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CAPITALISM, SOCIALISM AND STEADY GROWTH [1]

I. Introduction

The purpose of this paper is that of considering the choice of production techniques from the point of view of both the capitalist entrepreneur maximising the present value of his firm's assets at a given interest rate and the socialist planner maximising the consumption per head associated with the maintenance of a given growth rate.

A model of production is set up, in which output is made of a versatile consumption and production good, called *putty*, and of the *machines* which are made of putty and are necessary to assist labour in order to produce putty. It is assumed that technical choice is irreversible, *i.e.*, that putty is moulded and baked into clay machines of given specifications, which cannot be turned back into putty or into machines of different specifications. Also, their use is not affected by technical progress, which improves the design of new machines but not the operation of those already constructed.

This framework, which Phelps first named "putty-clay," [2] has been widely used in recent economic literature.[3] This paper, however, differs from other putty-clay models in that it does *not* contain two customary assumptions, namely that:

(i) the process of transforming this versatile consumption-production good into durable machines is costless, *i.e.*, no labour is needed to mould and bake putty into clay, and

(ii) putty is turned into clay-machines instantaneously, so that there are no gestation lags of investment. Both assumptions, as we shall see, reduce significantly the scope of technical choice.

The first assumption, that the transformation of putty into clay is costless, is necessary to keep a putty-clay model in the realm of a one-commodity world. Only in this case can gross investment be measured simply by the

[1] Acknowledgments are due to Maurice Dobb, Piero Garegnani, Richard Goodwin, Malcolm MacCallum, Joan Robinson, Luigi Spaventa and Piero Sraffa for helpful comments and criticisms on an earlier draft of this paper. Responsibility for any error, needless to say, rests solely with the writer.

[2] E. S. Phelps, "Substitution, Fixed Proportions, Growth and Distribution," *International Economic Review*, September 1963.

[3] L. Johansen, "Substitution versus Fixed Production Coefficients in the Theory of Economic Growth: a Synthesis," *Econometrica*, April 1959; W. E. G. Salter, *Productivity and Technical Change* (Cambridge, 1960); R. M. Solow, "Substitution and Fixed Proportions in the Theory of Capital," *Review of Economic Studies*, April 1966; M. C. Kemp and P. C. Thanh, "On a Class of Growth Models," *Econometrica*, April 1966; R. M. Solow, J. Tobin, C. C. von Weizsäcker and M. Yaari, "Neoclassical Growth with Fixed Factor Proportions," *Review of Economic Studies*, April 1966; C. J. Bliss, "On Putty-clay," *Review of Economic Studies*, April 1968.

amount of putty which is turned into clay in each period. If moulding and baking putty into clay requires labour the value of a new machine expressed in terms of putty depends on the interest rate (or the wage-rate). *Gross* output will be made up of that part of putty which is actually devoted to consumption *plus* the output of machines; in addition to the sector producing putty, one needs as many other sectors as there are units of time—in the course of the gestation period of machines—during which labour is needed to process putty into machines. To measure *net* output a number of other sectors are needed, in addition to the putty-producing sector, equal to the number of time units into which the lifetime of a machine can be broken, from the beginning of its construction to the end of its lifetime, because each machine at each different stage of its construction or its operation is a different commodity. We can look at the production process either as joint production of putty and machines or as joint production of dated putty. In this system, as Professor Kaldor once put it, " the inputs of different dates jointly produce the outputs of different dates; and it is impossible to separate out the contribution to the output of different dates of the input of a single date." [1] Output per head—whether gross or net—associated with a given technique would then depend both on the rate of interest—determining the price of each machine in terms of putty—and the growth rate, determining the relative proportion of putty and machines of all kind in total output. The assumption of the costless transformation of putty into clay and the use of *gross* measures evade this fundamental issue of capital theory.

The second assumption, of no gestation period of investment, which is also customary in putty-clay models, eliminates one of the possible dimensions of technical choice, namely the possibility of a trade-off between the length of the gestation period and the durability of fixed equipment.[2] Both assumptions, as we shall see, are relevant to the problem of " reswitching " of techniques, *i.e.*, the eligibility of the same technique at more than one level or range of the interest rate, with other techniques being eligible at intermediate levels.[3]

[1] N. Kaldor, " The Controversy on the Theory of Capital," *Econometrica*, July 1937, reprinted in *Essays on Value and Distribution* (1960), p. 159.

[2] A. Bhaduri has investigated this aspect of technical choice in a simple case, in : " An Aspect of Project-selection: Durability *vs.* Construction-period," ECONOMIC JOURNAL, June 1968. He finds that " on economic grounds (other things being equal) one may expect a combination of shorter durability and shorter construction period to be more advantageous in a fast growing economy " (p. 346). Here we shall treat gestation and durability more generally, as being only a partial aspect of technical choice—and not necessarily directly related—without the " other things being equal " assumption.

[3] This phenomenon was first noticed in the modern literature by Joan Robinson, Champernowne and Sraffa (J. Robinson, " The Production Function and the Theory of Capital," *Review of Economic Studies*, 1953; *The Accumulation of Capital* (London: Macmillan, 1956); D. G. Champernowne, " The Production Function and the Theory of Capital: a Comment," *Review of Economic Studies*, 1953; P. Sraffa, *Production of Commodities by Means of Commodities, Prelude to a Critique of Economic*

Neither assumption is made in this paper. A more flexible model will be used instead, which takes into account the labour cost of investment, and the gestation and durability of investment, and is designed to handle production techniques characterised by any possible time profile of output and inputs.

Within this framework conditions for reswitching of techniques are stated, and the problem is shown to be relevant both to the capitalist firm and the socialist planner. A version of the golden rule of accumulation is stated, with a second-best proposition. It is shown that the relevance of the reswitching phenomenon is not affected by technical progress. Relative prices of machines and consumption goods are introduced, and the conditions for macroeconomic equilibrium are examined under both capitalism and socialism. In the context of the model the concept of capital is shown to be dispensable under socialism.

II. Assumptions

There is a versatile commodity, putty, which can be either consumed directly or turned into machines by an irreversible process requiring labour. Time is divided into periods of equal length. Putty is perishable and lasts for one period only, unless it is turned into clay. Clay-machines last for more than one period; their durability depends on their shape, the amount and the time pattern of labour and putty which has gone into their making.

Putty is produced by labour and machines. Labour is homogeneous. The technical specifications of machines, *i.e.*, the pattern of the time flow of inputs and outputs associated with them, differ and cannot be altered after their construction. A " technique of production " is represented by a time-flow of putty-output, in which the putty to be moulded and baked into durable machines appears with the negative sign, and a time-flow of labour inputs. The sequence of the time pattern of putty-output is given by $\{a_i\}$, where $a_i \leqslant 0$ for $i = 0,1,\ldots k-1$ is the amount of putty which is needed initially to be handed over to the workers making machines during period i (the making of a machine can take more than one period; if one single period is needed, $k = 1$; if putty is being produced by labour only, then $k = 0$); $a_k > 0$, $a_i \geqslant 0$ for $i = k+1,\ldots n$ is the putty which is produced thereafter, during each of the subsequent $n-k+1$ periods. We assume that $\sum_{i=0}^{n} a_i > 0$, *i.e.*, total net putty output over the time of operation is strictly positive. The sequence of the time pattern of labour inputs required first to make machines, then to operate them to produce the flow

Theory (Cambridge University Press, 1960), and has been widely debated in a series of papers in the *Quarterly Journal of Economics*, October 1966. See also G. C. Harcourt, " Some Cambridge Controversies in the Theory of Capital," *Journal of Economic Literature*, June 1969.

of putty output, is given by $\{l_i\}$, where $l_0 > 0$ (because labour is always required to start the process), $l_i \geqslant 0$ for $i = 1; 2, \ldots n$. We also assume that l_n and a_n are both positive. There are constant returns to scale. The scale of a technique of production is taken so that $l_0 = 1$. Any convex combination of two techniques is also a technique, but the number of techniques which cannot be expressed as a convex combination of other techniques is finite. The length of k and n is not necessarily the same for all techniques. If a process does not have to be operated to the nth period, but can be stopped after a number of periods $m < n$, each length of operation of the same process is regarded as a separate process. We neglect "inferior" techniques, *i.e.*, such that they give an amount of output at some period lower than another technique, without having a higher output at some other period, and/or a lower labour input at the same or some other period.

