input–output analysis, even if he did not exert any direct or indirect influence upon Leontief.

All economists interested in the history of their discipline will be grateful to Emeric Lendjel for this superb discovery. The praise must be extended to Gilbert Abraham-Frois, since he and Emeric Lendjel have collected and edited a substantial amount of Potron's economic papers in *Les Œuvres Economiques de l'Abbé Potron* (2004), making his work more easily accessible to economists. Unfortunately, however, their

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reading of Potron (Abraham-Frois and Lendjel, 2004, 2005, 2006) and their interpretation of his use of the Perron–Frobenius theorem are misleading. They have been lured by the apparent analogy between a well-known productivity and profitability condition that stems from a simplified Sraffa model, when labour is replaced by a physical wage basket, and one of the conditions stated by Potron, namely the dominant root of some non-negative matrix is smaller than one. We first show, in the next section, that Potron does not fit the straitjacket that Abraham-Frois and Lendjel (from now on abbreviated as AF&L) have manufactured. An alternative interpretation of Potron's writings, proposed by Bidard *et al.* (2007), is based upon a set of vector inequalities that supposedly express the core of Potron's model. However, this interpretation could be challenged on the grounds that it diverges too much from what Potron actually wrote. The origin of the difficulty lies in Potron's exceptional lack of pedagogical talent and his taste for complications when writing down equations. This explains the puzzling remark by Abraham-Frois and Lendjel: 'Potron did apply the Perron–Frobenius theorem and explicitly referred to a square matrix; however, as indicated in one referee's report, he does not explain clearly how he obtains a square matrix' (AF&L, 2006, p. 371, n. 21).

The present paper proposes a scrupulous examination of Potron's reasoning process, mainly based upon his early writings. Potron started from an application of the Perron–Frobenius theorem to the existence of a positive solution to a system of equations (as we will argue in the third section of this paper). Next, he applied this mathematical result to an economic model. The application of the Perron–Frobenius theorem then becomes transparent, and the overall conclusion is the 'Potron theorem', as we call it, in which the notion of dominant eigenvalue (of well-identified matrices!) is prominent (which is shown in the fourth section). However, Potron was not completely satisfied with the statement of the Perron–Frobenius theorem when he elaborated his result. Because he intended to apply it to matrices of the input–output type, which are likely to have some zero entries, he introduced the now well-known distinction between reducible and irreducible square matrices and generalized the theorem, with the idea to introduce the minimal hypotheses compatible with the utmost generality of his result. Our discussion will show that he did not totally succeed in this respect (in the fifth section).

2. Potron as Seen by Abraham-Frois and Lendjel

A major problem with the AF&L interpretation of Potron concerns their treatment of one of Potron's crucial magnitudes: the maximum number N of working days per year. In previous works they either ignored it (2004), or claimed that Potron assumed that one day of labour would be sufficient for the reproduction of the economic system (2005). In their most recent publication (2006), however, they try to introduce it formally. A look at the quantity equations of their model suffices to reveal the weakness of their approach.

According to AF&L, Potron's system can be represented by a 'socio-technical' matrix $\mathbf{A}^*$, similar to the one used by Sraffa (1960, §§4–5) at the beginning of his book. Following their notation, let $\mathbf{A}$ be the $[k \times k]$ matrix of commodity inputs, $\mathbf{L}$ the $[m \times k]$ matrix of labour inputs, and $\mathbf{D}$ the $[k \times m]$ matrix of consumption baskets, one for each of the m categories of labour. AF&L define the $[k \times k]$ matrix $\mathbf{A}^*$as:

$$\mathbf{A}^* = \mathbf{A} + \mathbf{DL} \tag{1}$$

which, for simplicity, is assumed to be irreducible. For a given final demand vector **y**, AF&L then write the quantity system as:

$$\mathbf{x} = \mathbf{A}^{*}\mathbf{x} + \mathbf{y} \tag{2}$$

where **x** is the total output vector. The solution of **x** is defined as:

$$\mathbf{x} = (\mathbf{I} - \mathbf{A}^{*})^{-1}\mathbf{y} \tag{3}$$

According to AF&L, Potron applied the Perron–Frobenius theorem when he showed that a strictly positive solution **x** is obtained for any semipositive final demand vector **y** if and only if there exists 'a dominant eigenvalue of the indecomposable matrix $\mathbf{A}^{*}$ which is less than 1' (AF&L, 2006, p. 363). However, since this is 'a much simplified version of Potron's model (AF&L, 2006, p. 364), a modification is required to introduce Potron's magnitude *N*. This consists of replacing the socio-technical matrix $\mathbf{A}^{*}$ by the similar-looking matrix $\mathbf{A}^{**}$ defined as:[1]

$$\mathbf{A}^{**} = \mathbf{A} + \frac{1}{N}\mathbf{DL} \tag{4}$$

Claiming that the dominant eigenvalue of $\mathbf{A}^{**}$ would be *N* times that of $\mathbf{A}^{*}$, AF&L arrive at Potron's condition that 'the characteristic number of the socio-economic state be at most equal to the number of working-days per year' (AF&L, 2006, p. 364).

