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The Production Function and the Theory of Capital

INTRODUCTION

The dominance in neo-classical economic teaching of the concept of a production function, in which the relative prices of the factors of production are exhibited as a function of the ratio in which they are employed in a given state of technical knowledge, has had an enervating effect upon the development of the subject, for by concentrating upon the question of the proportions of factors it has distracted attention from the more difficult but more rewarding questions of the influences governing the supplies of the factors and of the causes and consequences of changes in technical knowledge.

Moreover, the production function has been a powerful instrument of miseducation. The student of economic theory is taught to write $O = f(L, C)$ where L is a quantity of labour, C a quantity of capital and O a rate of output of commodities.[1] He is instructed to assume all workers alike, and to measure L in man-hours of labour; he is told something about the index-number problem involved in choosing a unit of output; and then he is hurried on to the next question, in the hope that he will forget to ask in what units C is measured. Before ever he does ask, he has become a professor, and so sloppy habits of thought are handed on from one generation to the next.

The question is certainly not an easy one to answer. The capital in existence at any moment may be treated simply as "part of the environment in which labour works."[2] We then have a production function in terms of labour alone. This is the right procedure for the short period within which the supply of concrete capital goods does not alter, but outside the short period it is a very weak line to take, for it means that we cannot distinguish a change in the stock of capital (which can be made over the long run by accumulation) from a change in the weather (an act of God).

We may look upon a stock of capital as the specific list of all the goods in existence at any moment (including work-in-progress in the pipe lines of production). But this again is of no use outside the strict bounds of the short period, for any change in the ratio of capital to labour involves a re-organisation of methods of production and requires a change in the shapes, sizes and specifications of many or all the goods appearing in the original list.[3]

As soon as we leave the short period, however, a host of difficulties appear. Should capital be valued according to its future earning power or its past costs?

When we know the future expected rate of output associated with a certain capital good, and expected future prices and costs, then, if we are given a rate of interest, we can value the capital good as a discounted stream of future profit which it will earn. But to do so, we have to begin by taking the rate of interest as given, whereas the main purpose of the production function is to show how wages and the rate of interest (regarded as the wages of capital) are determined by technical conditions and the factor ratio.

[1] Throughout this essay we shall be abstracting from land as a factor of production, so we will not bother the student with it.

[2] Keynes, *General Theory*, p. 214.

[3] In Professor Robertson's example, when a tenth man joins nine who are digging a hole, nine more expensive spades are turned into nine cheaper spades and a bucket to fetch beer. (*Economic Fragments*, p. 47.)

Are we then to value capital goods by their cost of production ? Clearly money cost of production is neither here nor there unless we can specify the purchasing power of money, but we may cost the capital goods in terms of wage units, that is, in effect, to measure their cost in terms of a unit of standard labour.

To treat capital as a quantity of labour time expended in the past is congenial to the production-function point of view, for it corresponds to the essential nature of capital regarded as a factor of production. Investment consists, in essence, in employing labour now in a way which will yield its fruits in the future while saving is making current products available for the workers to consume in the meantime ; and the productiveness of capital consists in the fact that a unit of labour that was expended at a certain time in the past is more valuable to-day than a unit expended to-day, because its fruits are already ripe.

But here we encounter a fundamental difficulty which lies at the root of the whole problem of capital. A unit of labour is never expended in a pure form. All work is done with the assistance of goods of some kind or another. When Adam delved and Eve span there were evidently a spade and a spindle already in existence. The cost of capital includes the cost of capital goods, and since they must be constructed before they can be used, part of the cost of capital is interest over the period of time between the moment when work was done in constructing capital goods and the time when they are producing a stream of output. This is not just a consequence of capitalism, for equally in a socialist society a unit of labour, expended to-day, which will yield a product in five years' time, is not the same thing as a unit which will yield a product to-morrow.

Finally, even if it were possible to measure capital simply in terms of labour time, we still should not have answered the question : of what units is C composed ? When we are discussing accumulation, it is natural to think of capital as measured in terms of product. The process of accumulation consists in refraining from consuming current output in order to add to the stock of wealth. But when we consider what addition to productive resources a given amount of accumulation makes, we must measure capital in labour units, for the addition to the stock of productive equipment made by adding an increment of capital depends upon how much work is done in constructing it, not upon the cost, in terms of final product, of an hour's labour. Thus, as we move from one point on a production function to another, measuring capital in terms of product, we have to know the product-wage rate in order to see the effect upon production of changing the ratio of capital to labour. Or if we measure in labour units, we have to know the product-wage in order to see how much accumulation would be required to produce a given increment of capital. But the wage rate alters with the ratio of the factors : one symbol, C, cannot stand both for a quantity of product and a quantity of labour time.

All the same, the problem which the production function professes to analyse, although it has been too much puffed up by the attention paid to it, is a genuine problem. To-day, in country Alpha, a length of roadway is being cleared by a few men with bulldozers ; in Beta a road (of near-enough the same quality) is being made by some hundreds of men with picks and ox-carts. In Gamma thousands of men are working with wooden shovels and little baskets to remove the soil. When all possible allowances have been made for differences in national character and climate, and for differences in the state of knowledge, it seems pretty clear that the main reason for this state of affairs is that capital in some sense is more plentiful in Alpha than in Gamma. Looked at from the point of view of an individual capitalist, it would not pay to use Alpha methods in Gamma (even if unlimited finance were available) at the rate of interest which is ruling, and looked at from the point of view of society, it would need a prodigious effort of accumulation to raise all the labour available in Gamma even to the Beta level of technique. The problem is a real one. We cannot abandon the production function without an effort to rescue the element of common-sense that has been entangled in it.