We shall consider the full-employment growth of economies with access to this kind of technology, under institutional conditions corresponding to textbook capitalism, centralised and decentralised socialism. In all systems production is organised in productive units called firms, by managers who are all equally efficient. In each period total labour supply is given, and growing at a steady rate λ, $\lambda > -1$. Labour is hired by firms at a real wage w per man per period, paid at the end of the period. Managers are homogenous with the rest of the working force, and the input of their labour is included in the labour coefficients l_i. Economic systems differ in three respects: property relations, market conditions and criteria for technical choice.

Under centralised socialism physical productive assets belong to the State, which appropriates whatever is produced in excess of the payment of wages. It is a monopsonist in the labour market, and fixes the wage-rate w, to which labour supply is inelastic. Firms are simply administrative units, managers are state officers who are ordered to use the technique chosen by the central planner, and receive the necessary material inputs and wage fund (in excess of their current production of putty) free of charge as grants from the State.[1] Among the production techniques available, the central planner selects the technique maximising the rate of consumption per head associated with the maintenance of full-employment steady growth.

Under decentralised socialism physical productive assets belong to state firms. Firms have access to a perfectly competitive labour market, and have infinite power of borrowing and lending putty from and to the State, at a rate of interest r fixed by the State. They have built their assets by borrowing from the State in the past, they appropriate current output and pay wages and interest out of it. Among the production techniques available,

[1] Central fixing of the wage-rate, free investment funds granted from the state budget, central choice of production techniques, administrative orders to the managers of state firms: these are aspects typical of the pre-war Soviet planning system.

they select the technique maximising the present value of their assets at the ruling interest rate.[1] The socialist planner will still wish the technique maximising consumption per head to be chosen, but the only way he can affect technical choice is by choosing the interest rate r, which is the basis of the decisions of state managers.

Under capitalism, physical productive assets belong to individual capitalists, either directly or through shareholding. Firms have access to a perfectly competitive labour market, and have infinite power of borrowing and lending putty at a rate of interest r. Capitalists appropriate the excess of output over what is needed to pay managers and workers the competitive wage, consume part of it and accumulate the rest. Among the production techniques available, the one which maximises the present value of the assets of capitalists at the ruling interest rate is chosen.

Both under capitalism and decentralised socialism macroeconomic equilibrium requires that the production of putty in excess of current consumption requirements should be equal to the material input requirements in the construction of machines. The conditions for equilibrium will be examined in the next sections; we can imagine, provisionally, that the economy in question is connected with a perfect international capital market.

II. The "Wage-interest" Frontier

We shall first consider the implications of the present-value maximisation criterion for technical choice.

Suppose there is one technique only, and no technical progress. The present value v of starting a unit scale process, $\{a_i\}$, $\{l_i\}$, is given by

$$v = \sum_{i=0}^{n} (a_i - wl_i)(1 + r)^{-i}. \tag{1}$$

Since the labour market is competitive, as long as v is positive workers will be successful in demanding higher wages, from firms competing with each other trying to get hold of labour. Equilibrium in the labour market requires that

$$v = 0. \tag{2}$$

[1] These characteristics can be found, for instance, in the Czechoslovak economy in 1967. According to the documents of the 1967 economic reforms, wage guidelines were fixed centrally, but managers could pay additional bonuses to workers, out of an enterprise fund made of retained profits, subject to the payment of a tax on the wage fund, called "stabilisation" tax. See "General Guidelines for Enterprise Operation, Valid from January 1, 1967," in *New Trends in the Czechoslovak Economy*, Booklet No. 6, September 1966. The present value criterion for investment choice was introduced in April 1967 by the State Commission for Technology, *Zásady hodnocení ekonomické efektivnosti investic* (Criteria for the assessment of economic effectiveness of investment), Č.j. 16.653/42/67. See D. M. Nuti, "Investment Reforms in Czechoslovakia," *Soviet Studies*, January 1970.

At each level of the interest rate there is, for a given technique, a maximum wage-rate which firms, performing lending and borrowing operations, can afford to pay to workers and make no loss. This is given by the following equation, obtained from (1) and (2):

$$w = \frac{\sum_{i=0}^{n} a_i(1+r)^{-i}}{\sum_{i=0}^{n} l_i(1+r)^{-i}} \tag{3}$$

This we call the "wage–interest frontier." (The general form of this function, $w = w(r)$, is discussed in the mathematical appendix.) The function has the following properties:

(i) for $r = 0$, $w = \sum_{i=0}^{n} a_i / \sum_{i=0}^{n} l_i > 0$

(ii) there is only one value of r, r^*, for which $w(r) = 0$ because $\sum_{i=0}^{n} l_i(1+r)^{-i}$ is always positive, and because there is only one inversion of sign in the coefficients of the polynomial at the numerator.[1]

From (i) and (ii) it follows that $w(r) > 0$ for $0 \leqslant r < r^*$. (iii) the sign of the first derivative of $w(r)$ is negative for $r = r^*$, but for $0 < r < r^*$ does not have to be negative throughout, and the graph of $w(r)$ may present "bumps." The maximum number of bumps is shown in the appendix to be given by the number of alternations of sign of $\left(\frac{a_{i+1}}{l_{i+1}} - \frac{a_i}{l_i}\right)$, for $i = k, \ldots n$. Bumps therefore might occur if output per man fluctuates from the kth period onwards, for instance, if machines require periodical repairs and spare parts are made out of current output (a_i could even become negative for some $i > k$ if repairs requirements exceed current output, but we have assumed that this is never the case). The economic meaning of the bump is that, over some range of the rate of interest, a firm is a borrower in some periods and a lender in some other periods, and it gains from an increase of the interest rate as a lender more than it loses as a borrower, so that it is able to pay a higher wage-rate if it can perform lending–borrowing operations at a higher interest rate. The presence of bumps, however, is not essential to the following argument.

(iv) The only cases in which the $w(r)$ function is a straight line are ones in which $l_0 = 0$. This will never be the case under our assumptions, because we always have $l_0 > 0$.

Possible graphs of equation (3) are given in Fig. 1.

[1] The number of positive real roots of a real polynomial is equal to the number q of its variations of sign—after having suppressed all terms having zeros as coefficients—or is less than q by a positive even integer.

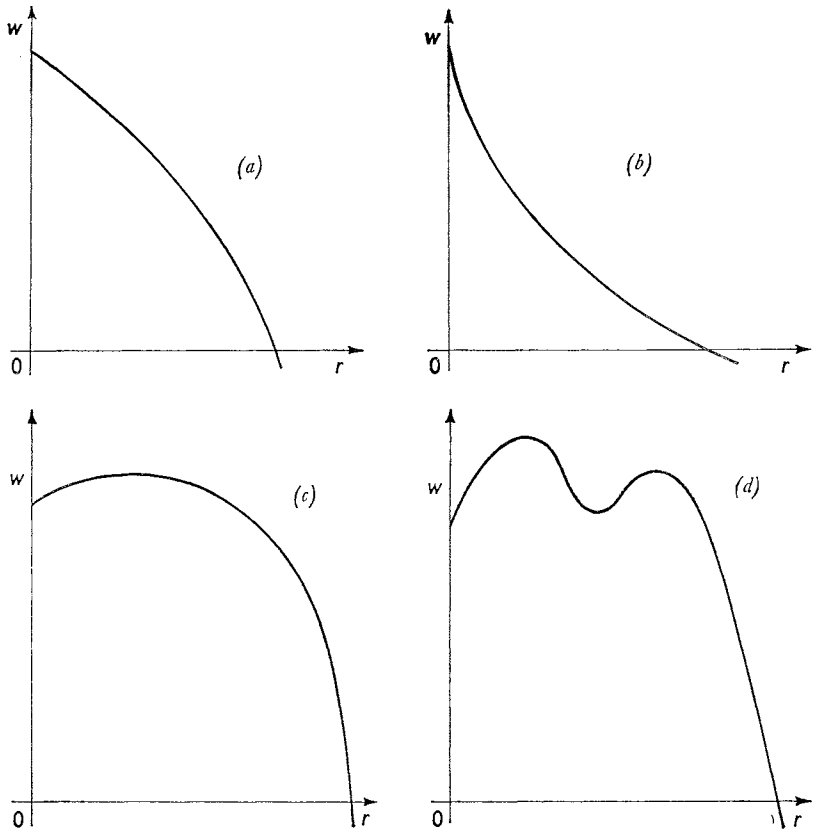

FIG. 1

If a given process does not have to be operated to the nth period but can be stopped before at no additional expense we draw the wage–interest frontier for each length of operation T such that $k \leqslant T \leqslant n$, and superimpose them on the same diagram.[1] Some of them might be inferior. For instance, if output per man is constant after the machine is built, *i.e.*, $\frac{a_i}{l_i} = \bar{a}$ for $i \geqslant k$, any length of operation $T < n$ will give a lower wage-rate than $T = n$ at all values of the rate of interest. If, however, output per man varies over the operation of a machine it might happen that different lengths of operation will be best over different ranges of the interest rate. If the wage frontier has bumps this procedure will smooth the bumps out of the external boundary of the frontiers.[2] If different lengths of operation of

[1] Of course there is no point is considering $T < k$, because $\sum_{i=0}^{T} a_i \leqslant 0$ for $T < k$, and at non-negative interest rates the wage would be negative.

