GEORG VON CHARASOFF AND ANTICIPATION OF VON MISES ITERATION IN ECONOMIC ANALYSIS

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Cahiers d'économie Politique n° 71, 2016

Dans son ouvrage principal, Das System des Marxismus. Darstellung und Kritik *1910, Georg von Charasoff a critiqué et reconstruit la théorie des prix de Marx et, ce faisant, a anticipé la plupart des résultats analytiques qui allaient être obtenus plus tard à l'occasion de la « controverse sur la transformation ». Cepen-dant, sa contribution concerne un champ plus étendu que la seule théorie marxienne. Cet article vise à montrer comment l'analyse écono-mique linéaire de Charasoff, et en particulier sa théorie de l' «Urkapital» (capital originel), du prix de production et de la valeur-travail, a anticipé le développement de la théorie des matrices de Richard von Mises et Hilda Pol-laczek-Geiringer (1929). Ils ont élaboré des procédures de calcul des valeurs propres et des vecteurs propres (la «Power Method» ou «von Mises Iteration») et proposé, en outre, une méthode itérative pour résoudre des systèmes d'équations linéaires non homogènes.*

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1. Introduction

In his main work *Das System des Marxismus. Darstellung und Kritik*, published in 1910, Georg von Charasoff criticized and reconstructed Marx's price theory and, in doing this, anticipated, at an advanced analytical level, most of the results that were to be achieved later in the course of the 'transformation controversy'. Since his main work was rediscovered more than seventy years after its publication, he has been acknowledged by many historians of economic thoughts as a forerunner of Leontief, Sraffa, von Neumann etc. (see Egidi and Gilibert, 1989; Kurz, 1989; Howard and King, 1992; Kurz and Salvadori, 1995: 1998; Egidi, 1998; Gehrke, 1998 and Stamatis, 1999). Recently, Mori (2011) attempted to consistently reformulate Charasoff's argument by a mathematical model to extract some analytical characteristics of his system. Besides, although there remain only a few original documents on his life, some biographical investigation has been already conducted (see Mori, 2007; Gehrke, 2013). The aim of this paper is to show that the contribution of Charasoff's linear economic analysis went beyond the particular range of Marxian theory or the economic theory of the time in general and can be furthermore duly acknowledged in the light of the development of matrix theory in the beginning of the 20th century. In particular, we will make explicit how his theory of "Urkapital (original capital)", price of production and labour value anticipated results of von Mises and Pollaczek-Geiringer (1929).

The linear algebraic development we would like to refer to as background is in particular the Perron-Frobenius theorem in 1907-12 on the one hand and the so-called Power Method (or von Mises Iteration) devised initially by Richard von Mises and Hilda Pollaczek-Geiringer in 1929 on the other. As well known, the Perron-Frobenius theorem (in its generalized version) proposed the existence of the *non-negative* absolutely largest eigenvalue with a *semi-positive* associated eigenvector for every non-negative square matrix (Perron,1907; Frobenius, 1908: 1909: 1912)). Twenty years later, without directly using this theorem, and by using iterative procedures unlike this theorem, von Mises, the brother of the economist Ludwig, together with Pollaczek-Geiringer devised and proved practical procedures to calculate eigenvalues and eigenvectors for any (not only non-negative) square matrix under some assumptions. For each square matrix, starting from a suitable vector, one can reach the matrix's eigenvector associated with the dominant eigenvalue by multiplying the initial vector iteratively by the matrix.

von Mises and Pollaczek-Geiringer (Mises/Geiringer in the following) developed their ideas of iterative procedure based on an iterative method

which approximately determines eigenvalues and eigenfunctions of boundary-value problems (see Vianello (1898), Stodola (1904), Pohlhausen (1921), Koch (1926))[2]. Therefore, Mises/Geiringer (1929) can be seen indeed as an extension of research on this historical context. However, their achievement can be acknowledged independently because the field of research is different between them and their precursors: linear equations for the former and differential equations for the latter. Furthermore, Mises/Geiringer (1929) proposed to use also an iterative procedure to solve *inhomogeneous* linear equation systems. Before them, there had been a forerunner in this subject, namely Seidel (1874), and they took over some of his ideas. However, one of their iterative procedures which matters in respect to Charasoff was developed independently of Seidel (1874).

In the following, in Section II we would like to rationally reconstruct Mises/Geiringer's argument about the procedure for finding eigenvalues and eigenvectors and reformulate their main proposition after correcting its ambiguities and inconsistencies. Section III shows how Charasoff's theory of "Urkapital" anticipated the "von Mises Iteration" and how he went beyond von Mises/Geiringer's results by establishing the invariance of the limit point and carrying out the dual iteration for determining prices of production and "dimensions". Next, we move on to the inhomogeneous system of equations and summarize Mises/Geiringer's procedure for solution in Section IV. Section V shows that Charasoff's recursive method to calculate the labour value is nothing but a special case of the iterative procedure which was to be proposed by Mises/Geiringer nineteen years later. Finally, Section VI concludes the paper.

2. Procedure for solving homogeneous linear equation systems

In 1927, Richard von Mises held a lecture[3] about "Praktische Analysis" and taught a series of calculation procedures for solving linear equation systems.

2. This so-called Vianello-Stodola method was used to compute the "natural frequency (Eigenschwingung)" of an elastic material. This calculation is important in the mechanics because if e.g. the speed of a rotating shaft of steam turbine reaches its natural frequency, the resonance occurs and the material could break down (so that it is called "critical speed (kritische Drehzahl)"). Just as the case of Stodola (1904) exemplifies, this research field was closely related to the development of steam turbines which just started to be used in ships and trains from the beginning of the 20th century (we recall that the first turbine ship was constructed in 1894 and the "unsinkable" Titanic sank in 1912).

3. According to his biography, it must have been at University of Berlin.

In 1929, he published some of these procedures with Hilda Pollaczek-Geiringer (Mises/Geiringer, 1929). The authors treated inhomogeneous equation systems in the first part of their paper, and homogeneous equation systems in the second. We will examine, for convenience, homogeneous systems first.

The authors consider the following homogeneous equation system:

$$\boldsymbol{x} = \lambda \mathfrak{A} \boldsymbol{x} \tag{1}$$

where $\mathfrak{A} = \{\alpha_{ij}\} \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Note that the non-negativity of constants and variables is not assumed in their paper. The problem is to solve a non-zero vector $\boldsymbol{x}$ and a scalar λ. It is a problem of finding eigenvalues of $\mathfrak{A}$ and their associated (column) eigenvectors. Throughout this part, they made the following assumption:

(MA.1) $\mathfrak{A}$ is symmetric and invertible.

This assumption implies that the matrix $\mathfrak{A}$ has n non-zero real eigenvalues (counting multiple ones if any) and n linear independent real eigenvectors because any symmetrical matrix is diagonalizable with real numbers. Therefore, the existence of a solution to Equation (1) is obvious, the problem here, however, is not to show the existence but to determine eigenvalues and eigenvectors specifically. Such a problem is trivial today and easy to solve by a computer, but it was very serious and important at that time because „the most voluminous calculation machine does not have enough digits to provide a result of, say, three digits“ (Mises/Geiringer, 1929, 62, translated by Mori).

