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MAURICE POTRON'S LINEAR ECONOMIC MODEL: A *DE FACTO* PROOF OF 'FUNDAMENTAL MARXIAN THEOREM'

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ABSTRACT

Maurice Potron was a French mathematician whose largely unknown contributions to economic analysis should be acknowledged as pioneering achievements. The purpose of this paper is to (1) reconstruct Potron's core model in terms of modern input–output analysis, (2) show that Potron proved *de facto* Fundamental Marxian Theorem (FMT) 48 years earlier than Morishima, Seton and Okishio by adapting the Perron–Frobenius theorems to economic problems, and (3) claim that Potron proved *de facto* FMT by considering heterogeneous labours 65 years earlier and even more generally than Bowles and Gintis.

1. INTRODUCTION

In their book published in 2004, G. Abraham-Frois and É. Lendjel reprinted the economics-related writings of the French mathematician Maurice Potron (1872–1942), which had previously been overlooked by researchers in the history of economics. This book contains 12 of Potron's articles published from 1911 to 1941, prefaced with a biography acknowledging his pioneering achievements.

However, the 'modern reformulation of Maurice Potron's model' by the editors of Abraham-Frois and Lendjel (2004, 2006) is *incomplete* because it considers only the existence theorem, and that only in the case of indecomposability. The reformulation is *incorrect* because it fails to distinguish

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between the existence conditions of some equilibriums[1] or to derive ν, Potron's 'characteristic number of the economico-social system'.[2] In doing so, they missed a close affinity between Potron's propositions and Fundamental Marxian Theorem (FMT).[3]

This affinity is surprising because Potron, as a Jesuit, had a different ideological background[4] from Marx, and he, as a mathematician, was obviously not versed in economic literature, much less in Marxian economics. He did not use even the very notions of 'rate of exploitation' or 'rate of profit'[5] (although he used the notion of 'bénéfice'), and it is more than possible that he did not intend to prove any propositions related to FMT.

In this paper, we examine an *analytical implication* of Potron's economic model. First, we focus on the *analytical* argumentation of his economic theory and disregard its political, ethical or metaphysical meanings and judgements. In doing so, we find a close affinity of purely analytical nature between his propositions and FMT. Second, we deduce the implications of his economic model. Thus, we may derive more implications than Potron. In the course of this deduction, however, we propose one substantial interpretation: 'characteristic number of the economico-social system' as 'necessary labour' in Marx's sense, i.e. the difference between working time and the 'characteristic number' as 'surplus labour'. Once this interpretation is established, FMT can be derived from his model in a logically compelling way, and the deduction can be done by trivially extending his original arguments. Therefore, we claim that Potron proved *de facto* FMT.

As for the above-mentioned interpretation, because Potron himself did not make any suggestions, we assume its full responsibility. However, because of the definition of the term, we think it is tenable to interpret 'the average working time that labourers must provide to produce their consumption' as 'necessary labour'.

For a better understanding of Potron's economic model in section 2, we present here some essential points of its characteristics that should be explained in more detail in section 3. His linear economic model is so general

[1] Contrary to Abraham-Frois and Lendjel (2004, n. 42, p. 42) and Abraham-Frois and Lendjel (2006, n. 15, p. 370), SSRPW is *not* formally identical to ESRPW. See sections 2.4.1 and 3.3.

[2] See the incorrect multiplication of α by N in Abraham-Frois and Lendjel (2006, p. 364).

[3] In this paper, it is not claimed that FMT is faithful to Marx. It must be noted that FMT does not require the labour theory of value as a component of the price theory, which also holds true for Potron's model.

[4] See Abraham-Frois and Lendjel (2004, 2006) and Bidard *et al.* (2006).

[5] As well known, the rate of profit in Marx's sense is independent of the secondary distribution of the profit to social members as various revenues. In this sense, it is not against the Marxian notion to classify 'bénéfice' as profit.

that (1) any number of heterogeneous categories of labourers, each having different professional skills and consumption preferences, can be considered, (2) any amount of working time can be considered, (3) the labour market need not be equilibrating, and (4) price making (including wage setting) is not restricted to the case of perfect competition, i.e. it allows different wage and profit rates for each sector. Such a high degree of generality of the model explains the possibility of deducing a wide range of implications from his propositions, not only for the case of perfect competition and full employment but also for other types of markets.