[2] Choosing the length of operation T might not always be possible, for instance, if putty is mined in open-cast mines requiring the replacement of topsoil with relatively large labour expenses towards the end of the operation of the process.

a technique appear in the outer boundary of its wage frontiers the optimum economic lifetime of plant is shown to depend on the interest rate.

If we perform the same operation for all techniques of production available, and superimpose all the $w = w(r)$ functions in the same diagram, we obtain a picture whose outer boundary gives the maximum wage-rate which firms confronted with a given range of techniques can afford to pay, given the rate of interest at which they can undertake lending and borrowing operations. Throughout this paper by $w(r)$ we shall always indicate this outer boundary, which is illustrated in Fig. 2.

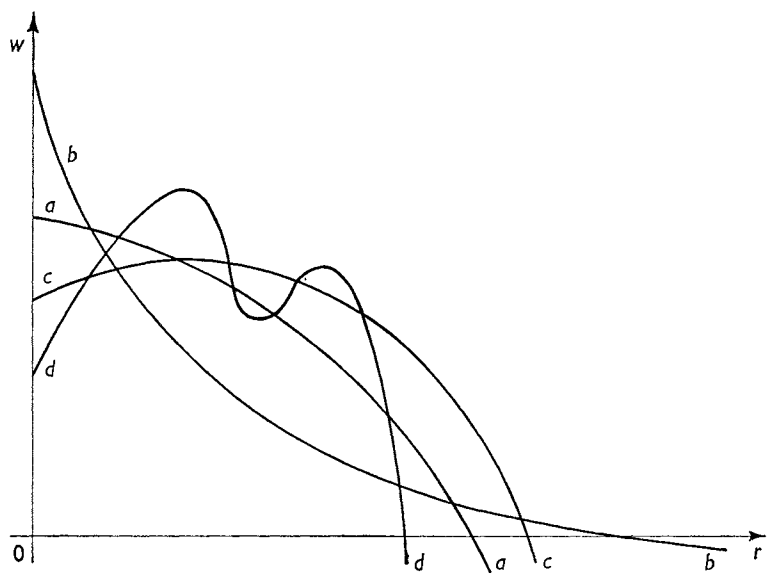

FIG. 2

It might be impossible to rank techniques of production so that each technique is associated with a single value or range of values of the interest rate. Reswitching of techniques might be observed in economies with access to the same technology and different values of the interest rate: the same technique might be in use at two different values of r, with another technique used at intermediate values of r. If there are two techniques, A and B, reswitching means that A affords the same wage-rate as B at more than one level of the interest rate. Suppose technique A is given by (a_{Ai}, l_{Ai}), where

$$a_{Ai} \leqslant 0 \text{ for } i = 0,1, \ldots k_A - 1$$
$$a_{Ai} \geqslant 0 \text{ for } i = k_A, \ldots n_A$$
$$l_{Ai} \geqslant 0 \text{ for } i = 0, \ldots n_A$$

and technique B is given by (a_{Bi}, l_{Bi}), and $k_A \gtreqless k_B$, $n_A \gtreqless n_B$. Reswitching will occur if the equation

$$(4) \qquad \frac{\sum_{i=0}^{n_A} a_{Ai}(1+r)^{-i}}{\sum_{i=0}^{n_A} l_{Ai}(1+r)^{-i}} - \frac{\sum_{i=0}^{n_B} a_{Bi}(1+r)^{-i}}{\sum_{i=0}^{n_B} l_{Bi}(1+r)^{-i}} = 0$$

has more than one positive root. This condition can be rewritten as

$$(5) \quad \sum_{i=0}^{n_B} l_{Bi}(1+r)^{-i} \sum_{i=0}^{n_A} a_{Ai}(1+r)^{-i} - \sum_{i=0}^{n_A} l_{Ai}(1+r)^{-i} \sum_{i=0}^{n_B} a_{Bi}(1+r)^{-i} = 0$$

having more than one positive root. There is no reason whatsoever to assume that this is not the case on grounds of realism. Suppose that the two techniques are such that $n_A = n_B$ and $l_{Ai} = l_{Bi}$ for all $i = 0,1, \ldots n$. The condition for reswitching becomes

$$(6) \qquad \sum_{i=0}^{n} (a_{Ai} - a_{Bi})(1+r)^{-i} = 0$$

having more than one positive root. The necessary (but not sufficient) condition for this being the case is that the sign of $(a_{Ai} - a_{Bi})$ should alternate more than once: there is nothing extravagant in assuming that output (investment counting as negative output) with one technique is higher in two periods and lower in an intermediate period, with respect to another technique, as in Fig. 3 below.

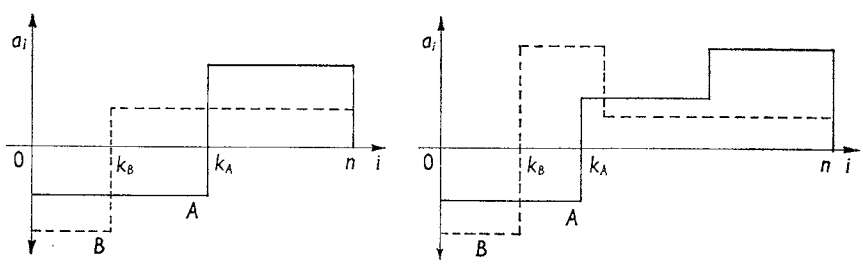

FIG. 3

The actual number of roots (and therefore of switching points) can be found by using Sturm's theorem.[1] When reswitching occurs, the available

[1] Let $f(x)$ be a polynomial with real coefficients such that $f(x) = 0$ has no multiple roots. Construct the identities

$$c_0 f = q_1 f' - f_2,\ c_1 f' = q_2 f_2 - f_3,\ c_2 f_2 - q_3 f_3 - f_4,$$
$$\cdots\cdots\cdots\cdots$$
$$c_{k-2} f_{k-2} = q_{k-1} f_{k-1} - f_k,$$

where q_r/c_{r-1} is the quotient of the division f_{r-1}/f_r; f_k is a constant $\neq 0$, and each f_r is of degree one less than its predecessor. Let a and b be real numbers neither of which is a root of $f(x) = 0$, while $a < b$. Then the number of real roots between a and b of $f(x) = 0$ is the excess of the number of variations of sign in the chain

$$f(x), f'(x), f_2(x), \ldots, f_{k-1}(x), f_k$$

for $x = a$ over the number of their variations of sign for $x = b$. Terms which vanish are to be discarded before counting the variations of sign.

blueprints cannot be so ordered in a book that at a higher interest rate a higher numbered page contains the " best " technique, unless the same blueprint can be inserted more than once. It should be noted that the actual number of reswitching points between the wage frontiers of two techniques is totally uninteresting: in a sense, we can say that the greater the number of reswitching points, the closer the two techniques can be considered to be, and therefore the less important the fact of reswitching. A better measure, however loose, of the importance of reswitching can be given by the maximum difference between the wage-rates afforded by the two techniques at the same rate of interest, because this is a measure of the maximum inefficiency which can result from a wrong choice of techniques (or otherwise some other statistics of the distribution of such differences, taken with the positive sign: $|w_A(r) - w_B(r)|$).

IV. The " Consumption-growth " Frontier

We shall now look at what determines, under the technical conditions already described, the level of consumption per head at different alternative steady growth rates, and its relation with the wage-frontier.