To be able to accomplish the task of calculation sufficiently precisely and conveniently, they proposed to use an iterative procedure because it has the advantage that „it shows in each stage an approximation, which can be further improved if necessary. Besides, errors do not spread continuously, but they are in general automatically corrected in the course of calculation“ (Mises/Geiringer, 1929, 62, translated by Mori). So, they define the following iteration.

$$\boldsymbol{z}^{(\nu+1)} = \mu^{(\nu)} \mathfrak{A} \boldsymbol{z}^{(\nu)} \ (\nu = 1, 2, \ldots) \tag{2}$$

where $\boldsymbol{z}^{(\nu)} := \begin{pmatrix} z_1^{(\nu)} \\ \cdots \\ z_n^{(\nu)} \end{pmatrix} \in \mathbb{R}^n$ and $\mu^{(\nu)} \in \mathbb{R}_{++}$ for $\nu = 1, 2, \ldots$

In their paper, they showed, using this iteration, how to calculate each of n eigenvalues and its associated eigenvector. Here, we concentrate especially on their procedure to calculate the dominant eigenvalue (i.e. eigenvalue of maximum modulus) $1/\lambda_1$ and its associated eigenvector[4]. The dominant eigenvalue may be simple or multiple (i.e a single root or multiple root of the characteristic equation). And it may be unique or not (i.e. a positive, negative and complex root may have the same modulus). From (MA. 1), however, only $-1/\lambda_1$ comes into question if the dominant eigenvalue $1/\lambda_1$ is not unique (because $\mathfrak{A}$ has no complex root). There are three cases to be considered according to the property of the dominant eigenvalue $1/\lambda_1$:

(i) $1/\lambda_1$ is simple and $-1/\lambda_1$ is not an eigenvalue

(ii) $1/\lambda_1$ is multiple and $-1/\lambda_1$ is not an eigenvalue

(iii) $-1/\lambda_1$ is an eigenvalue

Regarding the case (i), they conclude with the following proposition (the first part of „Satz 11").

> "For a homogenous linear equation system of the form (1) with parameter λ which can be written shortly as $\boldsymbol{x} = \lambda\mathfrak{A}\boldsymbol{x}$ ($\alpha_{ij} = \alpha_{ji}$), the smallest eigenvalue[5] and the associated eigensolutions can be found by setting an iteration $\boldsymbol{z}^{(\nu+1)} = \mu^{(\nu)}\mathfrak{A}\boldsymbol{z}^{(\nu)}$ ($\nu = 1, 2, \ldots$) and starting from an *arbitrary* vector $\boldsymbol{z}^{(1)}$ with suitable coefficients $\mu^{(\nu)}$. If one continues so far that $z^{(\nu+1)}$ is approximately parallel to $\boldsymbol{z}^{(\nu)}$, then the proportion of components of $\boldsymbol{z}^{(\nu)}$ to components of $\boldsymbol{z}^{(\nu+1)}$ provides the value λ_1; the common direction of $\boldsymbol{z}^{(\nu)}$ and $\boldsymbol{z}^{(\nu+1)}$ is an associated eigensolution." (Mises/Geiringer, 1929, translated by Mori, Italic for "arbitrary" not in original)[6].

As they paraphrased, $\boldsymbol{z}^{(\nu)}$ converges *except for a factor* to the required eigenvector, and each *quotient* of $z_i^{(\nu+1)}$ to $z_i^{(\nu)}$ converges to the required (inverse) eigenvalue λ_1 (Mises/Geiringer, 1929, 152), where $z_i^{()}$ is the i-th component of $z^{()}$.

Their phrases like "*converge except for a factor* (*bis auf einen Faktor konvergieren*)" and convergence of "*quotient*" of each component of two

4. It must be noted that their definition of eigenvalue was different from the usual one of today, i.e. the eigenvalue was defined by λ, *not* $1/\lambda$. Therefore, their eigenvalue is the inverse of today's one so that their "smallest eigenvalue" corresponds to the eigenvalue of maximum modulus in today's definition.

5. See footnote right before this.

6. Some symbols were adjusted to the usage of this paper. And see their poof of Proposition 1 (Satz 11) in Mises/Geiringer (1929, 153-154).

successive vectors $z_i^{(\nu)}$ and $z_i^{(\nu+1)}$ are intuitively understandable but need to be precisely formulated. First, the convergence of $z^{(\nu)}$ *except for a factor* can be formulated as follows[7].

$$\exists x \in \mathbb{R}^n, x \neq 0: \lim_{\nu\to\infty} \frac{z^{(\nu)}}{\bar{z}^{(\nu)}} = x, \tag{3}$$

where $\bar{z}^{(\nu)}$ is the component of $z^{(\nu)}$ with the largest modulus. Note that the sign of components might oscillate if λ_1 is negative.

Second, based on the above convergence of $z^{(\nu)}$ except for a factor according to (3), the convergence of *quotient* of each component of two successive vectors can be formulated as follows[8]:

$$\exists \lambda_1 \in \mathbb{R}, \lambda_1 \neq 0: \lim_{\nu\to\infty} \frac{\mu^{(\nu)} z_i^{(\nu)}}{z_i^{(\nu+1)}} = \lambda_1$$

for all i satisfying $\lim_{\nu\to\infty} \frac{\left|z_i^{(\nu)}\right|}{\left\|z^{(\nu)}\right\|} \neq 0,$ (4)

Furthermore, contrary to the above proposition (Satz 11), which states that starting from *any* arbitrary initial vector, the iteration (2) converges *except for a factor* to the dominant eigenvector (eigenvector associated with the dominant eigenvalue), there must be some vector from which the iteration (2) does not converge *except for a factor* to the dominant eigenvector. A very simple counterexample is $\mathfrak{A} := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, which satisfies Assumption (MA.1), i.e. $\mathfrak{A}$ is symmetric and invertible. Take a unit vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as the initial vector. Then, the iteration (2) leads never to the dominant eigenvalue 2 or the associated eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, the proposition is not valid

7. The convergence of $z^{(\nu)}$ *except for a factor* was originally formulated by the authors themselves as follows (ibid, p. 154):

$$z^{(\nu+1)} \to \frac{\mu(\nu)}{\lambda_1} z^{(\nu)} \quad \text{for } \nu\to\infty$$

However, this formulation is incorrect because if $z^{(\nu)}$ converges itself to null vector, this formulation is always valid even if $z^{(\nu)}$ does not converge *except for a factor.*

8. The convergence of *quotient* of each component of two successive vectors was formulated in the paper as follows (ibid, 154).

$$\lambda_1 = \lim_{\nu\to\infty} \frac{\mu^{(\nu)} z_1^{(\nu)}}{z_1^{(\nu+1)}} = \cdots = \lim_{\nu\to\infty} \frac{\mu^{(\nu)} z_n^{(\nu)}}{z_n^{(\nu+1)}}$$

This formulation is not precise because some component i of $z^{(\nu+1)}$ may vanish from some ν on, and therefore for this i after this ν, the quotient cannot be well-defined anymore.

for *every arbitrary* initial vector as suggested there but for *some suitable* initial vector.