THE QUANTITY OF CAPITAL

"Capital" is not what capital is called, it is what its name is called. The capital goods in existence at a moment of time are all the goods in existence at that moment. It is not all the things in existence. It includes neither a rubbish heap nor Mont Blanc. The characteristic by which "goods" are specified is that they have value, that is purchasing power over each other. Thus, in country Alpha an empty petrol tin is not a "good," whereas in Gamma where old tins are a source of valuable industrial raw material, it is.

The list of goods is quite specific. It is so many actual particular objects, called blast furnaces, overcoats, etc., etc. Goods grouped under the same name differ from each other in the details of their physical specifications and these must not be over-looked. Differences in their ages are also important. A blast furnace twenty years old is not equivalent to a brand new one of the same specification in other respects, nor is an egg twenty days old equivalent to a brand new one. There is another relevant characteristic of the goods. An overcoat requires one body to wear it, and an egg one mouth to eat it. Without one body, or one mouth, they are useless, and two bodies or mouths (at a given moment of time) cannot share in using them. But a blast furnace can be used by a certain range of numbers of bodies to turn iron ore into iron. Therefore the description of a blast furnace includes an account of its rate of output as a function of the number of bodies operating it. (Since we shall not discuss short-period problems, the number of bodies actually working each piece of equipment, in the situations with which we shall be concerned, is the number which is technically most appropriate to it.)

There is another aspect of the goods which is quite different. Of two overcoats, completely similar in all the above respects, one is on the body of Mrs. Jones, who is purring with inward delight at her fine appearance. Another is on the body of Mrs. Snooks, who is grizzling because, her husband's income being what it is, she is obliged to buy mass-produced clothes. In what follows we shall not discuss this aspect of goods at all. We take it that an overcoat (Mark IV) is an overcoat (Mark IV), and no nonsense.

Now, this enormous who's who of individual goods is not a thing what we can handle at all easily. To express it as a *quantity* of goods we have to evaluate the items of which it is composed. We can evaluate the goods in terms of the real cost of producing them—that is, the work and the formerly existing goods required to make them, or in terms of their value expressed in some unit of purchasing power, or we can evaluate them according to their productivity—that is, what the stock of goods will become in the future if work is done in conjunction with it.

In a position of equilibrium all three evaluations yield equivalent results ; there is a quantity which can be translated from one number to another by changing the unit. This is the definition of equilibrium. It entails that there have been no events over the relevant period of past time which have disturbed the relation between the various valuations of a given stock of goods, and that the human beings in the situation are expecting the future to be just like the past—entirely devoid of such disturbing events. Then the rate of profit ruling to-day is the rate which was expected to rule to-day when the decision to invest in any capital good now extant was made, and the expected future receipts, capitalised at the current rate of profit, are equal to the cost of the capital goods which are expected to produce them.

When an unexpected event occurs, the three ways of evaluating the stock of goods part company and no amount of juggling with units will bring them together again.

We are accustomed to talk of the rate of profit on capital earned by a business as though profits and capital were both sums of money. Capital when it consists of as yet uninvested finance is a sum of money, and the net receipts of a business are sums of money. But the two never co-exist in time. While the capital is a sum of money, the profits are not yet being earned. When the profits (quasi-rents) are being earned, the capital has ceased to be money and become a plant. All sorts of things may happen which cause the value of the plant to diverge from its original cost. When an event has occurred, say, a fall in prices, which was not foreseen when investment in the plant was made, how do we regard the capital represented by the plant ?

The man of deeds, who has decisions to make, is considering how future prospects have altered. He is concerned with new finance or accrued amortisation funds, which he must decide how to use. He cannot do anything about the plant (unless the situation is so desperate that he decides to scrap it). He is not particularly interested (except when he has to make out a case before a Royal Commission) in how the man of words, who is measuring capital, chooses to value the plant.[1]

The man of words has a wide choice of possible methods of evaluation but none of them is very satisfactory. First, capital may be conceived of as consisting either in the cost or in the value of the plant. If cost is the measure, should money cost actually incurred be reckoned ? It is only of historical interest, for the purchasing power of money has since changed. Is the money cost to be deflated ? Then by what index ? Or is capital to be measured at current replacement cost ? The situation may be such that no one in his senses would build a plant like this one if he were to build now. Replacement cost may be purely academic. But even if the plant is, in fact, due to be replaced by a replica of itself at some future date, we still have to ask what proportion of the value of a brand new plant is represented by this elderly plant ? And the answer to that question involves future earnings, not cost alone.

If the capital is to be measured by value, how decide what the present value of the plant is ? The price at which it could be sold as an integral whole has not much significance, as the market for such transactions is narrow. To take its price on the Stock Exchange (if it is quoted) is to go before a tribunal whose credentials are dubious. If the capital-measurer makes his own judgment, he takes what he regards as likely to be the future earnings of the plant and discounts them at what he regards as the right rate of interest for the purpose, thus triumphantly showing that the most probable rate of profit on the capital invested in the plant is equal to the most appropriate rate of interest.

All these puzzles arise because there is a gap in time between investing money capital and receiving money profits, and in that gap events may occur which alter the value of money.

To abstract from uncertainty means to postulate that no such events occur, so that the *ex ante* expectations which govern the actions of the man of deeds are never out of gear with the *ex post* experience which governs the pronouncements of the man of words, and to say that equilibrium obtains is to say that no such events have occurred for some time, or are thought liable to occur in the future.

The ambiguity of the conception of a quantity of capital is connected with a profound methodological error, which makes the major part of neo-classical doctrine spurious.

[1] " A man of words but not of deeds
Is like a garden full of weeds."
This is sadly true of the theory of capital.

The neo-classical economist thinks of a position of equilibrium as a position towards which an economy is tending to move as time goes by. But it is impossible for a system to *get into* a position of equilibrium, for the very nature of equilibrium is that the system is already in it, and has been in it for a certain length of past time.