Suppose there is only one technique available, the number of projects (of unit scale) started in each period has been increasing at a constant rate g per period in the last n periods, and the amount of labour currently employed on projects just started is L_t. The number of projects started this period therefore is given by $L_t/l_0 = L_t$; the number of projects started during the period $t-1$ is equal to $L_t(1+g)^{-1}$, and in general the number of projects started at time $t-i$ is equal to $L_t(1+g)^{-i}$. A project started at time $t-i$ will require l_i units of labour and will be associated with a_i units of output (or $-a_i$ units of investment, if $i < k$). Current employment on projects started at time $t-i$, L_{t-i}, is therefore given by equation (7):

$$L_{t-i} = L_t(1+g)^{-i} l_i, \quad i = 0,1, \ldots n \tag{7}$$

From this we can now determine total employment, N; total gross putty output, X; total putty needed as a material to make machines, J; and consumption, C. They are given by the following equations:

$$N_t = \sum_{i=0}^{n} l_i(1+g)^{-i} \,.\, L_t \tag{8}$$

$$X_t = \sum_{i=k}^{n} a_i(1+g)^{-i} \,.\, L_t \tag{9}$$

$$J_t = -\sum_{i=0}^{k-1} a_i(1+g)^{-i} \,.\, L_t \tag{10}$$

$$C_t = X_t - J_t = \sum_{i=0}^{n} a_i(1+g)^{-i} \,.\, L_t \tag{11}$$

From equations (8), (9) and (11) we can express gross putty output per head, $x = X/N$, and consumption per head $c = C/N$ as a function of the growth rate of investment:

$$x = \frac{\sum_{i=k}^{n} a_i(1+g)^{-i}}{\sum_{i=0}^{n} l_i(1+g)^{-i}} \tag{12}$$

$$c = \frac{\sum_{i=0}^{n} a_i(1+g)^{-i}}{\sum_{i=0}^{n} l_i(1+g)^{-i}} \tag{13}$$

Consumption and gross output of putty per head appear therefore to depend solely on the steady growth rate of investment, which will be also the growth rate of the whole economy (as long as investment has been growing at that rate for the last n periods). At full employment (and without technical progress as we have assumed so far) the rate of growth in investment g will have to be equal to the rate of growth of employment λ. Equation (13), expressing consumption per head c as a function of the growth rate g of investment, $c = c(g)$ is exactly identical to equation (3), the wage–interest frontier, with g instead of r and c instead of w. All we have said in relation to equation (3) applies also to equation (13), which we shall call the " consumption-growth " frontier, because each of its points indicates the maximum consumption per head corresponding to a given steady growth rate, and vice-versa, the growth rate (or rates, if there are " bumps ") achievable with a given level of consumption per head. This relation holds both in a socialist planned and in a capitalist economy, growing at a steady growth rate. If there is more than one technique, however, only under centrally planned socialism will the technical choice be determined with reference to the consumption per head maintainable at a given growth rate, whereas under capitalism and decentralised socialism maximisation of present value, as we shall see, might lead to the choice of a different technique.

If we draw the graph of equation (13) for all techniques of production available, the outer boundary will give the maximum level of consumption per head which is consistent with each growth rate. The picture is represented in Fig. 4, which looks exactly like Fig. 2, so that we can measure w, c on the vertical axes and r, g on the horizontal axes. We can now draw the functions also for $g < 0$ and for $c(g) < 0$: negative growth rates—unlike negative interest rates—are economically quite plausible, and the properties of a steadily declining economy can be explored. Negative consumable output per head at some growth rate indicates how much steady external aid per head is needed, on top of subsistence real consump-

tion per head, to maintain that growth rate.[1] However, in order to draw conclusions out of this framework, we need to know not only the outer boundary of the frontiers but also the whole network of frontiers and their interweaving. Under capitalism or decentralised socialism, where technical choice is based on the maximisation of present-value criterion, consumable putty-output per head c will be a function both of the interest rate, which determines the technique chosen, and of the rate of growth of investment. Let us call $a_{i,r}$ and $l_{i,r}$ the technical coefficients of the technique selected at an interest rate r. The function expressing consumable output per head as a function of the growth rate and the interest rate, $c = c(r, g)$ can be written as

$$c = \frac{\sum_{i=0}^{n} a_{i,r}(1+g)^{-i}}{\sum_{i=0}^{n} l_{i,r}(1+g)^{-i}} \tag{13'}$$

If the rate of interest differs from the growth rate, in such conditions consumption per head is not necessarily located on the outer boundary of the frontiers. We can now state the following propositions:

(i) All we have said about reswitching of techniques at alternative interest rates applies here to the reswitching of techniques at alternative steady state growth rates. (Hence, the same relation holds between T and g for each technique, as it holds for T and r.) If growth has been efficiently planned by socialist planners, one might find the same, consumption-maximising technique in two economies where investment grows at a different rate, and another technique in a third economy where investment grows at an intermediate rate.[2]

(ii) If the criterion for technical choice is present-value maximisation at a given interest rate, in a competitive labour market, we can state the following version of the "golden rule": [3] "For a given growth rate of

[1] The maximum number of bumps in the function $c = c(g)$ for $c < 0$ is given by the number of alternations of sign of

$$\left(\frac{a_{i+1}}{l_{i+1}} - \frac{a_i}{l_i}\right) \text{ for } i < k.$$

[2] If the consumption-growth frontier is increasing over a particular range of the growth rate the corresponding growth rates are in a sense inefficient, in that higher growth rates could have been attained, raising consumption per head instead of reducing it. The "bump" in the frontier did not matter for the firm, which had to take the interest rate as given, but matters for the planner to the extent to which he can control the rate of growth of labour supply through immigration and population policy.

[3] This is the mirror image of von Neumann's statement about the conditions to obtain the maximum growth rate corresponding to a given level of consumption per head, in: "A Model of General Equilibrium," *Review of Economic Studies*, 1945. Several versions of this rule have appeared since: see F. H. Hahn and R. C. O. Matthews, "The Theory of Economic Growth: a Survey," Economic Journal, December 1964. In the context of planned socialist growth the same rule is also stated by M. H. Dobb in *Welfare Economics and the Economics of Socialism* (Cambridge, 1969), Ch. 8.

investment, a sufficient condition for consumption per head to be the highest consistent with such growth rate is that the rate of interest should be equal to the rate of growth of investment. If the number of techniques available is infinite, and there is no reswitching, and the switching points are dense, this is also a necessary condition." From Fig. 4 we can see that for any given value of g, say $\bar{g}$: (*a*) If $r = \bar{g}$, the technique (or techniques if there is a switch point at $\bar{g}$) chosen is that yielding the maximum consumption per head attainable at that growth rate. (*b*) Let us call the switching values of the rates of growth and interest a, b, e and f; if the consumption-maximising technique switches at $g = b < \bar{g}$ and at $g = e > \bar{g}$, then as

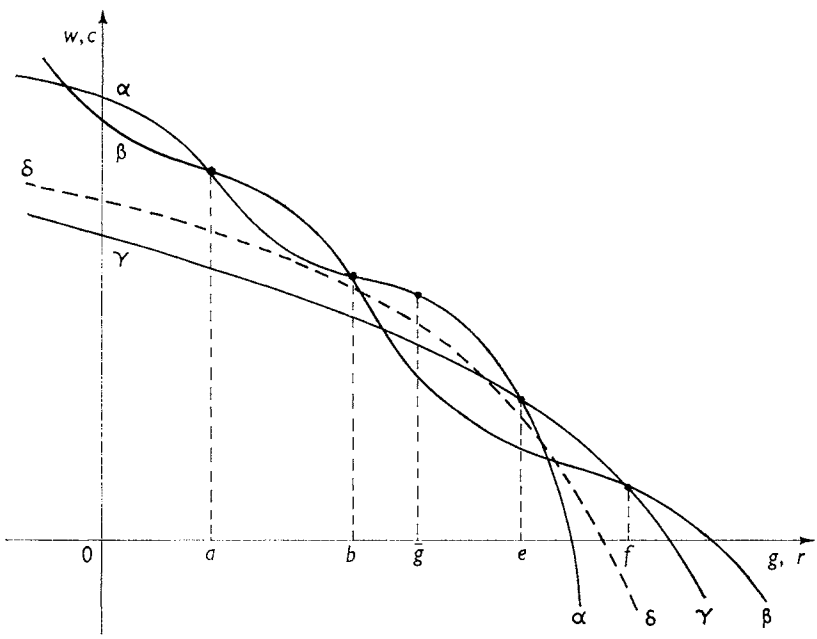

FIG. 4

long as $b < r < e$ the present-value-maximising technique and the consumption-maximising technique will be the same (at $r = b$ or $r = e$ present value could be maximised by linear combinations of two techniques, but this would not necessarily maximise consumable output per head). (*c*) If there is reswitching the technique which maximises consumable output per head at a rate of growth $\bar{g}$ might maximise present value also over some other range of r. In Fig. 4, for instance, the technique maximising consumable output at $\bar{g}$ is also chosen for $0 < r < a$ as well as $b < r < e$. This means that if $a < r < b$ firms can be induced to choose the consumption-maximising technique either by increasing the interest rate, bringing it closer to $\bar{g}$, or by reducing it further and bringing it closer to zero. The difference between g and r, in other words, cannot be taken as a measure

of inefficiency. (*d*) Suboptimality can take not only the form of the wrong plant but also of the wrong length of operation of the " right " plant.