Three comments are in order. First, the assertion about the relation between the dominant eigenvalues of $\mathbf{A}^{*}$ and $\mathbf{A}^{**}$ is flawed, and would certainly never have been made by a mathematician who published in prestigious journals like the *Comptes Rendus de l'Académie des Sciences* and the *Annales Scientifiques de l'Ecole Normale Supérieure*. Second, Potron indeed considered the matrix $\mathbf{A}^{**}$, but showed that its dominant eigenvalue must be at most equal to 1, not to *N*. And third, Potron defined the characteristic number of the socio-economic state as the dominant eigenvalue of another matrix, not of $\mathbf{A}^{**}$. The reason for AF&L's confusions lies in their attempt to re-establish Potron's conclusions on the basis of their a priori interpretation, thereby largely neglecting what he actually wrote. As a matter of fact, Potron never considered the matrix $\mathbf{A}^{*}$ and, to put it bluntly, Potron is not Sraffa.

3. From Perron–Frobenius to Linear Equations

We are convinced that we can only do justice to Potron by revisiting the original texts and by showing that he correctly applied the Perron–Frobenius theorem. In his first economic publication, Potron (1911a) explained his economic model but stated the main conclusions without proofs. For a full analysis of the model he referred to a forthcoming publication (Potron, 1913). In between, he summarized his results in two short notes published by the French Academy of Sciences, the first dealing with the purely mathematical aspects (Potron, 1911b) and the second showing how the results could be applied to an economic model (Potron, 1911c), and he presented his ideas less formally in Potron (1912a, 1912b). He repeatedly affirmed that the solution of his model was to a large extent based upon the results of Perron (1907) and Frobenius (1908, 1909).[2] Concentrating on Potron's early publications, we will show that this is indeed the case.

Potron's first reference to the Perron–Frobenius theorem on positive square matrices is in the opening sentence of Potron (1911b), where he briefly summarized its conclusions. Next he indicated how some of these results could be extended to non-negative square matrices, making use of a continuity argument partly outlined by Frobenius and a theorem by Minkowski. The new results could then be used to characterize the solutions of three systems of equations. One of these is system (II):

'Similarly, *when s and t are two independent parameters, and > 0, the system*

$$\text{(II)}\begin{cases} (1^*) & s\alpha_i - \sum_k a_{ki}\alpha_k = \sum_l b_{li}\beta_l \\ (4^*) & t\beta_l = \sum_i c_{li}\alpha_i \qquad\qquad (i, k = 1, \ldots, n;\ l = 1, \ldots, p), \\ (5^*) & \alpha_i > 0, \quad \beta_l > 0 \end{cases}$$

in which we have

$$a_{ik} \geqq 0,\ b_{li} \geqq 0,\ c_{li} \geqq 0,\ \sum_i c_{li} > 0 \quad (i = 1, \ldots, n;\ l = 1, \ldots, p),$$

admits solutions only, and, if $|a_{ik}|$ is not partially reduced, always if s is the characteristic root $\sigma(t)$ of maximum modulus of $\left|a_{ki} + \frac{1}{t}\sum_l b_{li}c_{lk}\right|$.' (Potron, 1911b, p. 1131)[3]

Moreover, Potron showed that if s exceeds the dominant root of matrix $|a_{ik}|$, system (II) is equivalent to another system (III), which is obtained by the substitution of:

$$t\beta_j - \sum_l p_{lj}\beta_l = 0 \quad (j, l = 1, \ldots, p) \tag{5}$$

for (4*) in system (II).[4] (We will return below to the definition of the p_{lj} coefficients.)[5] In particular, he stated that system (III) '*admits solutions only, and, if $|p_{lj}|$ is not partially reduced, always if t is the characteristic root $\tau(s)$ of maximum modulus of $|p_{lj}|$*' (Potron, 1911b, p. 1131).

It must be noted that Potron's wording reflects the state of linear algebra theory at the beginning of the 20th century, which is nowadays outdated. For instance, Potron never referred to the now common notions of inverse matrix, eigenvector, and even vector, although he did use the term matrix. His results can, however, easily be transcribed into the language of modern matrix algebra. With transparent notations, Potron's system (II) is written more compactly as:

$$s\boldsymbol{\alpha} - \boldsymbol{\alpha}\mathbf{A} = \boldsymbol{\beta}\mathbf{B} \tag{6}$$

$$t\boldsymbol{\beta} = \boldsymbol{\alpha}\mathbf{C} \tag{7}$$

$$\boldsymbol{\alpha} > \mathbf{0},\ \boldsymbol{\beta} > \mathbf{0} \tag{8}$$

in which the data are the $[n \times n]$ matrix $\mathbf{A} = |a_{ik}|$, the $[p \times n]$ matrix $\mathbf{B} = |b_{li}|$, and the $[n \times p]$ matrix $\mathbf{C} = |c_{il}|$. Potron assumed that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given and non-negative, and that no column of $\mathbf{C}$ is zero. The unknowns are the $[1 \times n]$ row vector $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_i]$ and the $[1 \times p]$ row vector $\boldsymbol{\beta} = [\boldsymbol{\beta}_l]$; the scalars s and t are two positive parameters.[6]