Time is unlike space in two very striking respects. In space, bodies moving from A to B may pass bodies moving from B to A, but in time the strictest possible rule of one-way traffic is always in force. And in space the distance from A to B is of the same order of magnitude (whatever allowance you like to make for the Trade Winds) as the distance from B to A ; but in time the distance from to-day to to-morrow is twenty-four hours, while the distance from to-day to yesterday is infinite, as the poets have often remarked. Therefore a space metaphor applied to time is a very tricky knife to handle, and the concept of equilibrium often cuts the arm that wields it.

When an event has occurred we are thrown back upon the who's who of goods in existence, and the " quantity of capital " ceases to have any other meaning. Then only that part of the theory of value which treats of the short period, in which the physical stock of capital equipment is given, has any application.

Nevertheless, some of the internal relations between the parts of a system can be most easily thought about by imagining it to be in equilibrium, and an examination of these relations is useful, provided that it is conducted with due regard to its limitations.

In what follows we are concerned with such internal relations, shown by the properties of an equilibrium situation—in particular with the interrelations between three of its properties—the quantity of capital, the labour force, and the state of technical knowledge.

ASSUMPTIONS AND DEFINITIONS

We must make certain drastic assumptions in order to isolate our problem.

I

(1) Labour is perfectly homogeneous. All men are alike, and each (when employed) performs a regular number of hours' work over a year.

(2) Land, including all non-produced means of production, is homogeneous. This involves that there are no specialised factors of production such as particular kinds of soil or mineral deposits, and no influence of geography upon production.

(3) All households consume commodities in the same proportions, irrespective of changes in their relative prices ; differences in average income per head and in the distribution of income between individuals have no effect upon the composition of demand for final output. We can then measure output simply in units of a composite commodity, representing each good in the proportion in which it is being produced. In so far as net investment is going on, capital goods are represented in the unit of final output.[1]

(4) There are no economies or diseconomies of scale for output as a whole or for particular commodities.

I call these assumptions the trick assumptions, because they are only a scarecrow to keep the index number birds off our fields until after the harvest.

[1] There is a certain awkwardness in assuming that the proportion of net investment in total income is independent of the distribution of income, but this difficulty does not impinge upon the questions that we have to discuss.

II

The argument is confined to a two-factor economy, and a two-class society.

(1) To isolate the problem of capital we abstract from land and all non-produced means of production. The " free gifts of nature " are completely free, plentiful and unappropriated. Space and air are necessary to production, but neither commands a price.

(2) We rule out assets, such as the goodwill of a business, which are wealth to an individual but not to society ; we mean by capital physical productive resources, which would have the same significance in an artizan, a capitalist and a socialist economy.

(3) To eliminate the influence of the entrepreneur, we assume that economies of scale internal to a productive unit are exhausted, in each line of production, at a moderate rate of output, and that an experienced entrepreneur has no advantage over a new hand. The know-how of production is widely diffused and differences in skill of management are unimportant. The only qualification required for employing labour is then the ownership of sufficient capital (or command of sufficient credit) to set up a productive unit of the minimum size. (The assumption of no economies of scale (I, 2) appears here as well as in the scarecrow.) In these conditions, the distinction between interest and profit ceases to be significant. There is no specific " reward of enterprise " apart from the reward of owning capital, and no owner of capital will provide finance for an entrepreneur at appreciably less than the rate of profit which he can expect to obtain by using his capital himself to employ labour. Thus in our system, profits and wages exhaust total net income.

The size of individual productive units, above the technical minimum, is not strictly limited in the long run, but there is no particular pressure towards a growth in size, and, when the total amount of capital is accumulating, the number of independent capitalists is conceived to multiply more or less in proportion to the amount of capital ; the average scale on which they operate is more or less constant, at a size which is small in relation to the markets supplied, so that conditions of atomistic competition can prevail. (The minimum size of a productive unit is, however, large enough to make it very difficult for a worker to become a capitalist, so that the system does not relapse into an economy of artizans.)

III

We call the stock of goods in existence at any moment *physical capital.* The value of these goods in terms of a unit of output we call *capital* simpliciter. Capital valued in terms of wage units we call *real capital*; though it must be observed that there is a slightly misleading flavour about this term, since the cost of capital goods, in terms of wage units, includes interest over the time required to construct them and to use them in production. Thus the same stock of physical goods represents a larger amount of *real capital* when the rate of interest is higher (and has been higher in the past) than when it is (and has been) lower.

We call the ratio of real capital to man hours of current employment per annum the *factor ratio.*

We take as the wage unit the price of an hour's labour in terms of the composite unit of product, no matter whether the worker who performs an hour's work is paid in cash or in peanuts.[1] In what follows we mean by " wages," the cost of labour to

[1] When Mr. (now Professor) Hicks eliminated the equation for money from the $n + 1$ value equations for n commodities, Mr. Lerner remarked, " If I eliminate the equation for peanuts, what then ? " I take peanuts as an example of a commodity chosen at random, in allusion to this extremely sapient contribution to the pure theory of value.

the employer, in terms of product. When we have occasion to relax the trick assumptions and to look at wages from the point of view of workers regarded as consumers, we call the purchasing power of the wage a worker gets his real wage. Thus we are using the " real wage rate " in its common or garden sense, and the " wage rate " simpliciter in a special sense.

LONG-PERIOD EQUILIBRIUM

Our argument treats of the relations between quantities of factors of production in existence. It cannot take into its purview the disturbances arising from the process of changing the quantities of factors. We must, therefore, rule out all problems of effective demand and confine our argument to positions of equilibrium. What does this imply ?