To make the proposition consistent, the domain of the initial vector $\boldsymbol{z}^{(1)}$ must be restricted in the following manner. Since $\mathfrak{A}$ is symmetric, there exist n linear independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, where $\boldsymbol{x}_i$ is associated with the dominant eigenvalue λ_i ($i = 1, \ldots, n$). The proposition is valid if and only if the initial vector $\boldsymbol{z}^{(1)}$ fulfills $\boldsymbol{z}^{(1)} \notin L(\boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$ where $L(\boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$ denotes the linear span of $\{\boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$.

Therefore, the proposition can be consistently reformulated as follows.

Proposition 1 (von Mises and Pollaczek-Geiringer)

If the dominant eigenvalue of $\mathfrak{A}$ is unique and simple, the iteration (2) fulfills (3) and (4) for any initial vector $\boldsymbol{z}^{(1)}$ satisfying $\boldsymbol{z}^{(1)} \notin L(\boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$.

As regards the case (ii), the authors propose to start the iteration with another initial vector in order to obtain, *in general* (i.e. often but not always), another eigenvector associated with the dominant eigenvalue.

Concerning the case (iii), they proposed the following procedure. Set $\mu^{(\nu)} = 1$ for all ν, and define a subsequence $\boldsymbol{z}^{(2\nu-1)}$, $i = 1, \ldots$ Then, each *quotient* of $z_i^{(2\nu-1)}$ to $z_i^{(2\nu+1)}$ converges to the square of the required (inverse) eigenvalue λ_1 (i.e. according to our reformulation (3), $\lambda_1^2 = \lim_{\nu\to\infty} \frac{z_i^{(2\nu-1)}}{z_i^{(2\nu+1)}}$, $\forall i : \lim_{\nu\to\infty} \frac{\left|z_i^{(2\nu-1)}\right|}{\left\|\boldsymbol{z}^{(2\nu-1)}\right\|} \neq 0$), and $\boldsymbol{z}^{'(\nu)}$ and $\boldsymbol{z}^{''(\nu)}$ is converging *except for a factor* to the eigenvector associated respectively with $1/\lambda_1$ and $-1/\lambda_1$, where

$$\boldsymbol{z}^{'(\nu)} := \boldsymbol{z}^{(\nu)} + \lambda_1 \boldsymbol{z}^{(\nu+1)}$$

$$\boldsymbol{z}^{''(\nu)} := \boldsymbol{z}^{(\nu)} - \lambda_1 \boldsymbol{z}^{(\nu+1)}.$$

From today's point of view, the above analysis in the cases (i) to (iii) seems quite obvious. At that time, however, it must have been an innovative discovery because this procedure has been named after the first author "von Mises Iteration" (or alternatively "Power Method"[9]).

9. See Bodewig (1959, 231). Proposition 1 is called there just "Theorem of von Mises". The same methode was reintroduced into economic analysis in 1953 by R.M. Goodwin (see Goodwin, 1983, 75-120).

3. Charasoff's theory of "Urkapital" and price of production

The main ideas of von Mises iteration had been, as stated in Introduction, anticipated by Georg von Charasoff nineteen years earlier in Charasoff (1910). Distinctive features of this anticipation are the following:

– Charasoff did not argue in a general algebraic style but by application to an economic context and by using numerical examples.

– He used (implicitly) another system of assumptions.

– He paid more attention to the invariance of limit among initial vectors.

– He paid more attention to the duality of equation system.

Just as Mises/Geiringer (1929), Charasoff set a linear equation system of the form (1) and solved it by employing an iteration of the form (2). As already mentioned above, however, this calculation procedure was not carried out by Charasoff in an abstract form, but by application to an economic context where the matrix $\mathfrak{A}$ was implicitly assumed to be an augmented input-coefficient matrix (i.e. including not only physical input but also physical wage in input-coefficients), and the column vector $\mathfrak{z}^{(\nu)}$ to be a quantity vector. The iteration according to (2) starting from an arbitrary initial vector $\mathfrak{z}^{(1)}$ and setting $\mu^{(\nu)} = 1$ for all ν is called "production series (Produktionsreihe)" of $\mathfrak{z}^{(1)}$ (Charasoff, 1910, 120; see also Mori, 2011). The iteration "production series" expressed the successive regression of the good vector $\mathfrak{z}^{(1)}$ to its input vector so that $\mathfrak{z}^{(2)}$ is input for $\mathfrak{z}^{(1)}$, $\mathfrak{z}^{(3)}$ is input for $\mathfrak{z}^{(2)}$, and so on. Charasoff shows that the "production series" converges except for a factor to an eigenvector and leads to its associated eigenvalue as the limit of quotient of components of two successive vectors just as the analysis in Mises/Geiringer (1929) showed[10].

Furthermore, Charasoff went in two respects beyond the explicit scope of Mises/Geiringer (1929). First, he paid more attention to the *invariance* of limit among initial vectors in the iteration (2) while the analysis of the latter was not confined only to the case of invariance (see the examples below). Charasoff showed namely that starting from any *arbitrary* good vector, the iteration (2) converges *except for a factor* to the same eigenvector (the *unique*

10. As the authors carefully mentioned, if there are several different eigenvalues with the same modulus (like case (iii) in section 2), the iteration does not converge even except for a factor.

normalized dominant eigenvector) and the quotient of components of two successive vectors converges to the same eigenvalue (the *unique* dominant eigenvalue) as well (For the proof, see Appendix 1). This invariance means that the limit is common for any arbitrary semi-positive initial vector (starting from non-positive good vector would be meaningless). Therefore, the invariant limit of vector sequences means the ultimate universal input and is called "original capital (Urkapital)" (Charasoff, 1910, 111). The invariant limit of the quotient sequences is interpreted as the growth factor (growth rate plus 1) of "original capital". As he wanted to see the proportion of components of each vector in the production series, he considered an associated iteration by setting $\mu^{(\nu)} = \frac{1}{\left\|\mathfrak{A}\, z^{(\nu)}\right\|}$ so that all vectors have the same length of one, i.e. $\left\|z^{(\nu)}\right\| = 1$ holds for all ν. The eigenvector attained by this associated iteration is called "*original type* (*Urtypus*)" (Charasoff, 1910, 124).