A first method to solve the system consists of calculating $\boldsymbol{\beta} = \boldsymbol{\alpha}\mathbf{C}/t$ by means of equation (7) and substituting it into equation (6). The transformed equation (6) becomes:

$$s\boldsymbol{\alpha} = \boldsymbol{\alpha}\left(\mathbf{A} + \frac{1}{t}\mathbf{CB}\right) \tag{9}$$

This equality means that $\boldsymbol{\alpha}$ is a row-eigenvector of the matrix $\mathbf{Q} = \mathbf{A} + \mathbf{CB}/t$, associated with the eigenvalue s. According to the Perron–Frobenius theorem, equality (9) with $\boldsymbol{\alpha} > \mathbf{0}$ requires that s is the dominant eigenvalue with maximum modulus of the matrix $\mathbf{Q}$. We will call it the dominant eigenvalue for short and denote it $dom(\mathbf{Q})$. Conversely, assume that the matrix $\mathbf{Q}$ is irreducible. The Perron–Frobenius theorem then asserts the existence of a unique (up to a factor) positive vector $\boldsymbol{\alpha}$, associated with the dominant eigenvalue s of $\mathbf{Q}$, which is a solution to equation (9). Then the vector $\boldsymbol{\beta}$ determined by means of equation (7) is positive (the hypothesis that no column of $\mathbf{C}$ is zero is used at this stage) and a solution to the system (6)–(8) is found. This is Potron's first statement above, relative to system (II).

A second method to solve the same system (6)–(7) consists of calculating $\boldsymbol{\alpha} = \boldsymbol{\beta}\mathbf{B}(s\mathbf{I} - \mathbf{A})^{-1}$ by means of equation (6) and substituting it into equation (7). The transformed equation (7) becomes:

$$t\boldsymbol{\beta} = \boldsymbol{\beta}\mathbf{B}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{C} \tag{10}$$

The elements of the matrix $\mathbf{P} = \mathbf{B}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{C}$ coincide with Potron's p_{lj} coefficients.[7] Let $r \equiv dom\,(\mathbf{A})$; the assumption $s > r$ ensures that the inverse matrix $(s\mathbf{I} - \mathbf{A})^{-1}$ exists and is non-negative – another consequence of the Perron–Frobenius theorem. The equality (10) means that $\boldsymbol{\beta}$ is a row-eigenvector of the non-negative matrix $\mathbf{P}$, associated with the eigenvalue t. Once more, the Perron–Frobenius theorem asserts that the positivity of $\boldsymbol{\beta}$ requires that t is the dominant eigenvalue of $\mathbf{P}$. Conversely, assume that $\mathbf{P}$ is irreducible. The Perron–Frobenius theorem asserts the existence of a unique (up to a factor) positive vector $\boldsymbol{\beta}$, associated with the dominant eigenvalue of $\mathbf{P}$, which is a solution to equation (10). If the matrix $\mathbf{B}$ has no zero column, the right-hand side row-vector $\boldsymbol{\beta}\mathbf{B}$ in equation (6) is positive, and, as a consequence of a sixth application of the Perron–Frobenius theorem, the row-vector $\boldsymbol{\alpha} = \boldsymbol{\beta}\mathbf{B}(s\mathbf{I} - \mathbf{A})^{-1}$ is also positive. Hence we arrive at Potron's second statement above, relative to system (III).

A comparison of the two proofs shows that, in the first, s is the dominant eigenvalue of $\mathbf{Q}$, whereas in the second t is the dominant eigenvalue of $\mathbf{P}$. This is a general property: '*One also sees that* [...] *the conditions: $t > 0$, s at least equal (or equal) to the characteristic root of maximum modulus of $\left|a_{ik} + \frac{1}{t}\sum_l b_{lk}c_{li}\right|$ amount to the conditions: $s > r$, t at least equal (or equal) to the characteristic root of maximum modulus of $|p_{lj}|$*' (Potron, 1911b, p. 1131).