Because we have focused on Potron's analytical contributions, we primarily investigated his writing of 1913, which stands out as the first of his economic studies that contain the full range of his mathematical reasoning. In other economics-related writings, Potron sets up a linear production model that is indeed identical in its content, except for its use of symbols,[6] numerical illustrations and practical applications, and repeatedly investigates the existence conditions of equilibrium solutions with respect to both quantity and the price system.

In the next section, we summarize Potron's arguments using modern expressions. While doing so, we do some justice to Potron's claim of extending Perron–Frobenius theorems on indecomposable matrices[7] to decomposable ones prior to Frobenius. Section 3 explains some unique characteristics of his economic model. In section 4, after introducing our interpretation, we deduce an important implication from Potron's propositions and show that he proved the generalized FMT *de facto*. Section 5 appraises the importance of his propositions in the history of Marxian mathematical economics.

2. POTRON'S ECONOMIC MODEL

Potron distinguishes constants and variables, and gives them the following definitions (we have substituted contemporary notations and symbols for his

[6] We can refer to Abraham-Frois and Lendjel (2004), who drew up a comparative table for symbols used in various versions of Potron's model.

[7] See Potron (1913, p. 53, footnote 1). One generally understands the Perron–Frobenius theorems to be the propositions stated by Frobenius (1908, 1909, 1912) and partly by Perron (1907). As we know, Frobenius had already extended his theorems to non-negative matrices in Frobenius (1908, section 4). Potron's claim of priority must be acknowledged with respect to his having investigated the general formulation of non-negative eigenvectors associated with the Frobenius root in the case of decomposable matrices (Potron 1913, section 6, pp. 56–58). Furthermore, it must be noted here that Potron had already articulated and proved the so-called Hawkins–Simon condition. See Potron (1913, section 9 and footnote 1, pp. 62–63) and also Bidard (2007).

original ones). Note that we use inequality signs for vectors and matrices in this paper so that $X > Y$, $X \geq Y$ and $X \geqq Y$ denote that $X - Y$ is positive, semi-positive and non-negative, respectively.

2.1 Constants

- input coefficient of good i for producing good j: $a_{ji} \in \mathbb{R}_+$
- input-coefficient matrix: $A := (a_{ji}) \in M(n \times n, \mathbb{R}_+)$
- Frobenius root, i.e. the real eigenvalue of maximal modulus, of A: $\alpha \in \mathbb{R}_+$
- labour input coefficient of category k for producing good j: $l_{jk} \in \mathbb{R}_+$
- quantity of good i consumed annually by a member of category k: $b_{ki} \in \mathbb{R}_+$
- number of annual working days: $N \in \mathbb{R}_{++}$
- augmented input-coefficient matrix: $C(N) := A + \frac{1}{N} LB \in M(n \times n, \mathbb{R}_+)$
- Frobenius root of $C(N)$: $\mu(N) \in \mathbb{R}_+$

In order to simplify our formulas in reconstructing Potron's model, we introduce, for convenience, the following definitions:

- labour-input-coefficient matrix: $L := (l_{jk}) \in M(n \times m, \mathbb{R}_+)$
- consumption-coefficient matrix: $B := (b_{ki}) \in M(m \times n, \mathbb{R}_+)$

2.2 Variables

- activity level of unit production process of good j: $x_j \in \mathbb{R}_+$
- price of good i: $p_i \in \mathbb{R}_+$
- time of unemployment of category k in production of good j: $u_{jk} \in \mathbb{R}_+$
- number of labours of category k employable for producing good j: $\lambda_{jk} := \frac{1}{N}(x_j l_{jk} + u_{jk}) \in \mathbb{R}_+$
- number of non-labourers of category k: $\kappa_k \in \mathbb{R}_+$
- number of consumers of category k: $\omega_k := \kappa_k + \sum_j \lambda_{jk} \in \mathbb{R}_+$
- value of annual consumption per member of category k: $\gamma_k := \sum_i b_{ki} p_i$
- daily money wage rate of labourers of category k in production of good j: $w_{jk} \in \mathbb{R}_+$
- savings from w_{jk}: $s_{jk} \in \mathbb{R}_+$
- profit contained in the price of good j: $\pi_j \in \mathbb{R}_+$

In order to simplify our formulas, we introduce, for the sake of convenience, the following definitions:

- activity vector: $x := (x_1, \cdots, x_n) \in \mathbb{R}_+^n$
- (column) price vector:[8] $p := (p_1, \cdots, p_n)' \in \mathbb{R}_+^n$
- vector of unemployment: $u := \left(\sum_j u_{j1} \cdots \sum_j u_{jm} \right) \in \mathbb{R}_+^m$
- vector of κ_k: $\kappa := (\kappa_1 \cdots \kappa_m) \in \mathbb{R}_+^m$
- vector of ω_k: $\omega := (\omega_1 \cdots \omega_m) \in \mathbb{R}_+^m$
- (column) vector of π_j: $\pi := (\pi_1 \cdots \pi_n)' \in \mathbb{R}_+^n$
- money-wage matrix: $W := (l_{jk} w_{jk}) \in M(n \times m, \mathbb{R}_+)$

2.3 Equilibrium conditions

Potron then sets up the following six equilibrium conditions:

(C.1) $x = xA + \omega B$
(C.2) $x > 0,\ \omega > 0,\ \kappa \geqq 0,\ \lambda_{jk} \geqq 0,\ u_{jk} \geqq 0$
(C.3) $p = Ap + W\iota + \pi,\ \iota = (1 \cdots 1)'$
(C.4) $w_{jk} = \left(\frac{1}{N} \gamma_k + s_{jk} \right)$
(C.5) $x_j l_{jk} w_{jk} = \lambda_{jk}(\gamma_k + N s_{jk}) = (x_j l_{jk} + u_{jk}) \left(\frac{1}{N} \gamma_k + s_{jk} \right)$
(C.6) $p > 0,\ \pi \geqq 0,\ \gamma_k > 0,\ w_{jk} > 0,\ s_{jk} \geqq 0$

2.4 Definitions

2.4.1 Definitions of the equilibriums

Depending on which equilibrium conditions are to be met, Potron classifies equilibriums into three types.[9]

(D.1) The state of variables x_j, ω_k, λ_{jk}, u_{jk}, κ_k ($j = 1, \cdots, n$; $k = 1, \cdots, m$) satisfying the conditions (C.1) and (C.2) is defined as a *satisfying regime of production and labour* (**SRPL**) ('régime satisfaisant de production et travail').

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[8] Prime applied to matrices and vectors denotes, as usual, their transposition.
[9] This classification had already appeared in Potron (1911b, p. 1459).

> We have to examine three problems successively [. . .]: 1° Whether it is possible to determine an allocation of labourers to diverse professional and social categories and an allocation of simple consumers to diverse social categories, such allocations that for each labour product, the production can be equal to the consumption without the requirement that a labourer has to work for more than the workable days: I will call all regimes of production and labour satisfying these two conditions *satisfying.* (Potron, 1913, p. 64)

(D.2) The state of variables p_i, γ_k, w_{jk}, s_{jk}, π_j (i, $j = 1, \cdots, n$; $k = 1, \cdots, m$) satisfying the conditions (C.3), (C.4) and (C.6) is defined as a *simply satisfying regime of price and wages* (**SSRPW**) ('régime simplement satisfaisant de prix et salaires').

> 2° Whether it is possible to determine the set of prices and wage rates in the manner that for every labour product, the price of exchange is at least equal to the cost price, and for every labourer, the wage corresponding to the maximum of performed labour is at least equal to the cost of life. I will call all regime of price and wages satisfying these two conditions *simply satisfying*. (Potron, 1913, p. 64)

(D.3) The state of variables x_j, ω_k, λ_{jk}, u_{jk}, κ_k, p_i, γ_k, w_{jk}, s_{jk}, π_j (i, $j = 1, \cdots, n$; $k = 1, \cdots, m$) satisfying the conditions (C.1), (C.2), (C.3), (C.5) and (C.6) is defined as an *effectively satisfying regime of price and wages* (**ESRPW**) ('régime effectivement satisfaisant de prix et salaires').

> 3° Given a satisfying regime of production and labour, whether it is possible to determine the set of prices and wage rates in the manner that the first condition of 2° is satisfied, and for every labour, the wage corresponding to the labour that is effectively demanded from him given the production to achieve is at least equal to the cost of life. I will call all regime of price and wages satisfying these double conditions *effectively satisfying*. (Potron, 1913, pp. 64–65)

2.4.2 Definition of the 'characteristic number of the economico-social system'

We now come to the most important notion characterizing Potron's economic ideas.

(D.4) Let ρ be a parameter greater than α, and the matrix $V(\rho)$ is defined as follows:

$$V(\rho) := B(\rho I - A)^{-1} L \in M(m \times m, \mathbb{R}_+)$$

(D.5) Let $\nu(\rho)$ be the Frobenius root of $V(\rho)$. $\nu(1)$ is defined as a *characteristic number of the economico-social system* ('nombre caractéristique du système économico-social'), which is denoted by v (Potron, 1913, p. 71).