(iii) We can also state the following " second-best " proposition (whether or not reswitching occurs). If $r \neq \bar{g}$, consumption per head might be higher for values of r farther away from $\bar{g}$ than for values closer to $\bar{g}$, and if for some reason the ranges of r over which the (consumption-maximising) technique is chosen are unattainable, there will be a range of values of r over which a " second-best " technique will be chosen, yielding the second highest consumption per head at a rate of growth $\bar{g}$ *among the techniques forming the frontier*. In Fig. 4 this is technique γ, which would be chosen over the range $c < r < f$. It appears, however, that, at the rate of growth $\bar{g}$, γ is inferior to a technique δ which does not appear anywhere along the frontier, and will never be chosen at any value of the interest rate. A typical case would be that of the steadily declining economy, where, if the rate of interest is not allowed to be negative, the consumption-maximising technique will never be chosen by firms (unless that technique is also the best at positive growth and interest rates). If wages and prices, however, are expressed in money terms and are expected to change in time at a steady percentage rate p the parameter relevant to technical choice would not be r, but $\left(\frac{1+r}{1+p} - 1\right)$. Even if there are constraints on the values of r, this " deflated " interest rate can be made equal to g, provided expectations can be generated of a steady percentage rate of price increase p such that

$$p = \frac{r - g}{1 + g} \tag{14}$$

The rule for obtaining optimal technical choice in conditions of steady state growth would now become $r = p + (1 + p)g$.

V. Technical Progress

Suppose technical progress takes place in time, exogenous, disembodied and neutral, in that it decreases labour inputs at all stages for all techniques by the same proportion $d < 1$. If the real wage increases at the rate $h = \frac{d}{1-d}$ the relative profitability of different processes is not altered, and the golden rule remains the same as before. If technical progress is neutral but, as we have have assumed in this model, is " embodied " in machines, which permamently have the input and output characteristics of the time of their construction, and real wages increase at the rate $h = \frac{d}{1-d}$ while labour inputs steadily decrease from one blue-print book to another at the

rate d, the present value of starting a unit scale project at time t, v_t, is given by equation (1′′):

$$(1'') \qquad v_t = \sum_{i=0}^{n} [a_i - w_t(1+h)^i l_i] \,.\, (1+r)^{-i}$$

and the maximum real wage-rate w_t which can be afforded at time t, on the understanding that it must increase at the rate h, is given by putting $v_t = 0$:

$$(3'') \qquad w_t = \frac{\sum_{i=0}^{n} a_i(1+r)^{-i}}{\sum_{i=0}^{n} l_i(1+r)^{-i}(1+h)^i}$$

For any state of knowledge at any given time t the real wage-rate will be lower, with respect to the situation without technical progress, if wages are expected to increase for all workers at the rate $h = \frac{d}{1-d}$ as labour inputs are reduced by technical progress at the rate d on machines whose construction is currently beginning. The graph of equation (3′′) is similar to that of equation (3), but the ranking of techniques and the number and position of switchpoints will differ at different values of h.

If a given technique does not have to be used to the end of its physical life, occurring in period n, but can be stopped earlier at no extra cost, we can again superimpose in the same diagram the wage–interest frontiers corresponding to different lengths of operation T of that technique, $k \leqslant T \leqslant n$. With wages rising at a rate h, the optimum economic lifetime might differ from n, even if $a_i/l_i = \bar{a}$ for $i \geqslant k$. Its actual length will depend on the interest rate. The same relation holding between T and r will hold also between T and g: given the technical coefficients and their rate of change in time, the best length of operation of a given technique from the point of view of maximisation of consumption per head will depend on the growth rate.

Given two techniques A and B, as described in Section III, the conditions for reswitching between them, which in the absence of technical progress was given by equation (5) having more than one positive root, becomes now that equation

$$(5') \quad \sum_{i=0}^{nB} l_{Bi}(1+r)^{-i}(1+h)^i \sum_{i=0}^{nA} a_{Ai}(1+r)^{-i} - \sum_{i=0}^{nA} l_{Ai}(1+r)^{-i}(1+h)^i \sum_{i=0}^{nB} a_{Bi}(1+r)^{-i} = 0$$

should have more than one positive root. Again, there is no reason whatsoever to assume that this is not the case on grounds of realism. Suppose that the two techniques are such that $n_A = n_B$ and $l_{Ai} = l_{Bi}$ for all $i = 0,1, \ldots n$. The condition for reswitching is still expressed by

$$(6) \qquad \sum_{i=0}^{n} (a_{Ai} - a_{Bi})(1+r)^{-i} = 0$$

having more than one positive root. This is exactly as in the case *without* technical progress: at each value of r such that $0 \leqslant r < r^*$ the real wage-rate, corresponding to a given technique if technical progress is expected to take place, will be lower, of course, than if technical progress were not expected, but r^* for each technique, and the switching values of r between techniques, will be the same. Suppose now that the two techniques A and B are such that $n_A = n_B$ and $a_{Ai} = a_{Bi}$ for $i = 0,1, \ldots n$, but differ for more than one labour coefficient. Without technical progress, the condition for reswitching between the two techniques is that equation

$$\sum_{i=0}^{n} (l_{Ai} - l_{Bi})(1 + r)^{-i} = 0 \tag{15}$$

should have more than one positive root. If there is technical progress the condition for reswitching becomes equation

$$\sum_{i=0}^{n} (l_{Ai} - l_{Bi})\left(1 + \frac{r - h}{1 + h}\right)^{-i} = 0 \tag{15'}$$

having more than one positive root. The number of switching points remains the same without or with technical progress, but the switching values of r are now different. If without technical progress there is reswitching between two techniques for values of r equal to r_1 and r_2, with technical progress the switching values of r become $[h + (1 + h)r_1]$ and $[h + (1 + h)r_2]$. It might happen that a switching point which without technical progress occurs at positive values of $w(r)$, with technical progress occurs at negative values of $w(r)$ and therefore loses economic significance. On the other hand, it might also happen that a switch point which without technical progress appears at negative values of r and has no economic significance appears now at non-negative interest rates and therefore acquires economic significance. Whenever techniques differ with respect to the sequence of labour inputs, whether or not they differ also with respect to the sequence of their a_i coefficients, there is no reason whatsoever to assume on the ground of realism that technical progress reduces the relevance of the reswitching phenomenon. (The same holds *a fortiori* if technical progress is of the " disembodied " kind, because in that case it does not alter at all the relative profitability of techniques.)