We will return to the proofs in the fifth section, but two points can be underlined immediately to explain the evolution of Potron's precise statements between 1911 and 1913. (i) Scrutinizing the first proof shows that the irreducibility hypothesis on the matrix $\mathbf{A}$ (matrix $\mathbf{A}$ is not 'partially reduced', in the above quotation) can be replaced by the weaker hypothesis that the matrix $\mathbf{Q}$ is irreducible, which is indeed the hypothesis retained in the 1913 version (Potron, 1913, p. 60). (ii) In the second proof, once the positivity of vector $\boldsymbol{\beta}$ has been established, the hypothesis that the matrix **B** has no zero column is used to transfer the property to $\boldsymbol{\alpha}$. In the 1911b version, Potron had forgotten this condition, but in the 1913 version he corrected this lacuna of the initial statement and introduced the missing hypothesis (Potron, 1913, p. 58).

4. From Linear Equations to Production

Potron concluded his first note for the Academy of Sciences by the sentence: 'These remarks immediately apply to the solution of important economic problems' (Potron, 1911b, p. 1132), which he very briefly explained in a second note (1911c). In his much more extensive 1913 paper he treated both the mathematical and economic aspects more thoroughly. The following presentation adheres to Potron's original notation, although we admit that for a comparison of the mathematical results and the economic model Potron could have used a more transparent notation.[8] The data of Potron's economic model consist of three matrices **A**, **B** and **T** and a scalar N (1911c, pp. 1458–1459; 1913, p. 65).

Let there be s goods and r types of labour. Every good is produced by means of goods and labour(s). The ith row of the $[s \times s]$ matrix **A** and of the $[s \times r]$ matrix **T** represent respectively the amounts of the various physical inputs and of the various types of labour used to produce one unit of good i (the unit of measurement for labour is the working day). Constant returns to scale are implicitly assumed. With regard to the now standard input–output analysis, note that the differences are: (i) the methods of production are described in rows, not in columns; (ii) labour is heterogeneous; and, most significantly, (iii) the coefficients of the input matrices represent physical magnitudes, a feature that allows Potron to use these matrices to deal with prices and wages, a part of the model that is not considered at this stage.

Each labourer of type or profession h ($h = 1, \ldots, r$) belongs to the social category h, and a member of that category (more precisely, his family) consumes a given and specific basket per year, which corresponds to his 'type of existence'. Each basket is represented by a $[1 \times s]$ row-vector, which stacked together define the $[r \times s]$ matrix **B** of yearly consumption baskets. N is the maximum number of working days per year, the same for any industry and any profession.

Potron's full model contains quite a number of additional variables. For simplicity, and in order to minimize the range of our disagreements with AF&L's interpretation of the model, we concentrate here on the limit case characterized by the absence of overproduction, of 'simple consumers', and of unemployment.[9] Given the data **A**, **B**, **T** and N which describe the 'socioeconomic state', Potron's first problem consists of defining a 'satisfactory regime of production and labour'. The question is to determine the activity level δ_i of every industry i ($i = 1, \ldots, s$), which coincides with the production of good i, and the number π_{ih} of workers of type h ($h = 1, \ldots, r$) employed in every industry i, such that the requirements of 'sufficient production' and 'right to rest' are met. Before we explain these requirements, let us observe that Potron introduced r auxiliary variables Π_h, the total number of workers of type h (see equation (16) in Potron, 1913, p. 65):

$$\Pi_h = \sum_i \pi_{ih} \quad (h = 1, \ldots, r) \tag{11}$$

The requirement of 'sufficient production' stipulates that the production of every commodity i must be equal to the sum of industrial consumption and final consumption. Hence the s equations (Potron, 1913, p. 65, equation (17)):

$$\delta_i = \sum\nolimits_k a_{ki}\delta_k + \sum\nolimits_h b_{ih}\Pi_h \quad (i = 1, \ldots, s) \tag{12}$$

According to the requirement of the 'right to rest', no labourer is obliged to work more than N days per year. Because of the full employment hypothesis, each labourer works exactly N days. This means that in every industry i the available amount of labour of type h, i.e. $N\pi_{ih}$ days, is exactly equal to what is needed for production at the activity level δ_i, i.e. $\delta_i t_{ih}$. Hence the sr equations (Potron, 1913, p. 65, equation (18)):

$$N\pi_{ih} = \delta_i t_{ih} \quad (i = 1, \ldots, s; \; h = 1, \ldots, r) \tag{13}$$

Of course the issue concerns the existence of an economically meaningful solution, i.e. positive activity levels and positive total numbers of workers (Potron, 1913, p. 66, equation (22)):

$$\delta_i > 0, \quad \Pi_h > 0, \quad \pi_{ih} \geq 0 \quad (i = 1, \ldots, s; \; h = 1, \ldots, r) \tag{14}$$

Does there exist a solution (δ_i, Π_h, π_{ih}) of the system (11)–(14)? Making use of the variables Π_h and summing over the industries in equation (13), Potron reduced this system of $r + s + sr$ equations and $r + s + sr$ inequalities to a system of only $r + s$ equations and $r + s$ inequalities (Potron, 1913, p. 67, system (IV)):

$$\delta_i - \sum\nolimits_k a_{ki}\delta_k = \sum\nolimits_h b_{ih}\Pi_h \tag{15}$$

$$\Pi_h = \frac{1}{N}\sum\nolimits_k \delta_k t_{kh} \tag{16}$$

$$\delta_i > 0, \quad \Pi_h > 0 \tag{17}$$

Suppose for a moment that we can find a solution (δ_i, Π_h) of this simpler system (15)–(17). Then let us define the numbers of labourers per industry π_{ih} by the equalities $\pi_{ih} = \delta_i t_{ih}/N$, which are compatible with equation (11) because equation (16) holds. Since these π_{ih} are all non-negative, the existence of a solution of the system (15)–(17) implies that of a solution of the system (11)–(14). Hence a satisfactory regime of production and labour is obtained.