2.5 Assumptions

Implicit in Potron's mathematical arguments are several assumptions, which are as follows.

(A.1) Single production

Each production process produces a unique product, which means that the output-coefficient matrix is an n-dimensional unit matrix.

(A.2) Productiveness of input-coefficient matrix

The economy produces any bundle of net products, which means that the Frobenius root of the input-coefficient matrix A is less than unit.[10]

(A.3) $\mu(N) > \alpha$

The Frobenius root of the augmented input-coefficient matrix $C(N)$ is greater than that of the input-coefficient matrix A for all values of N (>0).

(A.4) $\forall_j \sum_k l_{jk} > 0 \quad \text{and} \quad \forall_k \sum_j l_{jk} > 0$

Each production process employs a positive amount of labour of some category, and a positive amount of labour of each category is employed in some production process.[11]

2.6 Propositions

Based on the above-mentioned assumptions, Potron sets out the following propositions for which he provided a consistent proof. For Theorem 1, he

[10] This assumption was often explicitly stated by Potron (1913, pp. 70–71).

[11] Noting the dual character of the equation systems, we consider this assumption, besides assumption (A.5), to be suggested by the statements where the signs of 'b_{li}' and 'c_{li}' are specified. See Potron (1913, pp. 58–59).

carried out the essential part of the proof by applying the Perron–Frobenius theorems and extending them to non-negative matrices. For Lemma 2, he provided a complete proof from which Theorem 2 follows trivially. As for Lemma 1, although Potron skipped the proof, it can be completed using standard techniques of linear algebra.[12]

2.6.1 Theorem 1 (existence theorem of Potron)[13]

Let the following auxiliary assumption be valid (for this theorem only):

(A.5) $\forall_k \sum_i b_{ki} > 0 \quad \text{and} \quad \forall_i \sum_k b_{ki} > 0$

That is, a member of each category consumes a positive amount of some good and a positive amount of each good is consumed by a member of some category.[14] Then, in the case of the indecomposability of $C(N)$, SRPL and SSRPW exist if and only if $\mu(N) \leqq 1$. In the case of the decomposability of $C(N)$, they exist if $\mu(N) < 1$ and only if $\mu(N) \leqq 1$.

> A satisfying regime of production and labour—or a simply satisfying regime of price and wages—is possible only if [$\mu(N)$], the characteristic root of the matrix $\left[A + \frac{1}{N}LB\right]$, is $\leqq 1$. If $\mu(N) < 1$, there always exists a satisfying regime of production and labour in which one can obtain the numbers of simple consumers of each social category [κ] and the collective unemployment of each category of labourers [u] arbitrarily while letting the production remain equal to the consumption. At the same time, there always exists a simply satisfying system of price and wages in which one can obtain the benefits of enterprises [π] and the savings of labourers [s_{jk}] arbitrarily. Suppose [$\mu(N)$] = 1. If the matrix considered is not partially reduced, there exists again a satisfying regime of production and labour; however, in general, there can neither be simple consumers in any social category nor unemployment in any category of labourers. At the same time, there also exists a simply satisfying

[12] See, for example, Franklin (1968, pp. 191–192), 'theorem 1'.

[13] The essential part of the proof was given in Potron (1913, section 7 and the first two paragraphs of section 8, pp. 58–60).

[14] This auxiliary assumption, especially $\forall_i \sum_k b_{ki} > 0$, is so strong that it implies that every good, even if it is a capital good, be consumed by some consumer. We could, however, substitute the indecomposability of $C(N)$ for this strong assumption, which cancels the distinction between capital and consumption goods.

regime of price and wages; however, in general, there can neither be any benefits for the enterprises nor any savings for the labourers. (Potron, 1913, p. 68)[15]

2.6.2 Lemma 1 (property lemma of Potron)[16]

Let μ be a function defined by

$$\mu : (0, \infty) \to \mathbb{R}, N \mapsto \mu(N)$$

Then, $\mu(N)$ has the following properties:

(i) monotonically non-increasing
(ii) continuous
(iii) $\lim_{N \searrow 0} \mu(N) = \infty$
(iv) $\lim_{N \to \infty} \mu(N) = \alpha$

Next, let ν be a function defined as

$$\nu : (\alpha, \infty) \to \mathbb{R}, \rho \mapsto \nu(\rho)$$

Then, $\nu(\rho)$ has the following properties:

(v) continuous
(vi) $\lim_{\rho \searrow \alpha} \nu(\rho) = \infty$
(vii) $\lim_{\rho \to \infty} \nu(\rho) = 0$

2.6.3 Lemma 2 (inverse function lemma of Potron)[17]

$\mu(N)$ and $\nu(\rho)$ are the inverse function of each other, i.e.

$$N = \nu[\mu(N)] \quad \text{and} \quad \rho = \mu[\nu(\rho)]$$

[15] Square brackets in quotations denote our replenishments.
[16] The property lemma is stated in the last paragraph of Potron (1913, p. 60): 'Soit $\sigma(t)$ . . . grand'.
[17] The inverse function lemma is stated and completely proved in the second and third paragraphs of Potron (1913, p. 61), 'Je dit que . . . $s_1 > \sigma(t_1)$'. The lemma had been already articulated in Potron (1911a, p. 1131).

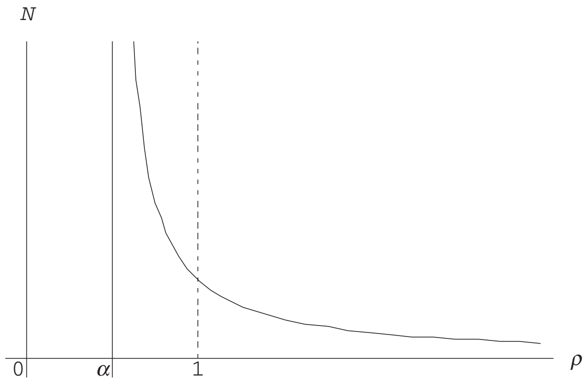

Figure 1. Graph of $N = \nu(\rho)$ and $\rho = \mu(N)$

2.6.4 Theorem 2 (equivalence theorem of Potron)

Let μ_1 be a number greater than α. Then, $\mu(N) < \mu_1$ if and only if $N > \nu(\mu_1)$; in particular, $\mu(N) < 1$ if and only if $N > \nu(1) = \nu$.

2.6.5 Graphic description

Potron explained the above propositions again by referring to a diagram he did not in fact draw; however, we have reproduced it according to his instructions, as shown in figure 1. The curve expresses both $N = \nu(\rho)$ and $\rho = \mu(N)$ (Lemma 2) and has $N = 0$ and $\rho = \alpha$ as its two asymptotes (Lemma 1); for the existence of SRPL and SSRPW, it is both necessary and, in the case of indecomposability, sufficient that the point (1, N) lies on the curve or above (Theorem 1). In addition, 'in order to know the position of the point (1, N) in comparison with the curve, that is, the sign of $[1 - \mu(N)]$, it is sufficient to compare N with the number $\nu(1) = \nu$' (Theorem 2) (Potron, 1913, p. 71).

3. CHARACTERISTICS OF POTRON'S MODEL

3.1 Classification of professional and social categories

According to Potron's economic model, a society consists of labourers and non-labourers ('simple consumers'); the former category includes unemployed labourers, while capitalists count among the latter (Potron, 1913,

p. 73). Both labourers and non-labourers are further classified into groups called 'professional and social categories' *ipso facto* according to the following four characteristics: (1) consumption demand, i.e. the members belonging to each category consume a different bundle of goods, (2) money wage rates, (3) propensity to save and consume, and (4) technological distinctions, such that the kinds of labour performed in each of the categories are considered to be heterogeneous and in principle not substitutable in the production process.

Especially, as for the last point, disputing contributions have piled up in Marxian economics on how heterogeneous labours can be treated in the framework of the labour theory of value or more widely in the linear production model. The problem in the latter sense was already raised by Potron in 1911, and one rational solution was given within the limits of the Leontief system (the equation system of single production).

3.2 Modelling of working time

Working time is explicitly included in Potron's model. It is assumed that all labourers work for a common time N. As Bidard *et al.* (2006) pointed out, working time plays a crucial role in Potron's model. The lifetime of a labourer has two dimensions: the unit of time of consumption is the year, while that of labour is the day; therefore, consumption coefficients in B and labour input coefficients in L are measured in different units of time.[18] There is no direct quantitative connection between the two coefficients. Therefore, the real wage rate (wage basket per unit of working time) is not independently given but is considered as a function of working time N (days); i.e. as the wage basket is determined *by the year* in the first instance, the level of *daily* wage basket (real wage rate) depends on how many days a labourer work in a given year. Correspondingly, the augmented input-coefficient matrix that includes the real wage rate as a variable is assumed to be a function of N, i.e. $C(N)$.