One notion of equilibrium is that it is reached (with a constant labour force) when the stock of capital and the rate of profit are such that there is no motive for further accumulation. This is associated with the idea of an ultimate thorough-going stationary state,[1] in which the rate of profit is equal to the " supply price of waiting." In this situation an accidental increase in the stock of capital above the equilibrium quantity would depress the rate of profit below this supply price, and cause the additional capital to be consumed ; while any reduction would raise the rate of profit, and cause the deficiency to be made good. Equilibrium prevails when the stock of capital is such that the rate of profit is equal to the supply price of that quantity of capital.

But this notion is a very treacherous one. Why should the supply price of waiting be assumed positive ? In Adam Smith's forest there was no property in capital and no profit (the means of production, wild deer and beavers, were plentiful and unappropriated). But there might still be waiting and interest. Suppose that some hunters wish to consume more than their kill, and others wish to carry consuming power into the future. Then the latter could lend to the former to-day, out of to-day's catch, against a promise of repayment in the future. The rate of interest (excess of repayment over original loan) would settle at the level which equated supply and demand for loans. Whether it was positive or negative would depend upon whether spendthrifts or prudent family men happened to predominate in the community. There is no *a priori* presumption in favour of a positive rate. Thus the rate of interest cannot be accounted for as the " cost of waiting ".

The reason why there is always a demand for loans at a positive rate of interest, in an economy where there is property in the means of production and means of production are scarce, is that finance expended now can be used to employ labour in productive processes which will yield a surplus in the future over costs of production. Interest is positive because profits are positive (though at the same time the cost and difficulty of obtaining finance play a part in keeping productive equipment scarce, and so contribute to maintaining the level of profits).

Where the " supply price of waiting " is very low or negative, the ultimate stationary equilibrium cannot be reached until the rate of profit has fallen equally low, capital has ceased to be scarce and capitalism has ceased to be capitalism. Therefore this type of equilibrium is not worth discussing.

The other way of approaching the question is simply to postulate that the stock of capital in existence at any moment is the amount that has been accumulated up to date, and that the reason why it is not larger is that it takes time to grow. This is the conception which is adopted in this essay. At any moment there is a certain stock of capital in existence. If the rate of profit and the desire to own more wealth are such

[1] Pigou, *The Economics of Stationary States*.

as to induce accumulation, the stock of capital is growing and, provided that labour is available or population growing, the system is in process of expanding without any disturbance to the conditions of equilibrium. (If two snapshots were taken of the economy at two different dates, the stock of capital, the amount of employment and the rate of output would all be larger, in the second photograph, by a certain percentage, but there would be no other difference.) If the stock of capital is being kept constant over time, that is merely a special case in which the rate of accumulation happens to be zero. (The two snapshots would then be indistinguishable.)

In the internal structure of the economy conditions of long-period equilibrium are assumed to prevail. Each type of product sells at its normal long-run supply price. For any one type of commodity, profit, at the rate ruling in the system as a whole, on the cost of capital equipment engaged in producing it, is part of the long-run supply price of the commodity, for no commodity will continue to be produced unless capital invested for the purpose of producing it yields at least the same rate of profit as the rest. (It is assumed that capitalists are free to move from one line of production to another.) Thus the " costs of production " which determine supply price consist of wages and profits. In this context the notion of a quantity of capital presents no difficulty, for, to any one capitalist, capital is a quantity of value, or generalised purchasing power, and under our trick assumptions, in a given equilibrium situation, a unit of any commodity can be used as a measure of purchasing power.

Since the system is in equilibrium in all its parts, the ruling rate of profit is being obtained on capital which is being used to produce capital goods, and enters into their " cost of production ". Profit on that part of the cost of capital represented by this profit is then a component of the " cost of production " of final output. A capitalist who buys a machine ready made pays a price for it which includes profit to the capitalist who sells it. The profit a capitalist who has the machine built in his own workshops will expect to receive, from sales of the final output, includes profit on the interest (at a notional rate equal to the ruling rate of profit) on the cost of having the machine built reckoned over the period of construction. For when he builds the machine himself he has a longer waiting period between starting to invest and receiving the first profit. If he could not earn profit on the notional interest cost, he would prefer to make an investment where there was a shorter waiting period, so that he could receive actual profit earlier. The actual profit he could plough into investment; thus acquiring (over the same waiting period) the same quantity of capital as in the case where he builds the machine for himself. (He would also have the advantage that he could change his mind and consume the profit, whereas in the first case he is committed to the whole scheme of investment once he begins.) Thus investments with a long gestation period will not be made unless they are expected to yield a profit on the element of capital cost represented by compound interest over the gestation period (if there were uncertainty, they would have to be expected to yield more, to compensate for the greater rigidity of the investment plan).

We need not go back to Adam to search for the first pure unit of labour that contributed to the construction of existing equipment. The capital goods in being to-day have mutually contributed to producing each other, and each is assumed to have received the appropriate amount of profit for doing so.

So much for the supply price of an item of new equipment. How are we to reckon the supply price of part-worn equipment ? Investment in new equipment is not made unless its gross earnings (excess of output over wages bill in terms of output) are expected to be sufficient to amortise the investment over its working life, allowing for interest at the ruling rate on accrued amortisation funds, as well as providing profit at the ruling rate. The supply price of an equipment which has been working for a certain time may be regarded as its initial cost accumulated up to date at compound interest, *minus* its gross earnings also accumulated from the dates at which they accrued up to the present, for this corresponds to the expectations which induced capitalists in the past to make the investment concerned.

Since initial cost is incurred at the beginning, and earnings accrue over time, the element of interest on cost in the above calculation exceeds the element of interest on earnings. Thus when an equipment has yielded a quarter of its expected total earnings, its supply price, in this sense, is somewhat more than three-quarters of its initial cost ; half-way through, somewhat more than half its initial cost, and so forth, the difference at any moment being larger the higher the rate of interest. Over its life the accumulated interest on its earnings, so to say, catches up upon the accumulated interest on its cost, so that at the end of its life it is fully paid off and its supply price (abstracting from scrap value) has fallen to zero.