Note that the invariance of limit is not guaranteed by Mises/Geiringer (1929). As already shown in section II, even in the case (i), the iteration (2) may converge *except for a factor* to a different eigenvector according to the initial vector. Let us take once again $\mathfrak{A} := \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously, starting from the initial vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the iteration converges except for a factor to different limits, i.e. to respectively $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The same is true for the case (ii) as an example $\mathfrak{A} := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ shows. In the case (iii), there must be some initial vector which does not even converge *except for a factor* at all. Let us take an example $\mathfrak{A} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and start from $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then, the sequence must oscillate. Therefore, in any case, the concept of "Urkapital" could hardly be obtained from the framework of Mises/Geiringer (1929). As the above examples show, the concept of "Urkapital" would lose its whole meaning ("Urkapital" means the ultimate universal input, i.e. the production series of any good vector converges *except for a factor* to the same input vector). Obviously, the above examples cannot be regarded as proper input matrices, which usually contain basic goods. The real point is, rather, that their logical structure, in particular Assumption (MA.1), was not made to properly specify characteristic features of input matrices. As shown below,

Charasoff adapted deliberately his assumptions to economic reality in order to inevitably result in the invariance of the limits.

Second, Charasoff was well aware of the *duality* of linear equation systems. After solving the system (1) as primal problem, he moved on to the dual problem, i.e.

$$\boldsymbol{p} = \lambda \boldsymbol{p} \mathfrak{A}, \tag{5}$$

and the iteration for solving the problem is

$$\boldsymbol{w}^{(\nu+1)} = \mu^{(\nu)} \boldsymbol{w}^{(\nu)} \mathfrak{A} \ (\nu = 1, 2, \ldots) \tag{6}$$

The iteration according to (6) was interpreted by Charasoff as "capitalistic competition"[11] (Charasoff, 1910, 134), which re-prices the products by leveling out different individual rates of profit among sectors. Take an arbitrary initial price $\boldsymbol{w}^{(1)}$. Then, as $\boldsymbol{w}^{(1)}$ is arbitrary, the rate of profit might be different from sector to sector. Therefore, by some common rate of profit $r^{(1)}$, the products must be re-priced so that the new price will be $\boldsymbol{w}^{(2)} = \mu^{(1)} \boldsymbol{w}^{(1)} \mathfrak{A}$ with $\mu^{(1)} = 1 + r^{(1)}$. He showed that the iteration starting from an arbitrary (positive) initial vector converges except for a scalar to the same eigenvector (the unique normalized dominant eigenvector). For the proof, see Appendix 1. The eigenvector attained by the iteration was interpreted as the equilibrium price in the sense that it equalizes sectoral rates of profit and therefore needs not to be re-priced anymore, and it is

11. Charasoff meant by the "capitalistic competition (kapitalistische Konkurrenz)" the free capital mobility across sectors for the purpose of maximizing profit as in the classical tradition. For example, Ricardo wrote: "Whilst every man is free to employ his capital where he pleases, he will naturally seek for it that employment which is most advantageous; … This restless desire on the part of all employers of stock, to quit a less profitable for a more advantageous business, has a strong tendency to equalize the rate of profit of all" (Ricardo, 1951, pp. 88-89). In the same way, Charasoff means by the capitalistic competition a tendency inherent in the capitalistic market "to set the commodity prices proportional to the capital prices, or to establish a general profit rate". *However*, his investigation on it consists "of course not in the description of a real process but in a schematic presentation that correctly captures the core of the problems and brings it to the correct light" (Charasoff 1910, 134, 135). The different feature of his "scheme" namely consists in modeling the process of equalizing profit rates as a *discrete-time* and *iterative* process. Specifically, this model can be characterized in the following way. Inputs are purchased at the beginning and outputs are sold at the end of each production period (which is implicitly assumed common to all sectors). For period *t*, e.g., the purchase of inputs occurs from outputs which were produced in the last period, i.e. period *t*-1, and valued at the *ex ante* current price *pt*-1, while the sale of outputs takes place in terms of the *ex post* current price *pt*. For period *t*, *temporal* profit rates are calculated based on input price in terms of purchase price *pt*-1 and output price in terms of sale price *pt*. Within each period, intersectoral capital movement targeting profit rate maximization and subsequent price adjustment due to changing output level are implicitly assumed to take place so smoothly that output price at the period end equalizes *temporal* profit rates for all sectors.

called "price of production (Produktionspreis)" (Charasoff, 1910, 137) The (inverse) eigenvalue λ is interpreted as the rate of profit (plus 1). Thus, the iteration starting from an arbitrary (positive) initial vector $\boldsymbol{w}^{(1)}$ represents an successive re-pricing of $\boldsymbol{w}^{(1)}$ to successively corrected price vectors $\boldsymbol{w}^{(2)}$, $\boldsymbol{w}^{(3)}$, … converging to the "price of production". (Since prices can be considered to be normalized, $\mu^{(\nu)} = \dfrac{1}{\left\| \boldsymbol{w}^{(\nu)} \mathfrak{A} \right\|}$ is implicitly assumed here so that all price vectors have the length of 1.) Taking particularly the vector of labour values as the initial vector, Charasoff identified this type of iteration as Marxian transformation of value to price of production (Charasoff, 1910, 138)[12].

Next, Charasoff drew a consequence from the duality of equation systems (1) and (5). He showed namely the way how one can solve the dual problem at the same time as the primal problem. Indeed, the rate of profit in the dual problem is automatically attained by finding the rate of growth in the primal problem because both rates are identical, i.e. λ–1 (the inverse eigenvalue of $\mathfrak{A}$ minus 1). But he also showed that the price of production as eigenrow of $\mathfrak{A}$ can be attained in the same procedure as the eigencolumn, i.e. "Urkapital".

According to Charasoff's theory of "Urkapital" stated above in this section, starting from an arbitrary good vector, the iteration (2) is converging except for a factor to the same dominant eigenvector. If we take n unit vectors $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \boldsymbol{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$ as initial vectors, their iterations are to converge *except for a factor* to the same eigenvector. Excepting for a factor means ignoring the size (or length) and sign of vectors. What is now at stake, however, is just the size (called "dimension" by Charasoff) of vectors, and it may be different between the n iterations at each stage ν.

Let us consider the ν-th terms of the iteration ("production series") of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, …, $\boldsymbol{e}_n$ and call them $\boldsymbol{e}_1^{(\nu)}, \boldsymbol{e}_2^{(\nu)}, \cdots, \boldsymbol{e}_n^{(\nu)}$ respectively. The vector of their sizes ("dimensions") at this stage is therefore $\eta^{(\nu)} := \left(\left\| \boldsymbol{e}_1^{(\nu)} \right\|, \cdots, \left\| \boldsymbol{e}_n^{(\nu)} \right\| \right)$. Thus, we

12. Twenty-three years later, Shibata (1933, 49–68) illustrated a similar iteration with a numerical example without referring to Charasoff, and Okishio (1972: 1973: 1974) then provided a formal proof of the convergence.

obtain a sequence of dimension vectors associated with the production series of $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Charasoff showed and exemplified by a numerical example that the normalized sequence of $\{\eta^{(\nu)}\}$ converges, i.e.

$$\exists p \in \mathbb{R}^n, p \neq 0: \lim_{\nu \to \infty} \frac{\eta^{(\nu)}}{\left\| \eta^{(\nu)} \right\|} = p \tag{7}$$

(note that $\eta^{(\nu)}$ is non-negative and $\left\| \eta^{(\nu)} \right\| \neq 0$ for all ν here, and therefore we can use $\left\| \eta^{(\nu)} \right\|$ for the denominator). Finally, the limit of the normalized sequence of $\{\eta^{(\nu)}\}$ is shown to be the dominant eigenrow of matrix $\mathfrak{A}$ and the (normalized) price of production. For the proof, see Appendix 2.