As mentioned above, the unit of working time N of Potron is the day. If we substitute the hour as the unit of time for labour and the day as that of consumption, as we will in fact do in the following sections, there is no need to change Potron's original model and propositions.

[18] The elements of B and L have *product unit/man-year* and *day/product unit*, respectively, as units of measurement.

3.3 Existence of unemployment

In Potron's model, unemployment is incorporated as a variable for each sector and each category. u_{jk} denotes time of unemployment for labourers of category k in production of good j. Thereby, the deviation of labour demand $x_j l_{jk}$ from its supply $N\lambda_{jk}$ is expressed. The condition (C.1), which represents an equilibrium of a quantity system, guarantees the production of consumption goods for all social members ω including non-labourer and unemployed labourers. However, as for the conditions expressing price equilibriums, two kinds of financial resources covering the consumption of unemployed labourers are distinguished. In (C.5), employed labourers' wages explicitly contain the consumption funds of unemployed labourers (including their savings). On the other hand, in the wages of (C.4), unemployed labourers' consumption funds are not included, which means that one (or several) of the following three situations applies: (1) there is no unemployment ($\forall_{j,k} u_{jk} = 0$), (2) employed labourers' savings provide the consumption funds of the unemployed, and (3) capitalists' profits fund the consumption of the unemployed. Therefore, the difference between ESRPW and SSRPW, which satisfy (C.5) and (C.4), respectively, is attributable to the difference in the methods used to fund the consumption of unemployed labourers.

One could interpret Potron's unemployment as proportional underemployment among all labourers, which would not influence the argument except for some trivialities. One would then need to only consider employed and unemployed labourers as employed and unemployed time of labourers, respectively.

3.4 Simple reproduction

Under SRPL, net products are not used for investment but are consumed by labourers or non-labourers. Especially, under ESRPW, which belongs to SRPL, in simultaneously realizing a price equilibrium, labourers' savings and capitalists' profits are exhaustively consumed by non-labourers. According to Marxian terminology, SRPL is simple reproduction—a stationary state without any growth in production.

3.5 Price making

In Potron's model, the price of output, based on full-cost principle, includes the price of input, wage payments and further profits called 'bénéfice'.

However, these profits do not need to occur at an equal rate for all sectors. In this respect, the price does not need to be the natural price in the sense of the classical school, nor the production price in the sense of Marx, but is rather a general one that includes them as special cases. In addition, the wage setting is not restricted to the case of perfect competition among labourers, i.e. it allows different wage rates for each sector.

4. IMPLICATIONS OF THE PROPOSITIONS

4.1 Formulization of 'necessary labour' under heterogeneous labours

We assume from now the absence of labourers' savings and the dominance of SSRPW.[19] We first consider the meaning of the 'characteristic number of the economico-social system' defined in section 2.4.2. With regard to the $n \times m$ matrix $(I - A)^{-1}L$, its (j, k)-element is the total sum of the labour of category k that is used directly and indirectly for producing good j. In other words, it is the embodied labour classified according to output and category and measured in terms of days. Therefore, the (k_1, k_2)-element of the $m \times m$ matrix $V(1)$ expresses the embodied labour of category k_2 in the annual wage basket of a labourer belonging to category k_1.

> [(k,i)-element of $B(I - A)^{-1}$] is the production of [good i] [directly and indirectly] necessary for the subsistence of a consumer of type [k]; [(k,k')-element of $B(I - A)^{-1}L$] represents the labour that this subsistence demands [directly and indirectly] from the category of labourers belonging to type [k']. (Potron, 1913, p. 70)

We now choose the non-negative row eigenvector z associated with the Frobenius root ν of $V(1)$, the sum of whose components is unity. We interpret the vector z as the 'standard labourer',[20] which represents the membership of the standard composition of categories. Then, the vector of embodied labours necessary to produce the annual wage basket of the standard labourer is expressed by $zV(1) = \nu z$. In Marxian terminology, the Frobenius root ν of $V(1)$ means the 'necessary labour' of the standard labourer. In order

[19] Even in the case of proportional underemployment, the argument remains valid by considering εN, εB, $\varepsilon V(1)$ and $\varepsilon \nu$ instead of N, B, $V(1)$ and ν, respectively, where we define ε as the uniform rate of employment.