The value of an equipment depends upon its expected future earnings. It may be regarded as future earnings discounted back to the present at a rate corresponding to the ruling rate of interest. In equilibrium conditions the supply price (in the above sense) and the value of an equipment are equal at all stages of its life.[1]

Equilibrium requires that the stock of items of equipment operated by all the capitalists producing a particular commodity is continuously being maintained. This entails that the age composition of the stock of equipment is such that the amortisation funds provided by the stock as a whole are being continuously spent on replacements. When the stock of equipment is in balance there is no need to enquire whether a particular worker is occupied in producing final output or in replacing plant. The whole of a given labour force is producing a stream of final output and at the same time maintaining the stock of equipment for future production. Nor is it necessary to inquire what book-keeping methods are used in reckoning amortisation quotas. These affect the relations between individual capitalists, but cancel out for the group as a whole.

In equilibrium the age composition of the stock of equipment is stable, but the total stock may be in course of expanding. The average age of the plants making up a balanced stock of stable age composition varies with the length of life of individual plants. If the total stock is remaining constant over time, the average age is equal to half the length of life. If the stock has been growing the proportion of younger plants is greater and average age is less than half the life span. (There is an exact analogy with the age composition of a stable population.)

The amount of capital embodied in a stock of equipment is the sum of the supply prices (reckoned as above) of the plants of which it is composed, and the ratio of the amount of capital to the sum of the costs of the plants when each was brand new is higher the greater the rate of interest.[2]

[1] The equalisation of the value of two annuities at any point of time entails their equalisation at any other point of time. If the cost of a new machine is equal, at the moment when it is brand new, to the discounted value of its expected gross earnings, it follows that, at any later point of time, the accumulated value of the original cost and gross earnings up to date will, if expectations have been proved correct up to date and are unaffected for the future, be equal to the present value of the remaining gross earnings expected over the future. Cf. Wicksell, " Real Capital and Interest," *Lectures* (English edition), Vol. I, p. 276.

[2] The order of magnitude of the influence of the rate of interest is shown by the formula provided in the Mathematical Addendum by D. G. Champernowne and R. F. Kahn. For this formula it is necessary to assume (*a*) that the total stock of capital is constant over time, (*b*) that earnings are at an even rate over the life of the plant. C is the capital value of an investment, K the initial outlay, r the rate of interest and T the period over which the asset earns. For values of rT less than 2 we use the approximation $C/K = \frac{1}{2}(1 + \frac{1}{6}rT)$.

On this basis, when the rate of interest is, for example, 6 per cent, a machine of ten years' life costing £100 when new must earn £13.3 per annum surplus over the current outlay on working it (including current repairs). The yield will then be 6 per cent on a capital value of £55. (*Continued overleaf.*)

Equilibrium requires that the rate of profit ruling to-day was expected to be ruling to-day when investment in any plant now extant was made, and the expectation of future profits obtaining to-day was expected to obtain to-day. Thus the value of capital in existence to-day is equal to its supply price calculated in this manner. The heavy weight which this method of valuing capital puts upon the assumptions of equilibrium emphasises the impossibility of valuing capital in an uncertain world. In a world where unexpected events occur which alter values, the points of view of the man of deeds, making investment decisions about the future, and of the man of words making observations about the past, are irreconcilable, and all we can do is botch up some conventional method of measuring capital that will satisfy neither of them.

THE TECHNIQUE OF PRODUCTION

How can we reduce the amorphous conception of a " state of technical knowledge " to definite terms ? Let us suppose that for any given line of production, we can draw up a list of actual techniques which could be used, with a given amount of current labour, to produce a flow of output of the commodity concerned, while maintaining the productive equipment required intact. Each technique is conceived to be specified in detail, and entails the use of particular items of equipment and a particular quantity of work-in-progress in the productive pipe line. Other things equal, a technique involving a longer production period (from clipping the sheep to selling the overcoat) requires a larger run-out of man hours embodied in work-in-progress. This is treated as part of the stock of capital goods required by this technique.[1] We then amalgamate the lists for particular commodities in such a way as to get a flow of output of commodities in the proportions dictated by the assumption that the composition of final output is given. We thus have a set of blue prints of techniques, each of which could be used to employ a given amount of labour to produce a flow of output.

The techniques are listed in a hierarchy, Alpha, Beta, etc., according to the rate of output which they produce with a given number of men. (The number of men must be a common multiple of the numbers required by a self-contained unit of each technique, to avoid " a ragged edge " when workers are allotted to plants.)

The internal description of a given technique is a purely engineering question, but the list of techniques cannot be drawn up in purely engineering terms, without

A capitalist who operates such a machine may amortise the initial investment by paying £10 every year into a sinking fund. Reckoning at simple interest only, he receives interest on the amortisation fund after one year of £0.6 ; in the sixth year £3 ; in the last year £5.4. Thus the annual return on his investment of £100 rises over the ten year period from approximately £3 to approximately £9 (the " ragged edge " is due to reckoning the amortisation quota as paid at the end of a year, and reckoning interest as paid annually instead of continuously). The undiscounted average annual income is, therefore, £6 (6 per cent on the initial outlay). Compound interest over the period compensates for discounting this income. Over the life history of the machine, the fall in capital value from £100 to zero is in step with the rise in the amortisation fund.

A group of ten such machines of ages zero to nine years have a pattern of values, at any moment, which corresponds to the pattern over time of a single machine. It requires an annual outlay on renewals of £100 permanently to maintain the stock of machines. They represent a capital value of £550 and yield a return of £33 per annum.

If the rate of interest were 10 per cent, rT would be equal to 1 and the capital value (abstracting from a higher initial cost of machines due to the higher interest rate) would be £583 ; the earnings of each machine would then have to be £15.8 to yield the required rate of profit.