Just as the concept of "Urkapital", the insight into the simultaneous determination of eigencolumn and eigenrow could hardly be obtained from Mises/Geiringer (1929) because the concept of "dimension" would lose its whole meaning if the concept of "Urkapital" could not be established, i.e. if initial vectors may converge to different limits.

Finally, in the following, we make explicit the logical structure which enables Charasoff to establish the concept of "Urkapital". For Proposition 1 in section 2 to be valid, the following condition is necessary and sufficient:

(A.1) There is a real and non-zero dominant eigenvalue.

Mises/Geiringer (1929) made Assumption (MA.1), i.e. symmetry and invertibility of $\mathfrak{A}$, to sufficiently guarantee (A.1). As we saw above in this section, however, their assumption (MA.1) is not suitable to found the theory of "Urkapital" and price of production à la Charasoff on it. Assumption (MA.1) does not guarantee in particular the invariance of limit of iteration among meaningful initial vectors. Contrarily, to guarantee the invariance and establish his theory consistently, Charasoff had deliberately elaborated a framework of his argument, particularly by effectively introducing the concept of basic and non-basic products / production ("Grund- und Nebenproduktion" (Charasoff, 1910, 81)). Note that the concept of basic and non-basic product is defined by Charasoff in terms of *augmented* input coefficients (i.e. including physical wage). (See his definition of basic and non-basic products in Mori, 2011, Section 2.1). Charasoff's framework of argument amounts to the following set of assumptions:

(CA.1) All physical input coefficients are non-negative.

(CA.2) The physical wage vector is non-negative and non-zero (i.e. semi-positive).

(CA.3) Labour is directly used (direct labour input is positive) in all sectors.

(CA.4) Non-basic products are not used as input in any sector.

The first three assumptions are almost self-evident from the economic point of view, and they imply in particular, that a positive amount of labour is directly used in all sectors and the physical wage contains a positive amount of at least one good, so that the (augmented) input matrix has at least one positive row, say i. According to the definition, good i is a basic product, and therefore the set of basic products is not empty. On the other hand, Assumption (CA.4) is somewhat restricting. As we see below, this assumption was deliberately used by Charasoff to guarantee a positive price for every good and also the uniqueness of the dominant eigencolumn ("Urkapital") except for a scalar. Although the assumption had the advantage of simplifying the argument, it could be obviously weakened if one does not want to lose the generality so much. For example, the following weaker assumption would be enough to guarantee the positive prices: the augmented input matrix should have a simple Frobenius root that is the only eigenvalue of maximum modulus. This is the case if non-basic goods are employed as inputs in a sufficiently small amount if any.

The above assumptions (CA.1) to (CA.4) as a whole imply the following properties of augmented input coefficient matrix $\mathfrak{A}$. For the proof, see Mori (2011, Section 2.2).

1) Matrix $\mathfrak{A}$ has a dominant eigenvalue which is unique, simple and positive

2) There is a positive dominant eigenrow.

3) There is a semi-positive dominant eigencolumn.

These properties of $\mathfrak{A}$ guarantee, unlike Assumption (MA.1), the invariance of limit of iteration among meaningful initial vectors (*semi-positive* $\boldsymbol{z}^{(1)}$s and *positive* $\boldsymbol{w}^{(1)}$s), and respectively the semi-positivity and positivity of this limit. For the proof, see again Appendix 1. This result means that starting from any semi-positive initial good vector, the production series converges to the same semi-positive "Urkapital" (except for a scalar), and that the dual iteration of re-pricing starting from any positive initial prices leads to the same positive equilibrium prices (except for a numéraire).

4. Procedure for solving inhomogeneous linear equation systems

In Mises/Geiringer (1929), the authors consider the following *inhomogeneous* equation system (Mises/Geiringer, 1929, 62):

$$\mathfrak{A}\boldsymbol{x}-\boldsymbol{r}=\boldsymbol{0} \qquad \boldsymbol{r}\neq\boldsymbol{0} \tag{8}$$

where $\mathfrak{A}=(\alpha_{ij})\in\mathbb{R}^{n\times n}$ is a coefficient matrix, $\boldsymbol{r}\in\mathbb{R}^n$ a constant, and $\boldsymbol{x}\in\mathbb{R}^n$ an unknown. Note that the non-negativity of constants and valuables is not assumed here too. The problem is to solve the inhomogeneous equation (8) by $\boldsymbol{x}$. And throughout this part, the authors made the following assumption (Mises/Geiringer, 1929, 62):

(MA.2) $\mathfrak{A}$ is *invertible*

The existence of a unique solution $\boldsymbol{x}$ of (8) is also obvious, the problem here, however, is not to show the existence but to specifically calculate the solution.

To be able to accomplish the task of calculation sufficiently precisely and conveniently, they proposed here to use the following iteration (Mises/Geiringer, 1929, 63):

$$\boldsymbol{x}^{(\nu+1)}=(\boldsymbol{I}+\boldsymbol{C}\mathfrak{A})\boldsymbol{x}^{(\nu)}-\boldsymbol{C}\boldsymbol{r}\ (\nu=1,2,\ldots) \tag{9}$$

where $\boldsymbol{x}^{(\nu)}\in\mathbb{R}^n$ for $\nu=1,\ 2,\ \ldots$ and $\boldsymbol{C}$ is a diagonal matrix, i.e. $\boldsymbol{C}:=\begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}\in\mathbb{R}^{n\times n}$.

Then, the authors conclude with the following proposition (which was originally "Satz 5" (Mises/Geiringer, 1929, 68)).

Proposition 2 (Mises and Pollaczek-Geiringer)

The iteration (9) converges to the unique solution of (8) if and only if all eigenvalues of the matrix $(\boldsymbol{I}+\boldsymbol{C}\mathfrak{A})$ *have a modulus less than unity.*

In other words, if we can choose such a matrix $\boldsymbol{C}$ that the modulus of the dominant eigenvalue of $(\boldsymbol{I}+\boldsymbol{C}\mathfrak{A})$ is less than one, then and only then $\lim_{\nu\to\infty}\boldsymbol{x}^{(\nu)}=\boldsymbol{x}$ holds, where $x^{(\nu)}$ and x are defined by (9) and (8) respectively .