As for the system (15)–(17), it is easily recognized that it is an example of Potron's system (II) studied in the third section, after appropriate changes in the designations of the data, the unknowns and the index ranges, and choosing the parameter values $s = 1$ and $t = N$ (Potron, 1911c). To facilitate the comparison with the system (6)–(8), let us write the system (15)–(17) into a matricial form, where $\boldsymbol{\delta}$ and $\boldsymbol{\Pi}$ are row vectors:

$$\boldsymbol{\delta} - \boldsymbol{\delta}\mathbf{A} = \boldsymbol{\Pi}\mathbf{B} \tag{18}$$

$$N\boldsymbol{\Pi} = \boldsymbol{\delta}\mathbf{T} \tag{19}$$

$$\boldsymbol{\delta} > \mathbf{0}, \quad \boldsymbol{\Pi} > \mathbf{0} \tag{20}$$

Under the hypotheses on the data mentioned in the third section, the previous mathematical result asserts the existence of a positive solution ($\boldsymbol{\delta}$, $\boldsymbol{\Pi}$), which is unique up to a factor (the normalization factor may be the size of the working population) provided that the following condition is met: $\sigma \equiv dom(\mathbf{A} + \mathbf{TB}/N) = 1$.[10] With the dominant eigenvalue of matrix $\mathbf{A}$ being assumed to be smaller than 1, this condition is equivalent to: $\nu \equiv \text{dom}\ (\mathbf{B}(\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{T}) = N$. Matrices $\mathbf{A} + \mathbf{TB}/N$ and $\mathbf{B}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{T}$ are both square, the first with a dimension equal to the number of goods, the second with a dimension equal to the number of types of labourers (or consumers).

Potron's analysis was not limited to the special case examined above. He generalized the notion of a 'satisfactory regime of production and labour' in order to take into account the presence of 'simple consumers' (i.e. rentiers) and the possibility of overproduction and unemployment (some labourers work less than the maximum number of days N). The activity levels and the distribution of the professions among industries must then be such that the overall consumption does not exceed the overall net production (principle of 'sufficient production') and no labourer works more than the maximum number of working days per year (principle of 'right to rest'). Formally, the equalities in equations (18)–(19) are replaced by vector inequalities of the type $\geq$. The existence proof for this general case relies on another ingenious application of the Perron–Frobenius theorem and the result is that the two eigenvalue equalities $\sigma = 1$ and/or $\nu = N$ are transformed into the equivalent inequalities $\sigma \leq 1$ and/or $\nu \leq N$.

Moreover, Potron extended the analysis to the value side of the model, introducing the notion of a 'satisfactory regime of prices and wages'. A satisfactory regime of prices and wages is defined by positive prices $\mathbf{p}$ and positive wages $\mathbf{w}$ such that the benefits in any production are non-negative (principle of 'justice in exchange') and the yearly wage of every labourer – more generally the yearly income of every household – is at least equal to the value of the consumption basket corresponding to his social category (principle of 'right to life'). In the limit case, these conditions are written as:

$$\mathbf{p} - \mathbf{Ap} = \frac{1}{N}\mathbf{Tw} \tag{21}$$

$$\mathbf{w} = \mathbf{Bp} \tag{22}$$

$$\mathbf{p} > \mathbf{0}, \quad \mathbf{w} > \mathbf{0} \tag{23}$$

with the vector of prices $\mathbf{p}$ and the vector of annual wages $\mathbf{w}$, both column vectors.[11] In the general case, these equalities are replaced by vector inequalities of the type $\geq$, which means that (some) firms have positive benefits and/or (some) workers positive economies. The existence proof for the value side is similar to the one for the physical side, with the understanding that a characterization of the latter determines the effective number of working days per year and is needed to know the yearly wage and solve the former. Potron's main economic results are summarized as follows (Potron, 1913, pp. 68, 70–71).