[20] Our concept of 'standard labourer' shares a common feature with the 'standard commodity' of P. Sraffa and the 'standard reduction' of U. Krause in that it relates to an eigenvector. The difference is that it relates to the matrix $B(I - A)^{-1}L$, while the latter two to A. See Sraffa (1960) and Krause (1979).

to understand this, we need to substitute hour as the unit of time of labour and day as the unit of time of consumption for the day and the year, respectively. This alteration of time units does not change Potron's original model and propositions, as we have already seen in section 3.2, and this substitution will be maintained in the rest of this paper. Potron conferred the special name characteristic number of economico-social system on this number ν in order to emphasize its importance.

> As to the characteristic root ν of maximal modulus, it represents the average number of normal working days that a labourer must provide in order that the products obtained annually exactly represent only the consumption of all the labourers. (Potron, 1913, p. 70)

4.2 Fundamental Marxian Theorem

By interpreting the characteristic number of the economico-social system as necessary labour as mentioned above, we can also understand the meaning connoted by the number $N - \nu$. According to the definition, N is the working time that every labourer actually works. If ν means the necessary labour of the standard labourer, $N - \nu$ means the 'surplus labour' of the standard labourer in the Marxian sense.

According to Theorem 2, the positive surplus labour ($N - \nu > 0$) is equivalent to the Frobenius root of the augmented input-coefficient matrix $C(N)$ being less than unit ($\mu(N) < 1$). If we now consider $\nu > 0$, and define the rate of exploitation e in the usual fashion as follows:

$$e := \frac{N - \nu}{\nu}$$

$N - \nu > 0$ and $e > 0$ are equivalent.

On the other hand, on condition that production prices are realized with the profit rate equalized for all sectors, it is obvious that the profit rate r must be determined by

$$\frac{1}{1+r} = \mu(N)$$

Therefore, $\mu(N) < 1$ and $r > 0$ are equivalent. In short, Theorem 2 implies the proposition that the positiveness of the rate of exploitation and that of the rate of profit are equivalent, which is precisely the proposition proved as 'Fundamental Marxian Theorem' by M. Morishima, F. Seton and N.

Okishio no less than half a century later. It can be said that Potron anticipated the equivalence of productiveness of the augmented input-coefficient matrix, its Hawkins–Simon condition,[21] the positive rate of profit and the positive rate of exploitation. An equivalence of the Leontievian and Marxian system was anticipated by obtaining the inverse functions $\mu(N)$ and $\nu(\rho)$. Furthermore, based on the argument in section 4.1, Potron's propositions solved the modern question of how FMT could be proved subject to the existence of heterogeneous labour. As we will see below, it is possible and necessary to rewrite the history of Marxian economics with respect to this finding, bringing it forward by more than 60 years.

5. RE-EXAMINATION OF THE HISTORY OF RESEARCH

5.1 Proof of FMT

According to the accepted history of Marxian economics, FMT is supposed to have been *first* proved by Morishima, Seton and Okishio and named by Morishima (Morishima and Catephores, 1978, p. 30). Speaking more concretely, Morishima and Seton (1961) and Okishio (1963) are said to have proved the positive rate of exploitation as *sufficient* and *necessary* conditions of the positive profit rate, respectively (Morishima, 1973, p. 53).

As we clarify in this paper, Potron proved Theorem 2. As FMT is implied thereby as a trivial corollary, one could say that it had been proved *ipso facto* by Potron in 1913. However, it is not claimed here that the *first* proof was given by Potron, as it is also possible to assert that a *de facto* proof had already been given in 1898, when Dmitriev formulated the relation $1 = (1 + r)d(I - (1 + r)A)^{-1}l$ (d is a wage basket vector, and l is a labour input-coefficient vector) (Dmitriev, 1986, pp. 103–104). This relation was furthermore taken over by Bortkiewicz (1907). More importantly, in 1910, and therefore prior to Potron, Charasoff showed the equivalence of the positive rate of exploitation and the positive profit rate by recognizing the price vector as an eigenvector of the input-coefficient matrix and determining the profit rate using the eigenvalue (Charasoff, 1910; see also Howard and King, 1992; Mori, 2007).

5.2 Generalization of the theorem by including heterogeneous labours

With Theorem 2, Potron proved *de facto* FMT first in a generalized form, where an arbitrary number of heterogeneous labours are permitted.

[21] See Potron (1913, section 9 and footnote 1, pp. 62–63) and also Bidard (2007).