If the length of life of machines was twenty years, and the rate of interest 5 per cent, capital value would again be £583, and each machine would have to yield £7.9 per annum (£5 for amortisation and £2.9 for interest) ; at 10 per cent, rT would be equal to 2 ; the capital value would then be £666, and each machine would have to yield £11.7 per annum.

[1] This way of looking at things is easier to understand than applying the notion of the " length of the period of production " to long-lived equipment. It is hard to treat, say, a loom as a length of time, but perfectly easy to regard a quantity of wool, a quantity of yarn and a quantity of cloth as part of the physical equipment required for producing a steady flow of output of overcoats.

regard to economic considerations. (From an engineering point of view, it is possible to use a steam hammer for cracking nuts.) We must therefore compare the costs of the equipments required by various techniques in order to be sure that we are nowhere using more capital to produce less product.[1]

The cost of an equipment can be reckoned in wage units (that is, labour time), but it includes an allowance for interest on costs incurred in the past to create the stock of goods in existence to-day (this, as we have seen, depends partly upon the time taken to construct capital goods and partly upon the time over which they are used). Thus cost in wage units must be expressed as a function of the rate of interest. Let us imagine that we proceed by first taking any reasonable value for the rate of interest for a preliminary run over the field. Take an outfit of equipment as a going concern, with all its pipe lines full of work-in-progress, composed of plants of the age distribution appropriate to the rate (which may be zero) at which the total stock of capital is expanding. Now imagine that our notional rate of interest represents the rate of profit on capital actually ruling to-day, and that the same rate of profit has been ruling as long as any item in the stock of capital goods now extant has been in existence. Then we reckon the supply price of the equipment at that rate of interest, assuming that each item in the stock has been earning profits at that rate since it came into service.

When the equipment required by the techniques has been costed in this way, any technique which involves a greater cost than another for the same or a smaller rate of output is ruled out, for it is uneconomic (at the assumed rate of interest), however beautiful it may be from an engineering point of view. Then an Alpha technique produces a higher rate of output with a given amount of labour, and involves a greater cost of equipment, than a Beta technique, Beta a higher output and greater cost of equipment than Gamma, and so on down the hierarchy. We shall describe the Alpha technique as "more mechanised" than the Beta technique, and so on down the list, for Alpha involves a greater quantity of real capital (capital in terms of past labour invested) per unit of labour currently employed, than does Beta, and this will normally show itself in a greater complexity of the equipment used. In so far as the advantage of the superior technique lies in a longer life for items of equipment, it shows itself in a smaller proportion of the given labour force being occupied at any moment in replacement of capital goods. The concept of "mechanisation" is also stretched to cover working capital. Thus, in the famous example of the wine cellar, a set of barrels one of every age from one to ten years is regarded as equipment for a more mechanised technique than one consisting of barrels of ages from one to nine years (each cellar being tended by one night watchman). But this example is sophisticated by the fact that the output of the "more mechanised" technique is superior in quality (the age of the wine representing final output), not quantity, to that of the "less mechanised."

We now repeat the costings at all rates of interest over a reasonable range. In the course of this process we may create gaps in the list of techniques, or find gaps formerly existing filled up, as the notional rate of interest alters. For example, if the man-hours required to construct a plant appropriate to Gamma technique are spread over a long time, or are heavily concentrated at the beginning of the gestation period, while those required to construct a Beta plant are spread over a short time or are bunched near the moment of completion, or if a Gamma plant is more durable, so that the average age of the items making up a balanced outfit of plants is greater, a rise in the rate of interest may raise Gamma's cost above Beta's. But Gamma's rate

[1] This seems to have been the point that Keynes had in mind when he compared the lengthiness of a productive process to its smelliness. (*General Theory*, p. 215.)

of output is lower. Thus at this notional rate of interest Gamma falls out of the hierarchy.[1]

Techniques may appear or disappear in the list as the notional rate of interest alters, but two techniques can never reverse their positions, for they were listed in the first place in order of rates of output with a given amount of current labour, and this is a purely engineering fact, independent of the rate of interest.

The difference between a more and less mechanised technique is not produced by adding some spoonfuls of investment to a *pot-au-feu* of " capital ". Each technique involves its own specific blue prints, and there may be no recognisable items in common between one and any other. There is, therefore, no reason why the hierarchy should consist of small steps in output per man. It may do so, or it may consist of a series of jumps with appreciable gaps between each technique and the next. It seems obvious, for instance, that large jumps occur between techniques involving different sources of power.

The individual capitalist is assumed to choose between possible techniques in such a way as to maximise the surplus of output that a given amount of capital yields over wages cost in terms of his own product, and thus to obtain the highest rate of profit on capital that the available techniques make possible.[2]

Given the hierarchy of techniques, the higher is the wage rate the more mechanised is the technique which is chosen. This principle is usually described in a somewhat mystifying way in terms of a " substitution of capital for labour " as the cost of labour rises. The essential point, however, is very simple. An Alpha plant involves a greater capital cost and yields a higher rate of output with a given amount of current labour than a Beta plant. At a higher wage rate both plants yield a smaller profit per man employed than at a lower wage rate, but a given difference in the wage rate reduces the excess of output over wages (that is, profit) in a smaller proportion where output is higher.

This can be illustrated by means of a crude numerical example, comparing the profitability of three techniques at two wage rates. To keep the arithmetic simple we take very large differences between techniques. The difference in the capital cost of the same plant at two different wage rates is not proportional to the difference in the wage rate, for at a higher wage there is a lower rate of profit prevailing and consequently a smaller element of interest in cost. Again to keep the example simple, we assume that the capital costs of the three plants are affected in the same way by a difference in the rate of interest, so that their relative costs are the same at both wage rates.