The Potron Theorem. Consider a socioeconomic system described by the $[s \times s]$ material input matrix $\mathbf{A}$, the $[s \times r]$ labour input matrix $\mathbf{T}$, the $[r \times s]$ matrix of yearly consumption baskets $\mathbf{B}$, and the maximum number of working days N. Let us assume that $\mathbf{A}$ is non-negative and has a dominant eigenvalue smaller than 1, that every column of $\mathbf{T}$ is semipositive, that every column of $\mathbf{B}$ is semipositive, and that N is positive. The dominant characteristic value σ of $\mathbf{Q} \equiv \mathbf{A} + \mathbf{TB}/N$ and that ν of $\mathbf{P} \equiv \mathbf{B}(\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{T}$ are such that

$\sigma > 1$ (respectively $\sigma < 1$, $\sigma = 1$) if and only if $\nu > N$ (respectively $\nu < N$, $\nu = N$). A satisfactory regime of production and labour and a satisfactory regime of prices and wages both exist if $\sigma < 1$ (or $\nu < N$). Neither of them exists if $\sigma > 1$ (or $\nu > N$). They exist in the limit case $\sigma = 1$ (or $\nu = N$) if either matrix **Q** or matrix **P** is irreducible.

The eigenvalue ν is what Potron called the 'characteristic number of the socioeconomic state'. It admits a simple economic interpretation: in Potron's own words, it stands for the '*average number of normal working days that a labourer must provide so that the annual production obtained represents exactly the exclusive consumption of all labourers*' (Potron, 1913, p. 70). Potron organized the economic discussion in terms of a comparison between the values of ν and N (in the 1913 paper, he proposed $N = 313$ days as the maximum number of working days per year).

5. Discussion

Some general comments on Potron's statements, technical treatment and hypotheses are in order. For the sake of clarity, we maintain the requirement that the physical and value unknowns (activity levels, employment, prices and wages) are positive, even if Potron also introduced the notion of a 'semi-satisfactory' system to deal with the limit case $\nu = N$ when the matrices **P** and **Q** are reducible.

We have already noticed at the end of the third section that Potron modified his hypotheses from one paper to the other, either in order to generalize the results and have statements fitting with the proofs (replacement of the irreducibility of **A** by that of **Q**), or to stick a patch on a logical weakness (introduction of a semipositivity condition on the rows of **C**). It must also be remarked that the 1913 paper contains no isolated statement in which the assumptions and the conclusions are grouped together in order to facilitate the discussion. These are but minor examples of the complications introduced by Potron's notations and writing style in general.

Our discussion starts from the observation that the notion of satisfactory regime of production and labour is defined by the equations (18)–(20), and that the use of either the matrix **P** or the matrix **Q** corresponds to two different ways to solve the system. It suffices that one of them works to ensure the existence result. We will first examine the validity of the results under Potron's hypotheses; next, we return to the initial problem and show the existence of *three* parallel approaches; we finally use these reflections to alleviate and unify Potron's hypotheses.

5.1. Existence and Duality Results

Let us first notice that Potron introduced irreducibility hypotheses only in the limit case $\sigma = 1$ (or $\nu = N$). But suppose that the strict inequality $\sigma < 1$ (or $\nu < N$) holds. The existence of a satisfactory regime of production and labour amounts to discussing the existence of positive solutions $\mathbf{x}$ to a vector inequality of the general type $\mathbf{xF} \leq \mathbf{x}$, or to a vector equality of the type $\mathbf{xF} + \mathbf{d} = \mathbf{x}$, $\mathbf{d}$ being a semipositive vector. In economic terms, the components of vector $\mathbf{x}$ are what Potron called the 'primary unknowns', namely the levels of 'production' and 'the distribution of workers', whereas vector $\mathbf{d}$ is obtained as a mix of 'secondary unknowns', namely 'the numbers of non-workers' (i.e. simple consumers), 'unemployment' and the levels of 'overproduction' (Potron, 1935, p. 63).

To ensure that '*there always exists a satisfactory regime of production and labour in which [...] one can take arbitrarily the numbers of simple consumers in each social category and the collective unemployment in each category of workers*' (Potron, 1913, p. 68), it must be the case that the vector $\mathbf{x} = \mathbf{d}(\mathbf{I} - \mathbf{F})^{-1}$ is positive for any semipositive vector **d**, and this requires the irreducibility of **F**. A similar conclusion holds for the dual model in terms of value, where the primary unknowns are the prices and the wages, whereas the secondary unknowns are 'the firms' benefits and the workers' economies' (Potron, 1913, p. 68). We must therefore extend the irreducibility hypotheses to the case $\sigma < 1$ (or $\nu < N$).

As for the duality result, Potron underlined the formal similarities between the notions of satisfactory regimes of production and labour, on the one hand, and that of prices and wages, on the other hand: '[T]he matrices in the first members are transposed from each other' (Potron, 1913, p. 67). He seems however to have forgotten that the existence of a positive solution must be checked for each regime separately. For the physical system, Potron introduced the hypotheses that **B** has no zero column (in connection with the use of matrix **P**), and that **T** has no zero column (in connection with the use of matrix **Q**). To guarantee an existence result for the dual problem, he should have introduced the symmetric hypotheses that **B** and **T** have no zero row.