In corollaries, Mises/Geiringer (1929) showed some *sufficient* conditions for the iteration (9) to be successful.

The first sufficient condition consists in $\sum_{i=1}^{n} \frac{|\alpha_{ij}|}{|\alpha_{ii}|} - 1 < 1$ for all j (Mises/Geiringer, 1929, 64). And the second sufficient condition is $\sum_{i,j} \frac{\alpha_{ij}^2}{\alpha_{ii}^2} - n < 1$ (Mises/Geiringer, 1929, 67). If one of both is valid, then the iteration (9) leads to the solution of (8) by taking $c_i = -\frac{1}{\alpha_{ii}}$ (i = 1, …, n).

5. Charasoff's theory of "Reproduktionsbasis" and labour value

Just as in the case of homogeneous equation systems, Charasoff anticipated the main ideas of the iterative procedure proposed by Mises/Geiringer (1929) to solve *inhomogeneous* equation systems. Also here, his style of argument is characterized by exemplifying the procedure with numerical examples applied to an economic context. The economic problem to which he applied the iterative procedure for solving inhomogeneous equation systems was the problem of calculating labour values of commodities. What was very characteristic of his calculation, is that he intentionally presented two different procedures of calculation, namely a simultaneous method, i.e. solving the value equation directly on the one hand, and a recursive method, i.e. counting retroactively a whole series of past expended labour on the other.

For this part of analysis, Charasoff made implicitly the following assumption.

(CA.5) Positive net product in all sectors is possible. In other words, the dominant eigenvalue of the input coefficient matrix $\boldsymbol{A}$ is less than unity.

Note that the matrix A here does not contain physical wage unlike the augmented input coefficient matrix before[13].

Let us begin with the first procedure. Charasoff considered a 3-sector economy which consists of sectors of means of production (Sector I), means of subsistence (Sector II) and luxuries (Sector III) and has the following

13. Note here that we explicitly distinguish Fraktur $\mathfrak{A}$ and roman $\boldsymbol{A}$ as notations of matrices.

production technique (Charasoff, 1910, 94-95). And the physical wage rate is 1 unit of means of subsistence.

70 unit of means of production ⊕ 30 unit of labour → 100 unit of means of production

20 unit of means of production ⊕ 20 unit of labour → 100 unit of means of subsistence

10 unit of means of production ⊕ 50 unit of labour → 100 unit of luxuries

Note that the arrow means the production process and its LHS is input and its RHS output. And ⊕ denotes the union of inputs (the notation is taken over from Kurz and Salvadori (1995)). Obviously, the calculation of labour values is very easy to be carried out in such a case. Let w_1, w_2 and w_3 be the labour value of one unit of means of production, means of subsistence and luxuries respectively. Then, Charasoff's procedure of calculation is to solve the following equations (Charasoff, 1910, 94-95):

$$70\ w_1 + 30 = 100\ w_1 \tag{10}$$

$$20\ w_1 + 20 = 100\ w_2 \tag{11}$$

$$10\ w_1 + 50 = 100\ w_3 \tag{12}$$

Solving (10)(11)(12) , he conclude $w_1 = 1$, $w_2 = 0.4$ and $w_3 = 0.6$. Accordingly, he calculated the rate of surplus value, i.e. (1–0.4)/0.4 = 3/2.

It must be noted here that Charasoff calculated the labour values in a *simultaneous* manner and not in a recursive manner, in other words, he calculated them by solving the equations (10) – (12) directly and not by counting retroactively a series of past labour. We can verify this fact by seeing that his equation system (10) – (12) is equivalent to the usual value equation[14]:

$$\boldsymbol{w} = \boldsymbol{wA} + \boldsymbol{l} \tag{13}$$

where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is an input coefficient matrix in the usual sense (i.e. without physical wage), $\boldsymbol{l} \in \mathbb{R}^n$ is a labour input coefficient vector and $\boldsymbol{w} \in \mathbb{R}^n$ is a vector of labour values. In his example, Charasoff set $\boldsymbol{A} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\boldsymbol{l} = \begin{pmatrix} 0.3 & 0.2 & 0.5 \end{pmatrix}$.

14. The value equation had been already published in 1904 by Dmitriev (1974).

However, Charasoff is well aware that this simultaneous method of calculation would be very difficult to be carried out in the reality with a large number of sectors "because the value of each means of production contains the value of those means of production which had to be used for producing it. Thus, one has a series of equations in which unknowns appear on both sides of equation. Just as if one moves around in an unsolvable circle" (Charasoff, 1910, 147). He then proposed another concept as a solution by saying: "Only the concept of reproduction basis (Reproduktionsbasis) solves this wrong cycle" (Charasoff, 1910, 147).

The second procedure for calculating labour values Charasoff presented was indeed a procedure using the concept of "Reprodutionsbasis". The reproduction basis was defined by Charasoff in the following manner.

Let $\boldsymbol{x}^{(1)}$ be an arbitrary good vector. Make the "production series" of $\boldsymbol{x}^{(1)}$ by ignoring physical wage in input and denote it $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$, $\boldsymbol{x}^{(3)}$, … The "reproduction basis" of $\boldsymbol{x}^{(1)}$ is defined by $\boldsymbol{x}^{(1)} + \boldsymbol{x}^{(2)} + \boldsymbol{x}^{(3)} + \ldots$

Let us reformulate Charasoff's definition of reproduction basis. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be an input coefficient matrix in the usual sense i.e. an input coefficient matrix including only physical input coefficients and not physical wage. For any semi-positive $\boldsymbol{x}^{(1)} \in \mathbb{R}^{n}_{+}$, $\tilde{\boldsymbol{x}}$ is called the reproduction basis of $\boldsymbol{x}^{(1)}$ if and only if

$$\tilde{\boldsymbol{x}} = \lim_{\nu\to\infty} \sum_{k=0}^{\nu-1} \boldsymbol{A}^k \boldsymbol{x}^{(1)} \tag{14}$$

holds. Then, Charasoff proposed to calculate the labour value of $\boldsymbol{x}^{(1)}$, denoted by $w(\boldsymbol{x}^{(1)})$, by multiplying the production basis of $\boldsymbol{x}^{(1)}$ by the labour input coefficient vector, i.e.

$$w(\boldsymbol{x}^{(1)}) = \boldsymbol{l}\tilde{\boldsymbol{x}} = \lim_{\nu\to\infty} \boldsymbol{l} \sum_{k=0}^{\nu-1} \boldsymbol{A}^k \boldsymbol{x}^{(1)} \tag{15}$$

Let $\boldsymbol{w}$ be, as before, the vector of labour values of product units. Then, we have

$$\boldsymbol{w} = \left(w(\boldsymbol{e}_1), \cdots, w(\boldsymbol{e}_n)\right)$$

where $\boldsymbol{e}_i$ is the i-th unit vector. According to Chrasoff's retroactive procedure (15), we obtain

$$\boldsymbol{w} = \lim_{\nu\to\infty} \boldsymbol{l} \sum_{k=0}^{\nu-1} \boldsymbol{A}^k. \tag{16}$$

As we have just seen, Charasoff proposed to replace the simultaneous method (13) by the retroactive procedure (16) as the only way to avoid the

"vicious cycle" of the simultaneous method. We can interpret this proposal of Charasoff in two ways. First of all, Charasoff's equivalence of double procedures, (13) and (16), can be interpreted as a *de facto* application of Frobenius' equation in Frobenius (1908), i.e.

$$(\boldsymbol{I}-\boldsymbol{A})^{-1}=\lim_{\nu\to\infty}\sum_{k=0}^{\nu-1}\boldsymbol{A}^k,$$

to the problem of labour value although it is not ascertained yet that Charasoff knew Frobenius' paper.