At the wage rate of 1 per man, Gamma and Beta techniques yield the same rate of profit, and Alpha technique a lower rate. At the wage 1.1, Beta and Alpha yield the same rate of profit, and Gamma a lower rate (in fact, zero). Thus when wages are one unit of product per man year, the individual capitalist is indifferent between Gamma and Beta technique—52 units of capital in terms of value purchase one Beta plant yielding a profit of 10 per annum, or two Gamma plants yielding 5 each. (If there were any uncertainty about future profits, the Gamma technique would be preferred, since an investment which is technically divisible is more flexible than one which is an integrated whole.) Alpha technique is out of the question. Similarly, when the wage rate is 1.1, Beta and Alpha are indifferent and Gamma is out of the question.

[1] An example of this phenomenon is illustrated in Fig. I.

[2] For simplicity of exposition we are postulating integrated production, so that raw materials, power, etc., bought by one capitalist from others do not appear as costs.

Number of Men per Plant : 50

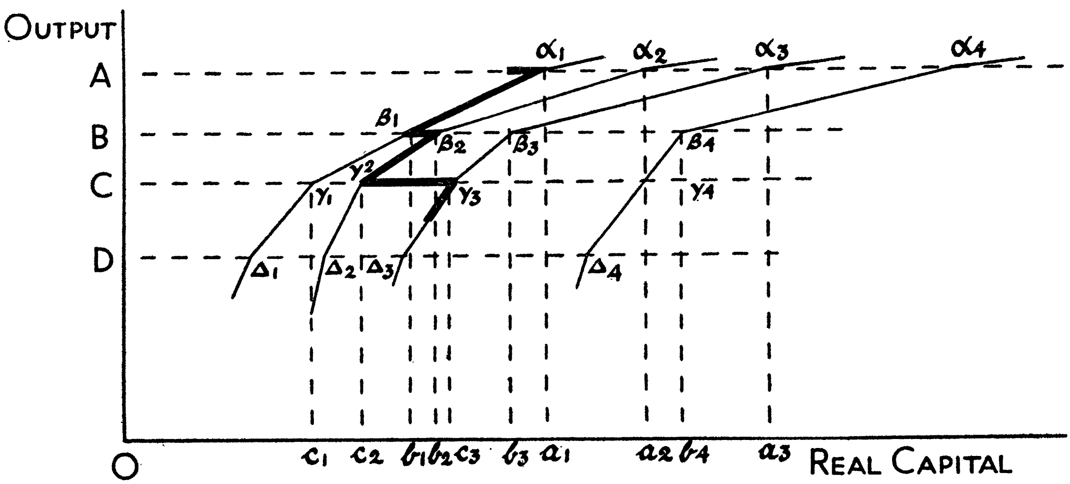
Plant	γ	β	α	γ	β	α
Wage rate ..	1	1	1	1.1	1.1	1.1
Capital .. Product .. Wage bill .. Profit	26 55 50 5	52 60 50 10	104 65 50 15	27.5 55 55 0	55 60 55 5	110 65 55 10
Rate of Profit (approximate)	19%	19%	14%	0	9%	9%

In an equilibrium position the technique of production throughout the economy has been chosen according to this principle,[1] and the factor ratio (real capital per man employed), given the technical possibilities, is governed by the wage rate.

THE RATIO OF CAPITAL TO LABOUR

When the hierarchy of techniques has been specified we can draw a factor-ratio curve connecting real capital per man employed with the rate of output. The reader is warned that it has a somewhat bizarre appearance compared to the smooth sweep of the usual text-book production function.

FIG. I



OA is the rate of output of a constant number of men using Alpha technique, *OB* the output with Beta technique, *OC* with Gamma technique, and *OD* with Delta technique. Oa_1 is the cost in terms of wage units of a balanced outfit of equipment

[1] For a possible case of multiple equilibrium see Appendix, p. 103.

required by Alpha technique, calculated at a certain rate of interest, Oa_2 the cost of the same equipment at a higher rate of interest, and so for each of the techniques at ascending interest rates.[1]

However large the jump may be between one technique and the next there is a continuous relationship between output per head and the factor ratio. Between, say, outputs OC and OB, Gamma and Beta techniques are both being employed (as in the numerical example, when the wage rate was 1) and between OB and OA, Beta and Alpha techniques are both being employed (as, in the example, at the wage rate of 1.1). A rise in the factor ratio from Oc_1 to Ob_1, or from Oc_2 to Ob_2, is due to a gradual increase in the proportion of Beta plants in use, which causes a rise in the average of real capital per man and a rise in average output per head.

Thus we can draw, for each rate of interest, a *productivity curve*, consisting of a series of straight lines of changing slope, which exhibits the rise of output due to ascending the hierarchy of techniques.

Each curve purports to show the engineering characteristics of the techniques, but it would be of no use to ask an engineer how they should be drawn. He does not understand the meaning of a given state of technical knowledge, for he is learning something fresh everyday as he works. " Given knowledge " is a drastic abstraction (though it may have some relevance for a " backward " country which can use a survey of past and present techniques operated in " advanced " countries as a catalogue of possibilities to choose from). It implies some absolute upper limit to the rate of output that a given labour force can produce. Our curves, therefore, must be drawn with a maximum output as an asymptote and their general shape is concave to the x axis[2]. To avoid complicating the exposition we will postulate that they are concave throughout.[3]

The relation between one curve and the next depends upon the reaction of the cost of various outfits of equipment to differences in the rate of interest, and this depends, as we have seen, in a complicated way, upon the gestation period and length of life of items of equipment. There is little to be said about it *a priori*,[4] though it is reasonable to suppose that the most mechanised techniques are the most sensitive to the rate of interest, so that the family of curves fans out laterally as it rises.[5]

The data embodied in this system of curves (if only we had any data !) provide a complete description of all the quantities of capital, valued in wage units, which can be used, in a given state of knowledge, to employ a constant labour force.