5.2. *The Three Potron Matrices*

We have noticed that Potron used two independent methods of resolution of the system (15)–(17): the **P**-approach first eliminates the activity levels and matrix **P** allows us to determine employment; then the activity levels are obtained. The **Q**-approach proceeds the other way round: it first eliminates the employment variables, then determines the activity levels by means of matrix **Q**, and finally calculates employment. A third method of resolution, or **R**-approach, is more natural: since the system we are studying is linear, one may introduce a matrix from the very beginning. It is immediately seen that, for the non-negative square matrix **R** of dimension $r + s$ defined as:

$$\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \frac{1}{N}\mathbf{T} & \mathbf{A} \end{bmatrix} \tag{24}$$

the existence of a satisfactory regime of production and labour and that of a satisfactory regime of prices and wages are respectively written as:

$$(\boldsymbol{\Pi}, \boldsymbol{\delta})\mathbf{R} = (\boldsymbol{\Pi}, \boldsymbol{\delta}), \quad \boldsymbol{\Pi} > \mathbf{0}, \quad \boldsymbol{\delta} > \mathbf{0} \tag{25}$$

and

$$\mathbf{R}\begin{pmatrix} \mathbf{w} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ \mathbf{p} \end{pmatrix}, \quad \mathbf{w} > \mathbf{0}, \quad \mathbf{p} > \mathbf{0} \tag{26}$$

The general definitions of satisfactory systems are obtained by replacing the equalities by inequalities of the type $\leq$. Clearly, condition *dom* $(\mathbf{R}) \leq 1$ is necessary for the existence of satisfactory regimes, and that condition is sufficient if **R** is irreducible. Incidentally, Potron

(1937, first conference, §3) did consider a non-negative matrix akin to **R,** but without exploring its mathematical properties. This $[(r+s) \times (r+s)]$ matrix is defined as:

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{T} & \mathbf{A} \end{bmatrix} \tag{27}$$

simply derived from **R** by replacing the submatrix $\mathbf{T}/N$ by **T**. Potron emphasized the 'fundamental' role of this 'tableau' – a type of comprehensive input–output matrix, which includes the physical input–output coefficients in the submatrix **A**, the labour coefficients in the sub-matrix **T**, and the yearly consumption baskets in the submatrix **B** – and urged economists to collect the statistical data necessary for its construction. Matrices **M** and **R** are very close, and in particular, are simultaneously reducible or irreducible.

When one examines the relationships between the three Potron matrices **P**, **Q** and **R**, the basic idea is that they are used to solve a unique problem in three different ways, but the conclusions we arrive at only characterize the conditions of existence of satisfactory regimes and do not depend on the way to solve the problem. A first illustration of this point of view is the equivalence between the conditions $dom(\mathbf{P}) = N$, $dom(\mathbf{Q}) = 1$ and $dom(\mathbf{R}) = 1$ (or $dom(\mathbf{P}) < N$, $dom(\mathbf{Q}) < 1$ and $dom(\mathbf{R}) < 1$ respectively). Potron did show this equivalence as far as matrices **P** and **Q** are concerned, and the property extends to **R**.

5.3. The Irreducibility Hypotheses

Apart from the quantitative restrictions on the dominant eigenvalues, the existence and duality results also rely on hypotheses relative to the distribution of the zero components in some vectors or matrices. Conditions of that type have already been mentioned, but Potron's treatment is loose and partly misleading.

Potron did not give an economic interpretation of these conditions. In the **Q**-approach, the technical assumptions that **T** has no zero column and **B** no zero row mean that every type of labour is used in some industry and that each consumption basket is non-zero. From an economic point of view, these hypotheses are natural, and we accept them without further ado. By contrast, the assumptions associated with the **P**-approach are that **B** has no zero column and **T** no zero row: they respectively mean that every good enters the consumption basket of some workers, thus discarding pure production goods like furnaces, and that every method of production uses some labour, thus discarding the ageing process for wine. None of these hypotheses is welcome in economics. To see if they can be alleviated, we return to the system (18)–(20). The **P**-approach amounts to calculating $\boldsymbol{\delta}$ as a function of $\boldsymbol{\Pi}$ in equation (18) and substituting the expression in equation (19) in order to characterize $\boldsymbol{\Pi}$ as the Perron-Frobenius eigenvector of the matrix $\mathbf{P} = \mathbf{B}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{T}$. Vector $\boldsymbol{\Pi}$ is positive by the irreducibility hypothesis on **P**. Then, a sufficient condition for the positivity of $\boldsymbol{\delta} = \boldsymbol{\Pi}\mathbf{B}(\mathbf{I} - \mathbf{A})^{-1}$ is that **B** has no zero column (this is the condition retained by Potron), but a weaker condition is:

$$\mathbf{B}(\mathbf{I} - \mathbf{A})^{-1} \quad \text{has no zero column} \tag{28}$$

Similarly, in the value problem, the condition that **T** has no zero row can be replaced by:

$$(\mathbf{I}-\mathbf{A})^{-1}\mathbf{T} \quad \text{has no zero row} \tag{29}$$

Condition (29) means that the direct and indirect labour contents of any commodity are non-zero, whereas the initial condition on **T** only considered direct labour.