Second, we can also show that Charasoff's proposal of the two alternative procedures for calculating labour values anticipated a special case of Mises/Geiringer's iterative procedure (9). As we can easily see, if we substitute ($\boldsymbol{I}$–$\boldsymbol{A}$) and $\boldsymbol{l}$ for respectively $\mathfrak{A}$ and $\boldsymbol{r}$, then the value equation (13) belongs to inhomogeneous equation systems of form (8). On the other hand, according to (9), the iterative procedure proposed by Mises/Geiringer for solving the value equation (8) would be, taking $\boldsymbol{C}=-\boldsymbol{I}$, the following iteration:

$$\boldsymbol{w}^{(\nu+1)}=\boldsymbol{w}^{(\nu)}(\boldsymbol{I}-(\boldsymbol{I}-\boldsymbol{A}))+\boldsymbol{l}=\boldsymbol{w}^{(\nu)}\boldsymbol{A}+\boldsymbol{l}$$

$$\boldsymbol{w}^{(\nu+1)}=\boldsymbol{w}^{(1)}\boldsymbol{A}^{\nu}+\boldsymbol{l}\sum_{k=0}^{\nu-1}\boldsymbol{A}^k \qquad \nu=1,2,\ldots$$

By Assumption (CA.5), we obtain

$$\lim_{\nu\to\infty}\boldsymbol{A}^{\nu}=0. \tag{17}$$

And recalling $\boldsymbol{C}=-\boldsymbol{I}$ and $\mathfrak{A}=\boldsymbol{I}-\boldsymbol{A}$, Assumption (CA.5) implies that all eigenvalues of ($\boldsymbol{I}+\boldsymbol{C}\mathfrak{A}$) ($=\boldsymbol{A}$) have a modulus less than unity, which meets the (necessary and sufficient) condition of Proposition 2 of Mises/Geiringer (1929). Therefore, the iteration (9) is guaranteed to succeed, and from (17),

$$\boldsymbol{w}=\lim_{\nu\to\infty}\boldsymbol{w}^{(\nu+1)}=\boldsymbol{w}^{(1)}\lim_{\nu\to\infty}\boldsymbol{A}^{\nu}+\lim_{\nu\to\infty}\boldsymbol{l}\sum_{k=0}^{\nu-1}\boldsymbol{A}^k=\lim_{\nu\to\infty}\boldsymbol{l}\sum_{k=0}^{\nu-1}\boldsymbol{A}^k \tag{18}$$

holds. This iterative procedure of Mises and Pollaczek-Geiringer provides just the same result (18) as Charasoff's retroactive procedure (16). Therefore, Charasoff's presentation of two alternative equivalent procedures for calculating labour values can be interpreted as the anticipation of a special case ($\boldsymbol{C}=-\boldsymbol{I}$ and $\mathfrak{A}=\boldsymbol{I}-\boldsymbol{A}$) of Mises/Geiringer (1929)'s procedure for solving inhomogeneous equation systems.

6. Conclusion

Georg von Charasoff's linear algebraic analysis was not carried out in an abstract form, but by application to an economic context where the matrix

was implicitly assumed as an input-coefficient matrix, and column and row vectors respectively as quantity and price vectors. According to the duality, the vector iteration converging to the eigenvector expresses as the primal problem the iterative regression of an (arbitrary semi-positive) initial good vector to its input vector converging to the ultimate input vector i.e. "original capital (*Urkapital*)", while it expresses as the dual problem the iterative progression of an (arbitrary positive) price vector to its successively corrected price vector converging to the equilibrium price vector i.e. "price of production".

Furthermore, Charasoff's linear economic analysis uses mainly numerical examples (and this only at most three-dimensionally) and therefore cannot be seen to contain an algebraic general proof. However, it exemplifies *de facto* the existence of the Frobenius root and its semi-positive eigenvector. It is indeed unknown whether Charasoff knew the papers of Perron or Frobenius, however, the earliness of his publication (one or two years after Frobenius) is as itself already remarkable. Besides, the characteristic feature of this exemplification consists in anticipating those procedures that were to be discovered nineteen years later by Mises/Geiringer. The main ideas of both works are quite similar. Furthermore, going beyond Mises/Geiringer (1929), Charasoff elaborated deliberately his framework of argument in order to guarantee the invariance of limit of iterations, and in doing so, enabled a profound insight to the duality of equation systems and the relationship between eigencolumn and eigenrow.

Acknowledgement

Earlier versions of this paper were presented at the Conference "The Pioneers of Linear Models of Production", January 17-18, 2013, University of Paris-Ouest, Nanterre and at the 21st International Input-Output Conference, July 7 – 12, 2013, Kitakyushu, Japan. I would like to thank the participants for useful comments. Special thanks go to Christian Bidard, Guido Erreygers, Christian Gehrke, Wilfried Parys, Bertram Schefold, Albert Steenge and Richard van den Berg. The usual caveat applies.

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Appendix 1. Uniqueness of limits in Charasoff's iterations

We first introduce the following symbols in the following appendices:

– input coefficient of good i for Sector j: $a_{ij} \in \mathbb{R}$

– input coefficient matrix: $\boldsymbol{A} := (a_{ij}) \in \mathbb{R}^{n \times n}$

– labour input coefficient for Sector j: $l_j \in \mathbb{R}$

– vector of labour input coefficients: $\boldsymbol{l} := (l_1, \cdots, l_n) \in \mathbb{R}^n$

– vector of physical wage (wage basket) per labour unit: $\boldsymbol{d} \in \mathbb{R}^n$

– augmented input coefficient matrix: $\boldsymbol{B} = (\boldsymbol{b}_{ij}) := \boldsymbol{A} + \boldsymbol{dl} \in \mathbb{R}^{n \times n}$

Note that we use inequality signs for vectors and matrices in this paper so that $\boldsymbol{X} > \boldsymbol{Y}$, $\boldsymbol{X} \gneq \boldsymbol{Y}$ and $\boldsymbol{X} \geq \boldsymbol{Y}$ denote that $\boldsymbol{X} - \boldsymbol{Y}$ is positive, semi-positive and non-negative, respectively.

According to the above notations, Charasoff's assumptions can be reformulated as follows.