Surveying the history of Marxian mathematical economics, it turns out that such a generalization of the theorem was formulated 64 years later. Bowles and Gintis asserted the equivalence of the positive profit rate and the positive rate of exploitation in 1977 using the premise of heterogeneous labour of m kinds, and expressing the labour input coefficient and labour value of each good as an m-dimensional vector. They emphasized the ‘novelty’ of their treatment in this respect (Bowles and Gintis, 1977, p. 185). They argued that corresponding to the m kinds of heterogeneous labour, m kinds of rates of exploitation should be defined, and tried to prove the equivalence of the positiveness of profit rate and that of all rates of exploitation.

In 1978, immediately after the publication of Bowles and Gintis (1977), Morishima acknowledged their theorem as a ‘novel idea’ at the same time pointing out some errors of their mathematical proof (Morishima, 1978). In response to this comment, Bowles and Gintis (1978) corrected their theorem as follows and proved it again; in particular, they proved that the positiveness of *all* rates of exploitation implies the positiveness of profit rate, and that the latter implies the positiveness of at least one of the rates of exploitation (Bowles and Gintis, 1978). However, we can show that this ‘novel’ theorem of Bowles and Gintis (1978) was only an almost trivial corollary of the more generalized theorem, which had been proved by Potron 65 years earlier, and that the problem, insofar as it was raised by them, had already been solved (see the Appendix).

APPENDIX

Proof of Bowles–Gintis Theorem

According to the symbols of this paper and the change of time units mentioned in section 4.1, Bowles and Gintis defined the rate of exploitation for the labour of category k and for the activity $x \geq 0$ in the following way.

$$\sigma_k(x) := \frac{xL^k}{\frac{1}{N} xLB(I-A)^{-1}L^k} - 1$$

where L^k denotes the kth column of the $n \times m$ matrix L. If $V(\rho)^k$ denotes the kth column of the matrix $V(\rho)$, we obtain directly from its definition

$$\sigma_k(x) = \frac{xL^k}{\frac{1}{N} xLV(1)^k} - 1$$

From this follows

$$\sigma_k(x) > 0 \Leftrightarrow NxL^k > xLV(1)^k$$

It is noted here that $\forall_{x\geq 0} xLV(1) > 0$ is assumed in addition in Bowles and Gintis (1977, 1978) in order that the rates of exploitation are well defined.[22]

Now, the main theorem of Bowles and Gintis (1978) (Theorem 2′) is as follows:

$$\textit{Bowles–Gintis Theorem}: \exists_{x\geq 0} \forall_k \sigma_k(x) > 0 \Rightarrow r > 0 \Rightarrow \forall_{x\geq 0} \exists_k \sigma_k(x) > 0$$

Proof: As for the first half of this theorem, because $\exists_{x\geq 0} \forall_k \sigma_k(x) > 0$ implies $\exists_{x\geq 0} NxL > xLV(1)$, by post-multiplying both sides of the latter inequality by $y(\geq 0)$, a non-negative eigenvector associated with the Frobenius root $\nu(1)$ of $V(1)$, we obtain

$$NxLy > xLV(1)y = xLy\nu(1)$$

Because $xLy > 0$, we have $N > \nu(1)$. Then, from Theorem 2, $\mu(N) < 1$, where $\mu(N)$ is the Frobenius root of the augmented input-coefficient matrix $C(N)$. Therefore,

$$r = \frac{1}{\mu(N)} - 1 > 0$$

can be established.

In order to prove the second half of the Bowles–Gintis theorem, we assume the contrary, i.e. $\exists_{x\geq 0} \forall_k \sigma_k(x) \leqq 0$, and show that it leads to a contradiction. Then, $\exists_{x\geq 0} NxL \leqq xLV(1)$ follows immediately. By post-multiplying both sides of the latter inequality by $y(\geq 0)$, a non-negative eigenvector associated with the Frobenius root $\nu(1)$ of $V(1)$, we obtain

$$NxLy \leqq xLV(1)y = xLy\nu(1)$$

Because of the assumption $xLV(1) > 0$, the right side of the inequality is positive. Then, we have $N \leqq \nu(1)$, and from Theorem 2, $\mu(N) \geqq 1$ and $r \leqq 0$ (qed).

[22] This assumption is guaranteed by the assumption of the 'quasi-irreducibility' of the input-coefficient matrix in Bowles and Gintis (1977, 1978). Of course, it can be also guaranteed by the additional assumptions, (A.5) and the indecomposability of A.

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