When technical progress occurs, the same relation between w and r holds again between c and g. Let us again call L_t the amount of labour employed on projects currently being started, and define the scale of projects in *to-day's* book of blueprints so that $l_0 = 1$. Let the number of projects started in every period increase, as in the case without technical progress, at a rate g per period. Labour employed on projects started in the previous period, L_{t-1}, is equal to $L_t(1 + h)(1 + g)^{-1}l_1$, and in general labour employed on projects started in the period $t - i$, L_{t-i}, is given by equation (7′):

$$L_{t-i} = L_t(1 + h)^i(1 + g)^{-i}l_i \tag{7'}$$

Gross putty output, X, total material inputs needed to make machines, J, and total consumption, C, are still given by equations (9), (10) and (11), but employment N_t is now given by equation (8′):

$$N_t = \sum_{i=0}^{n} l_i(1+g)^{-i}(1+h)^i \,.\, L_t \tag{8′}$$

which means that the proportion of total employment devoted to starting new projects, L_t/N_t, varies inversely with the rate of technical progress. Consumption per head at time t is accordingly given by equation (13′):

$$c_t = \frac{\sum_{i=0}^{n} a_i(1+g)^{-i}}{\sum_{i=0}^{n} l_i(1+g)^{-i}(1+h)^i} \tag{13′}$$

If the rate of growth of employment is equal to that of the labour force, λ, we have now

$$\lambda = \frac{1+g}{1+h} - 1 \tag{16}$$

i.e., $g \simeq \lambda + h$.

It should be noticed that the relation between equations (3″′) and (13′) is the same as that holding between equations (3) and (13), namely $w_t(r) = c_t(g)$ for $r = g$ so that the golden rule is not altered by the presence of technical progress of this kind.[1]

VI. Income and Capital

So far we have discussed the problems of growth and technical choice without having to measure the value of " machines " in terms of consumption goods (except that we have stipulated that the value of an investment option, *i.e.*, of a machine not yet built, must be zero). If we want to measure " income " according to international statistical conventions, however, the relative prices of machines of all ages in terms of consumption goods are needed, as the income produced in one period is a collection of heterogeneous objects, made of whatever happens to be in existence at the end of the period, *minus* whatever was in existence at the beginning of the period, *plus* what has been withdrawn from the productive system in the form of consumption.

Call v_j the value in terms of consumption goods (putty) of a machine used in a given process of a unit scale at the beginning of period j of its existence (or, more generally, the value at time t of having " access to " a unit scale process started at time $t - j$). Suppose there is no technical

[1] If real wages increase at a rate different from $h = \frac{d}{1-d}$, or if technical progress is not neutral in the sense defined above, of course there can be no steady state growth.

progress, wages are paid at the end of the period, and either there is no money or prices are constant in time. The value of a machine is given by

$$(17) \qquad v_j = \sum_{i=j}^{n} [a_i - l_i w(r)](1 + r)^{-i}, \quad j = 0, \ldots n$$

The value v of a piece of equipment embodying a given technique depends on its age j and the rate of interest r. We know that $v_0 = 0$ for the technique which is best at any given interest rate, by the very definition of $w(r)$ (see equation (3)). For a given technique, however, the "price" Wicksell effect $\frac{dv_j}{dr}$ and the "ageing" effect $[v_{j+1}(r) - v_j(r)]$ can in principle take either sign. When there are many techniques the level of the interest rate will determine *which* of the techniques is in use as well as the relative value of the different processes at each period of their operation.

From equation (7) we can obtain the number of machines of all ages in existence, so that the value of the capital stock of an economy will be given by

$$(18) \qquad V_t = L_t \cdot \sum_{j=1}^{n} v_j (1 + g)^{-j}$$

which from (17) can also be written as

$$(18') \qquad V_t = L_t \cdot \sum_{j=1}^{n} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j}$$

In steady growth net investment I_t undertaken during period t is given by

$$(19) \qquad L_t = g \cdot L_t \sum_{j=1}^{n} v_j (1 + g)^{-j}$$

which can also be written as

$$(19') \qquad I_t = g \cdot L_t \sum_{j=1}^{n} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j}$$

Income produced during period t, $Y_t = C_t + I_t$, from (11) and (19′) can be written as

$$(20) \qquad Y_t = L_t \left\{ \sum_{i=0}^{n} a_{i,r}(1 + g)^{-i} + g \sum_{j=1}^{n} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j} \right\}$$

Income per head, $y = y(r, g)$, can be obtained from (20) and (8):

$$(21) \qquad y = \frac{\sum_{i=0}^{n} a_{i,r}(1 + g)^{-i} + g \cdot \sum_{j=1}^{n} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j}}{\sum_{i=0}^{n} l_{i,r}(1 + g)^{-i}}$$

The value of output per man in an economy with access to a given technology depends on the interest rate, which determines the technique chosen

(if many are available) and the relative prices of machines and consumption goods, and on the growth rate, which determines the weight of each kind of commodity in output.

If there is only one technique we have that if $g = 0$, $y = c(0)$; if $r = 0$, $y = w(0) = c(0)$, so that we can say that $y(0, g) = y(r, 0)$. If the rate of interest is zero the value of output per man does not vary with the growth rate; if the growth rate is zero the value of output per man does not vary with the interest rate; and the value of output per man is the same in both cases.

If there are many techniques this is not necessarily the case. If $g = 0$, $y = c(r, 0)$; if $r = 0$, $y = w(0) = c(0,0)$. If the interest rate is zero the value of output per head still does not vary with the growth rate; but if the growth rate is zero the value of output per head will vary with the interest rate, and the two will be the same only if r is in the range for which $c(r, 0) = w(0)$.

The value of " capital per man " in the economy is given by (8) and (18′):

$$\frac{V_t}{N_t} = \frac{\sum_{j=1} \sum_{i=j}^{n} [a_{i,r} - l_{i,r} w(r)](1 + r)^{-i}(1 + g)^{-j}}{\sum_{i=0}^{n} l_{i,r}(1 + g)^{-i}} \tag{22}$$

As we saw in Section III, unless one has *faith* that the nature of technology is such that reswitching of techniques does not occur there is no reason to assume that each technique will be associated with a single value or range of values of the interest rate. But even if there is no reswitching, for a given growth rate the same *value* of capital per man can occur at more than a single level or range of the interest rate; or, conversely, for a given interest rate the same value of capital per man can occur at more than a single level or range of the growth rate.[1]

The concept of " value of capital " therefore does not add anything to the analysis of the problems of choice of production techniques for the capitalist firm and the socialist planner. The values associated with a given technique of production *depend on* the criterion and parameters of technical choice, and therefore *cannot provide* themselves any criterion or parameters on which technical choice could be based.

The analysis of the notions of income and capital could be easily extended to the cases where there is technical progress, wages are paid at the beginning of the period and price level is not constant, but the nature of the problem would remain unchanged.

[1] This has been pointed out by L. Spaventa, " Realism without Parables in Capital Theory," in CERUNA, *Recherches récentes sur la fonction de production* (Namur, 1968); *Rate of Growth, Rate of Profit, Value of Capital per Man* (mimeographed); and P. Garegnani, *Heterogeneous Capital, the Production Function and the Theory of Distribution* (mimeographed).

VII. Macroeconomic Equilibrium under Socialism and Capitalism

If we rule out international borrowing and lending the maintenance of equilibrium growth requires that actual consumption per head should be equal to consumable output $c = c(r, g)$, whatever the actual relation between r and g. Equilibrium relations must therefore hold between growth rate, interest rate and saving propensities. This, however, poses different problems under socialist and capitalist conditions.

The socialist planner will provide a certain amount of collective consumption per head, $z > 0$; will collect the voluntary savings of workers who will save, say, a fraction s_w of their net wages; will collect a fraction b of workers' wages in taxes, or pay out a corresponding subsidy of $b < 0$. As long as the planner can choose b and z, he can ensure that the condition is satisfied

$$z + (1 - s_w)(1 - b)w(r) = c(r, g) \tag{23}$$

and obtain simultaneously equilibrium growth and the desired balance between private and collective consumption. This is true whether or not he sticks to the "golden rule," whether he chooses the technique himself, or instructs state managers to use the present-value maximisation criterion. As long as equation (14) is satisfied, the excess of current putty output per head over c will be exactly equal to the amount required to maintain the rate of growth g, because this is exactly how we have defined c in equations (11) and (13). The interest rate workers get on their savings is presumably negligible, because the socialist planner does not want them to turn into *rentiers*, but even if they get the full rate r, the planner can always adjust z and b to obtain (14). If $w > c$, out of what is collected by the planner from the workers in the form of savings and taxation, $(s_w + b - bs_w)w$, an amount $(w - c)$ per man employed will have to be lent each period to firms via the credit system. If $c > w$ the planner will use the excess of firms' repayments and interest payments over current loans to firms, equal to $(c - w)$ per man employed, to finance collective consumption or to subsidise wages. From one period to another, if $g \neq 0$ the stock of machines of all ages (in gestation, new, used) will grow (or decline) at a rate g, the machine-mix depending on g, but unless he has to comply with international statistical agreements, the planner does not have to assess the "value" of the State's capital stock and its net change in time (net investment). All he might want to know is the sum of gross output which is due to come in the future from the stock of machines already existing in the economy. Let us call ρ the rate at which he discounts future output (this can be equal to zero, or to the interest rate he charges state firms, or it can take some other value). At the beginning of time t there are $L_t \,.\, (1 + g)^{-j}$ machines of age j in existence.