It can be shown that under the natural economic conditions mentioned above, the irreducibility of **P** combined with equations (28) and (29) is equivalent to the irreducibility of **Q**, which itself is equivalent to that of **R** (see Bidard and Erreygers, 2007, for a formal proof). Clearly enough, the irreducibility of **R** implies the positivity of production and employment in the physical system, and that of prices and wages in the value system. The overall conclusion is that Potron's existence and duality result relies on a unique condition on the level of a dominant eigenvalue and a unique irreducibility condition, even if these conditions take slightly different (but equivalent) forms according to the matrix **P**, **Q** or **R** in terms of which they are expressed.

6. Conclusion

Abraham-Frois and Lendjel fail to reveal the truly essential connection between Potron's economic model and the Perron–Frobenius theorem. On the basis of a superficial resemblance between one possible form of the Potron condition and the productivity requirement which appears in a simplified Sraffa model, they proceed to the identification of both models. Our disagreement with this approach is primarily methodological. It leaves no room for some important aspects of Potron's model and its specific concepts, such as the number of working days per year, the reflection on unemployment and the role of 'simple consumers'. From an analytical point of view, it makes the application of the Perron–Frobenius theorem unclear and it blurs the distinction that Potron introduced between 'primary' and 'secondary' unknowns. The only reason, though not justification, of this shortcut method is that the original model is rather badly explained. As we have tried to show, however, Potron's text is devoid of ambiguity and, if one blows off the dust, the link with the Perron–Frobenius theorem comes shining through.

In a nutshell, we can say that Potron was a non-economist with no common sense in his approach of social and economic problems, a good mathematician with a special taste for off-putting calculations, and a writer with scarcely any didactical skills. What remains is that this amateur with admirable intuitions has made important contributions to input–output analysis and economics in general, well in advance of his time. We dare hope that readers will be convinced that it is worth reading Potron attentively.

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Notes

[1]Please note that $\mathbf{A}^{**}$ is our notation, which we introduce for reasons of clarity; AF&L use $\mathbf{A}^{*}$ both for $\mathbf{A}+\mathbf{DL}$ and for $\mathbf{A}+\mathbf{DL}/N$.

[2]See, for example, Potron (1912a, p. 291); curiously, he designated Perron as a Swedish mathematician, when he was in fact just as German as Frobenius.

[3]All citations from Potron have been translated by us. We are preparing an English edition of Potron's collected economic papers (Bidard and Erreygers, 2008).

[4]In 1911b, Potron mistakenly substituted equation (5) for (1*), but the new system (4*)–(5) would then have $2p$ equations and $n + p$ unknowns. In Potron (1913, p. 60) he wrote down correctly the system (1*)–(5), with $n + p$ equations and $n + p$ unknowns.

[5]In order to avoid confusion with the b_{li} coefficients, we diverge here from Potron's original notation.

[6]The scrupulous reader may notice that, here and there, we permute the indices with regard to Potron's notations. These implicit transpositions simplify and clarify Potron's formalization without affecting the results.

[7]Since the notion of an inverse matrix was unusual at the beginning of the 20th century, Potron referred to these coefficients by means of the matrix of cofactors.

[8]This choice facilitates the reader's task when checking the faithfulness to the original texts, which is the question at stake. A more subtle goal is to give a flavour of Potron's noticeable lack of pedagogical talent: his pupils found his lectures obscure and, on one occasion, they protested against his use of counter-intuitive notations. We would like to let the reader benefit from the richness of these notations (e.g. note that letter *b* refers to consumption).

[9]In terms of the 1913 paper, the first assumption implies $\rho_i = 0$ $(i = 1, \ldots, s)$, the second $\bar{\omega}_h = 0$ $(h = 1, \ldots, r)$, and the third $\omega_{ih} = 0$ $(i = 1, \ldots, s;\ h = 1, \ldots, r)$.

[10]Incidentally, note that Potron's units of measure are complex but consistent: the entries of matrix $\mathbf{B}$ are measured in terms of consumption per year and per labourer. Those of matrix $\mathbf{T}$ are measured in terms of working days per labourer but, after division by the number N of working days per year, $\mathbf{T}/N$ is measured in years per labourer. On the whole, the matrix $\mathbf{TB}/N$ and the input matrix $\mathbf{A}$ have the same physical nature and can be added.

[11]Here we slightly deviate from Potron, who wrote the system with daily rather than annual wages. If $\mathbf{s}$ is the vector of daily wages, Potron's original equations can be obtained by substituting $N\mathbf{s}$ for $\mathbf{w}$.

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