(CA.1) $\boldsymbol{A} \geq \boldsymbol{0}$

(CA.2) $\boldsymbol{d} \gneq \boldsymbol{0}$

(CA.3) $\boldsymbol{l} > \boldsymbol{0}$

(CA.4) If $\boldsymbol{B}$ is decomposable, it can be transformed into the following form by suitable simultaneous substitutions of rows and columns:

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ 0 & 0 \end{pmatrix}$$

where $\boldsymbol{B}_{11}$ ($\gneq \boldsymbol{0}$) is indecomposable and each column of $\boldsymbol{B}_{12}$ is semi-positive.

Mori (2011, Section 2.2) shows, as stated in Section 3 of this paper, that the above assumptions (CA.1) to (CA.4) imply that (i) Matrix $\boldsymbol{B}$ has a dominant eigenvalue (denoted by λ_1) which is unique, simple and positive, (ii) there is a positive dominant eigenrow ($\boldsymbol{p}$), and (iii) there is a semi-positive eigencolumn ($\boldsymbol{x}$).

Next, define a matrix $\overline{\boldsymbol{B}}$ as $\overline{\boldsymbol{B}} := \boldsymbol{B} / \lambda_1$. Then, Mori (2011, Section 22.) shows as well, that the properties (i) to (iii) imply that the power sequence

$\{\bar{B}^t\}$ converges for $t \to \infty$ to a semi-positive matrix of rank one, denoted by $\bar{B}^*$, where $\bar{B}^* = \frac{1}{px} xp$, $p > \mathbf{0}$ and $x \gneq \mathbf{0}$.

Now, take an arbitrary semi-positive vector $z^{(1)} \in \mathbb{R}^n_+$ and define a sequence $\{z^{(t)}\}$ by $z^{(t)} := B^{t-1} z^{(1)}$. Then, because of the definition of $\bar{B}$ and $\bar{B}^*$, we have

$$\lim_{t\to\infty} \frac{z^{(t)}}{\left\|z^{(t)}\right\|} = \lim_{t\to\infty} \frac{B^{t-1} z^{(1)}}{\left\|B^{t-1} z^{(1)}\right\|} = \lim_{t\to\infty} \frac{\lambda_1^{t-1} \bar{B}^{t-1} z^{(1)}}{\left\|\lambda_1^{t-1} \bar{B}^{t-1} z^{(1)}\right\|}$$

$$= \lim_{t\to\infty} \frac{\bar{B}^{t-1} z^{(1)}}{\left\|\bar{B}^{t-1} z^{(1)}\right\|} = \frac{\bar{B}^* z^{(1)}}{\left\|\bar{B}^* z^{(1)}\right\|} = \frac{\frac{1}{px} xpz^{(1)}}{\left\|\frac{1}{px} xpz^{(1)}\right\|} = \frac{x}{\|x\|}.$$

Therefore, for any $z^{(1)} \gneq 0$, the normalized sequence of $\{z^{(t)}\}$ ("production series" of $z^{(1)}$) converges to the unique normalized dominant eigencolumn $\frac{x}{\|x\|}$.

And since $\bar{z}^{(t)}$ is continuous with respect to $z^{(t)}$, we have also $\lim_{t\to\infty} \frac{z^{(t)}}{\bar{z}^{(t)}} = \frac{x}{\bar{x}}$., where $\bar{z}^{(t)}$ and $\bar{x}$ are respectively the component of $z^{(t)}$ and x with the largest modulus.

Similarly, Take an arbitrary positive vector $w^{(1)} \in \mathbb{R}^n_{++}$ and define a sequence $\{w^{(t)}\}$ by $w^{(t)} := w^{(1)} B^{t-1}$. Then, we have

$$\lim_{t\to\infty} \frac{w^{(t)}}{\left\|w^{(t)}\right\|} = \frac{\frac{1}{px} w^{(1)} xp}{\left\|\frac{1}{px} w^{(1)} xp\right\|} = \frac{p}{\|p\|}.$$

Therefore, for any $w^{(1)} > \mathbf{0}$, the normalized sequence of $\{w^{(t)}\}$ ("capitalistic competition" starting from $w^{(1)}$) converges to the unique normalized dominant eigenrow $\frac{p}{\|p\|}$.

Appendix 2. Convergence of the sequence of "dimensions" to a eigenrow

Now, let us take unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, …, $\boldsymbol{e}_n$ as initial vectors of "production series" $\{\boldsymbol{z}^{(t)}\}$ in Appendix 1. Then, we can define the sequence of *dimensions* associated with "production series" of n unit vectors, $\{\eta^{(t)}\}$, as follows:

$$\eta^{(t)} := \left(\left\|\boldsymbol{B}^{t-1}\boldsymbol{e}_1\right\|,\cdots,\left\|\boldsymbol{B}^{t-1}\boldsymbol{e}_n\right\|\right).$$

Then, because of the definition of $\overline{B}$ and $\overline{B}^*$, we have

$$\lim_{t\to\infty}\frac{\eta^{(t)}}{\left\|\eta^{(t)}\right\|}=\lim_{t\to\infty}\frac{\left(\left\|\boldsymbol{B}^{t-1}\boldsymbol{e}_1\right\|,\cdots,\left\|\boldsymbol{B}^{t-1}\boldsymbol{e}_n\right\|\right)}{\left\|\left(\left\|\boldsymbol{B}^{t-1}\boldsymbol{e}_1\right\|,\cdots,\left\|\boldsymbol{B}^{t-1}\boldsymbol{e}_n\right\|\right)\right\|}=\lim_{t\to\infty}\frac{\left(\left\|\overline{\boldsymbol{B}}^{t-1}\boldsymbol{e}_1\right\|,\cdots,\left\|\overline{\boldsymbol{B}}^{t-1}\boldsymbol{e}_n\right\|\right)}{\left\|\left(\left\|\overline{\boldsymbol{B}}^{t-1}\boldsymbol{e}_1\right\|,\cdots,\left\|\overline{\boldsymbol{B}}^{t-1}\boldsymbol{e}_n\right\|\right)\right\|}$$

$$=\frac{\left(\left\|\overline{\boldsymbol{B}}^{*}\boldsymbol{e}_1\right\|,\cdots,\left\|\overline{\boldsymbol{B}}^{*}\boldsymbol{e}_n\right\|\right)}{\left\|\left(\left\|\overline{\boldsymbol{B}}^{*}\boldsymbol{e}_1\right\|,\cdots,\left\|\overline{\boldsymbol{B}}^{*}\boldsymbol{e}_n\right\|\right)\right\|}=\frac{\frac{1}{\boldsymbol{px}}\|\boldsymbol{x}\|\boldsymbol{p}}{\frac{1}{\boldsymbol{px}}\|\boldsymbol{x}\|\|\boldsymbol{p}\|}=\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}$$

Thus, we can verify that the normalized sequence of $\{\eta^{(t)}\}$ converges to the dominant eigenrow, and, therefore, the price of production.