The cumulative discounted putty-output A_j of a machine of age j is given by

$$A_j = \sum_{i=j}^{n} a_i(1 + \rho)^{-i} \tag{24}$$

Total cumulative gross putty output A_t from the stock of machines already existing in the economy at the beginning of time t is therefore given by

$$A_t = \sum_{j=0}^{n} \sum_{i=}^{n} a_i(1 + \rho)^{-i} \,.\, L_t(1 + g)^{-j} \tag{25}$$

He might want to calculate A_t excluding unfinished machines, in which case the sum is taken only for $j = k, \ldots n$. He has no reason to subtract wage costs from future putty output: if, however, he wants a measure of discounted future *surplus* of output over *necessary* labour inputs he will subtract the *subsistence wage* rather than $w(r)$. All these measurements have no interest for the managers of state firms. If they happen to exchange machines and putty with each other they will assess the value of a machine in the same way as a capitalist manager would (*i.e.*, subtracting from future output the expected wage costs as in equation (17)). Their measure, in turn, is of no interest for the planner: if they have followed his instructions of maximising the present value of their assets, in a competitive labour market, the value of their assets assessed from their point of view is equal to their outstanding liabilities to the State. The planner knows this magnitude from his books, but it is a purely accounting notion of no operational significance from *his* point of view.

The planner is " making profits " in the sense that if $g > 0$ production of machines in each period exceeds the replacement of machines which have come to the end of their physical lifetime; if $g < 0$ he is only making a " gross profit." Since profits are only the measure of investment undertaken, and in this sense are " reinvested " by definition, there is no need for measuring profits, *i.e.*, the net change in time of the capital stock. Within the framework outlined in this paper, this is true even in a socialist economy where " profits " are used as a source of bonus payments (to the managers and workers) and investment finance, because if all managers are equally efficient, profits in equilibrium should be maximum and equal to zero. If managers are not homogeneous, and managerial abilities need material rewards to come forward, infra-marginal managers would secure quasi-rents to their firms. At the ruling interest rate they would be able to pay a wage higher than that offered by the marginal manager, but they will actually pay the same rate as he does. Given whatever limits the size of their undertakings, infra-marginal managers will obtain quasi-rents equal to the numbers of workers they employ times the difference between the wage-rate they could afford to pay and the wage-rate offered by the marginal manager. The value of their assets, again, would not have to be assessed

to compute their " profits." Even under this form of decentralised socialism, which we could call " managerial socialism " to stress the role of managers in the decision-making process and the enjoyment of profits, the socialist planner could still make sure that actual total consumption does not exceed nor fall short of the level consistent with the maintenance of full-employment growth. In addition to the usual instruments of economic policy (namely, the choice of the level of collective consumption and wage taxation of subsidising), the planner could lay down rules about the share of profits retained by enterprises and the way they should be divided among managers and workers and between consumption and investment.

The problem of macroeconomic equilibrium and the role of profits and capital are, of course, entirely different in a capitalist economy. Whatever is produced in excess of what is needed to pay wages accrues to the capitalists in the form of profits; the evaluation of profits requires the evaluation of machines; capitalists might consume part of their profits; workers will get an interest rate on their savings comparable to that of capitalists. Unless there is state intervention, additional equilibrium relations will have to hold between saving propensities, output and consumption per head, rates of interest and growth. Let us suppose that all investment has to be financed out of profits, *either* because the workers' propensity to save is zero *or* managers of firms have the power to retain part of the profits and distribute the rest to shareholders, and both workers and shareholders have a zero propensity to save (so that s is equal to the retention ratio); *or* workers have a propensity to save $s_w > 0$, but this entitles them to control over a share of total profits equal, in steady state, to their share in current savings.[1] When this is the case we can write the equilibrium condition as

$$(1 - s)[y(r, g) - w(r)] = c(r, g) - w(r) \tag{26}$$

where s is the propensity to save out of profits. Whenever $y > w$, the equilibrium value of s, s^*, corresponding to a given pair of values of r and g is given by

$$s^* = \frac{y(r, g) - c(r, g)}{y(r, g) - w(r)} \tag{27}$$

Suppose a capitalist economy is organised according to the golden rule of accumulation so that $r = g$: in this case $c = w$, and it follows from (27) that the only equilibrium value of the saving propensity of capitalists is

[1] The relation between growth rate, saving propensities, profit rate and distributive shares has been put forward by N. Kaldor, " Alternative Theories of Distribution," *Review of Economic Studies*, 1956; J. Robinson, *The Accumulation of Capital* (1956); and generalised by L. L. Pasinetti, " Rate of Profit and Income Distribution in Relation to the Rate of Economic Growth," *Review of Economic Studies*, 1962. Pasinetti has shown that if workers receive an interest payment on their savings equal to that of capitalists, under certain conditions the propensity to save of workers does not affect the determination of the profit rate and the distributive shares. This proposition has been further discussed by P. A. Samuelson and F. Modigliani, N. Kaldor, J. Robinson and L. L. Pasinetti in *The Review of Economic Studies*, 1966.

unity. It follows that capitalist exploitation takes two forms: one is the capitalists' acquisition of consumption of goods through straightforward command over other people's labour; the other, more subtle form of exploitation is the lower average level of consumption per head associated with a suboptimal technical choice, whenever consumption out of profit prevents the fulfilment of the golden rule. (It should be emphasised again, perhaps, that the golden rule yields optimal technical choice only in conditions of steady state growth, if the criterion of optimality is taken to be the highest rate of steadily growing consumption per head; out of steady state or with a different optimality criterion the rule would not necessarily hold.)

Whenever the saving propensity of capitalists is less than unity, for each steady growth rate there will be one, or possibly many pairs of values of r and s^*. Given the constraint $1 \geqslant s \geqslant 0$, if w is a decreasing function of r we have $c(r, g) < w(r)$ for all $r < g$: for the constraint to be satisfied the growth rate must not exceed the interest rate.

In a capitalist as in a socialist economy, the notion of " value of capital " is not necessary to determine technical choice. In a planned socialist economy the only relevant parameters are the consumption per head—and its behaviour in time if there is technical change or the economy is out of a steady state—and the growth rate of employment. The concept of " value of capital," however, is indispensable to the political economy of capitalism because it performs two fundamental roles, one practical and one ideological.

At a practical level the evaluation of machines of different kinds and different ages in terms of output is needed to settle transactions among capitalist firms, to determine the value of the legal exclusive right to use machinery, and the value of the pieces of paper embodying such rights. It is necessary to determine distribution of income not between the haves and the have-nots but among the haves.

The ideological role of " the value of capital " is that of breaking the direct actual link between the *time pattern* of labour inputs and the *time pattern* of output in which any technology can be resolved, and establishing instead a relation between *current* output and *current* labour. To this purpose the *current* " value of the capital stock " is needed; a mythical conceptual construction in which the past and the future of the economy are telescoped into the present. Attention is focused not on past labour but on the present value of the embodiment of past labour, and its current productiveness can be taken to provide a justification for the attribution of the surplus of current output over the wage bill to those who have appropriated the embodiment of past labour, thereby providing the current basis of future appropriation.

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Mathematical Appendix

Equations (3) and (13) have the form

$$f(x) = \frac{a(x)}{l(x)} = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{n} l_i x^i}$$

The equations differ in that in (3) $x = \frac{1}{1+r}$, $r \geqslant 0$, so that x lies in (0, 1), while in (13) $x = \frac{1}{1+g}$, $g > -1$ and x lies in $(0, \infty)$. We shall analyse[1] $f(x)$ under the following conditions, common to both (3) and (13):

I: x lies in $(0, \infty)$

II: $l_0 = 1$, $l_n > 0$, $l_i \geqslant 0$ for $i = 1, \ldots (n-1)$

III: $a_i \leqslant 0$ for $i = 0, \ldots (k-1)$, $a_k > 0$, $a_n > 0$

and

$a_i \geqslant 0$ for $i = (k+1), \ldots (n-1)$; $k \geqslant 1$

IV: $\sum_{i=0}^{n} a_i > 0$

If no a_i is negative, $a(x) > 0$ for all $x > 0$ and $a(0) = 0$.

Suppose q is the largest i such that $a_i < 0$. Then if $a'^{(p)}$ is the pth derivative of a, and $p \leqslant q$, $a'^{(p)}(x) < 0$ for small x and $a'^{(p)} \to \infty$ as $x \to \infty$, together with Descartes' rule of signs, show that $a'^{(p)}$ has one, and only one, zero in $x > 0$. Also its turning-point (*i.e.*, the solution of $a'^{(p+1)} = 0$), if it exists, must occur at smaller x than its zero (the solution of $a'^{(p)} = 0$). For $p > q$, $a'^{(p)}$ has no zero or turning-point.

Similarly, for all p, $l'^{(p)}$ has no zero or turning-point in $x > 0$, and $l'^{(p)} \to \infty$ as $x \to \infty$, except for $l'^{(n)} = n!\, l_n$. $l'^{(p)} > 0$ for $x > 0$.

Now consider $g_p = \frac{a'^{(p)}}{l'^{(p)}}$ (Note $f = g_0$). This is defined and finite for all $x > 0$. At $x = 0$, $g_p = \frac{a_p}{l_p}$, or if $l_p = 0$ but $l_m \neq 0$ (m being the least number greater than p which satisfies this condition), then as $x \to 0$, g_p is approximately proportional to $x^{-(p+1-m)}$. As $x \to \infty$, $g_p \to \frac{a_n}{l_n}$.

$g_p = 0$ if, and only if, $a'^{(p)} = 0$, so g_p has one, and only one, zero, for $p \leqslant q$, and the zero of g_{p+1} occurs at smaller x than that of g_p.

$g'_p = 0$ if and only if $\frac{(g_{p+1} - g_p) l'^{(p+1)}}{l'^{(p)}} = 0$, *i.e.*, if, and only if, $g_p = g_{p+1}$,[2] and g'_p has the same sign as $g_{p+1} - g_p$. Thus g_p cannot cross g_{p+1} from below (above) when g_{p+1} is increasing (decreasing). If g_{p+1} has a maximum or mini-

[1] I am greatly indebted to Malcolm MacCallum, who provided this analysis, including the result on the maximum number of turning-points of $f(x)$ and its proof.

[2] For the case $p = 0$, this was pointed out to us by the Hon. C. Taylor.

mum, and g_p were to meet it there and hence have a maximum or minimum, this would violate the condition that g'_p has the same sign as $g_{p+1} - g_p$, since one would change sign and the other not. For the same reason if g_p were to have a point of inflexion at the crossing of g_{p+1}, then g_{p+1} must also have one, and by repetition so must g_{n-1} and g_n. But g_n is constant, and so g_{n-1} is either constant or monotone. Thus the only exceptional case is where all g_p are constant, which is ruled out by III.

Thus we see that between any two turning-points of g_p there must be a turning-point of g_{p+1}, so g_p has at most one more turning-point than g_{p+1} (if this were not so, the condition that g'_p and $g_{p+1} - g_p$ are of the same sign is violated).

For this to happen we must have g_p initially increasing if g_{p+1} is initially decreasing, and vice-versa. This is to say that $g_{p+2}(0) - g_{p+1}(0)$ and $g_{p+1}(0) - g_p(0)$ must be of opposite sign. Note that $g_p(0) = a_p/l_p$. There are two exceptional cases, one when $l_p = 0$ and one when $g_{p+1}(0) = g_p(0)$.

A. If $g_{p+1}(0) = g_p(0)$, then $g'_p(0) = 0$ and $g'_{p+1}(0)$ has the same sign as $g_{p+2}(0) - g_{p+1}(0)$. Hence $g_p \lesseqgtr g_{p+1}$ for sufficiently small x according as $g_{p+1}(0) \lesseqgtr g_{p+2}(0)$. By repetition of this argument we see that zeros in the sequence are to be ignored.

B. If $l_p = 0$, $g_p \to \pm\infty$ as $x \to 0$, and so we count $g_{p+1}(0) - g_p(0)$ as positive if a_p is negative, and negative if a_p is positive. Since when $l_r = 0$ $(r = h \ldots p)$ and $l_{p+1} \neq 0$, we have $g_p \approx x^{-(p-h)}$ for small x, we must count $g_{r+1}(0) - g_r(0)$ as negative if a_r positive, and positive if a_r negative.

Theorem 1. The number of turning-points of $f(x)$ under conditions I–IV has a maximum s, s being the number of alternations of sign of $g_p(0) - g_{p-1}(0)$ as p decreases from n to 1, exceptional cases being covered by A and B above.

The proof is above. The extension to the case $l_n = 0$ is easy.

We know g_k has at most $(m - 1)$ turning-points, where $(m - 1)$ is the number of alternations of sign of $g_p(0) - g_{p-1}(0)$ in $p = n, \ldots (k + 1)$. If $g_{k+1}(0) < g_k(0)$, g_q can have at most m turning-points, all being at positive values of g_q. Since g_{q-1} has its zero at a larger x than g_q, g_{q-1} has at most m turning-points at positive g_{q-1}, and repeating we have:

Theorem 2. The number of turning-points of $f(x)$ under conditions I–IV above which occur at positive values of $f(x)$ is m, where m is the number of alternations of sign of $g_{p+1}(0) - g_p(0)$ (using rules A and B) in $p = n, \ldots q$.

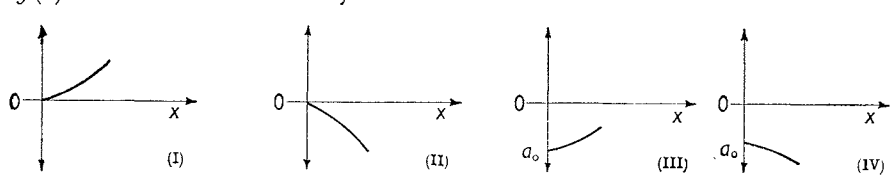
Corollary. The number of turning-points of $f(x)$ at negative values of $f(x)$ is at most $(s - m)$.

Examples. 1. If a_p/l_p increases steadily for $p = 1, \ldots n$, $f(x)$ has no turning-points at positive $f(x)$.

2. If a_p/l_p increases steadily for $p = q, \ldots v$ and decreases for $p = v, \ldots n$ $(q < v < n)$, $f(x)$ has one turning-point at positive $f(x)$.

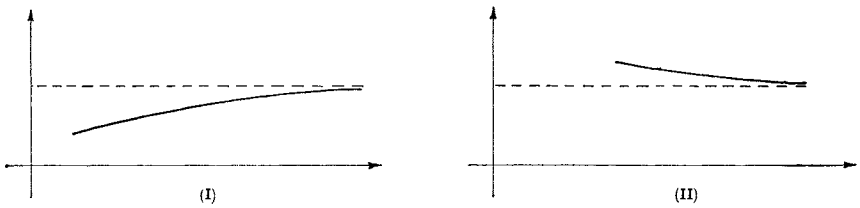
Thus the properties of $f(x)$ are as follows:

1. $f(x)$ starts in one of four ways:



2. In cases II–IV it has one, and only one, zero, in (0,1) as $\sum_{i=0}^{n} a_i > 0$ and $f(1) = \sum_{i=0}^{n} a_i$. In case I it has no zero.

3. It has two ways of ending:



4. The number of turning-points of $f(x)$ has a maximum given by the theorems above.

Once we know the properties of the function $f(x)$, we can infer the properties of f as a function of r, say $z(r)$, and/or g, which are the actual variables we want economically. We note that $r = \frac{1}{x} - 1$ or $g = \frac{1}{x} - 1$ as appropriate.

$$z(0) = f(1) > 0$$
$$z_{r \to \infty} = f(0)$$
$$z(-1) = f_{x \to \infty}$$

If x^* is a zero of $f(x)$, $z\left(\frac{1}{x^*} - 1\right) = 0$.

The number of turning-points of $z(r)$ for r in $(-1, \infty)$ or $(0, \infty)$ is the same as the number of turning-points of $f(x)$ for x in $(0, \infty)$ or $(0, 1)$ respectively.