Now, the conditions of equilibrium require that the rate of interest which enters into the cost of equipment is equal to the rate of profit actually ruling (for that rate

[1] To illustrate the point made above, Gamma technique is shown as becoming uneconomic at the fourth interest rate.

[2] This expresses " diminishing returns to capital " as the ratio of real capital to labour rises. It is important not to confuse diminishing returns in this sense with the Classical law of diminishing returns. Classical diminishing returns arise from an increase of population relatively to constant natural resources, which may well correspond to the facts of life, whereas the diminishing returns shown in the production function are the result of the artificial assumption of a given state of knowledge.

[3] There is no reason, from an engineering point of view, why they should not be convex over particular ranges, but if the slope, say, between a γ point and a β point is less than the slope between β and the corresponding α, it would indicate that the increase in output due to substituting an Alpha for a Beta plant would be more than proportionate to the increase, at a given wage rate, in the cost of equipment involved, so that Beta technique would never be profitable to use. This possibility is not ruled out by the process of eliminating uneconomic techniques from the curves (see p. 91). E.g. in Fig. 1, if γ_4 lay between the perpendicular $\beta_4 b_4$ and the line $\delta_4\beta_4$, the Gamma technique would not have been eliminated, but the curve $\delta_4\gamma_4\beta_4$ would be convex, and there could be no equilibrium on this section.

[4] The Mathematical Addendum indicates the lines on which it would be possible to work out the influence of the rate of interest on the cost of equipments having various characteristics in respect to length of life, etc. Cf. p. 89, note 2.

[5] For a " perverse " case which may occur when this is not true see Appendix, p. 106.

of profit has been ruling over the period when the existing stock of capital goods was being constructed). We must, therefore, imagine that, by a process of trial and error, we find a position, for each factor ratio, where the two are congruent. The productivity curve, says, $\gamma_2\beta_2\alpha_2$, is drawn up on the basis of a rate of interest equal to the rate of profit which would obtain if the wage rate were such as to make Gamma and Beta techniques equally profitable. The thick line in the diagram is the factor-ratio curve.

At γ_2 the factor ratio is Oc_2, and all men are employed with Gamma technique. An increase in the factor ratio from Oc_2 towards Ob_2 and a rise of output from OC towards OB would come about by substituting Beta for Gamma plants. When the factor ratio has risen to Ob_2 all workers are employed with Beta technique, and output is OB. A further increase in the factor ratio can come about only by the introduction of Alpha plants, but this requires a rise in the wage rate and entails a fall in the rate of profit. We, therefore, jump horizontally, from β_2 to β_1, onto the productivity curve corresponding to the rate of profit which obtains when the wage rate is such that Alpha and Beta techniques are equally profitable. The factor ratio increases from Ob_1 to Oa_1 by the substitution of Alpha for Beta plants until, at Oa_1 all men are employed with Alpha technique. A further increase in the factor ratio then requires a rise in the wage rate. And so on, up the hierarchy of techniques, from one productivity curve to another. At the final upper limit, where a further increase in the factor ratio cannot further increase output per man, we reach the state of Bliss, where wages absorb the whole product and capital has ceased to be scarce relatively to the state of knowledge.

The foregoing analysis shows that the relation of capital to labour, in an equilibrium position, can be regarded as the resultant of the interaction of three distinct influences : the wage rate, the rate of interest and the degree of mechanisation.

The influence of the wage rate upon the value in terms of product of given physical capital was emphasised by Wicksell,[1] and has been called the *Wicksell effect.*[2] When we regard a stock of capital as the result of accumulation brought about by saving—that is, refraining from consuming income—we measure the saving in terms of consumption forgone, and the accumulated capital as a sum of value in terms of product. The influence of the Wicksell effect (leaving aside for the moment the influence of interest on the cost of capital goods) determines the amount of physical capital which a given amount of past accumulation has brought into existence.

This influence can be distinguished from the effect of mechanisation by considering a case in which there is only one technique known. Suppose that to employ a man requires a specific set of capital goods, which we may call for convenience a machine, though it includes work-in-progress as well as long-lived equipment. Without just this machine, a worker can produce nothing, and no other kind of equipment has ever been thought of. Then comparing two situations in one of which, Beta-one, the wage rate is higher than in the other, Beta-two, a given amount of capital in Beta-one corresponds to a smaller number of machines and provides less employment than in Beta-two. This has evidently nothing to do with a " substitution of capital for labour " for, in a technical sense, no substitution is possible.

The operation of the Wicksell effect is counteracted by the operation of the *interest effect.* A higher wage rate entails a lower rate of profit and, therefore, in equilibrium, requires that a lower rate of interest has been ruling in the past. The cost of given physical capital in terms of wage units is less the higher the wage rate.[3]

[1] Loc. cit., p. 292.
[2] C. G. Uhr, " Knut Wicksell. A Centennial Evaluation," *American Economic Review,* December, 1951.
[3] This is shown in the diagram by the backward jump of the factor-ratio curve from β_2 to β_1. See also Appendix.

It is evidently possible that the interest effect should also outweigh the Wicksell effect, so that the value of given physical capital in terms of product is smaller at a higher wage rate. This would occur if, first, the cost of capital goods in terms of wage units reacts strongly to changes in the rate of interest (their gestation period *plus* their working life is long), and second, the wage rate is already high relatively to output per man, so that a given rise in the wage rate produces a large proportionate fall in the share of profit in product, and so in the rate of profit on capital.[1] Where the interest effect more than offsets the Wicksell effect we see the apparently paradoxical result that a given amount of capital (in terms of product) provides a smaller amount of employment at a lower than at a higher wage rate.

The higher factor ratio associated in equilibrium with a higher wage rate, due to the use of more mechanised technique, has been called the " Ricardo effect."[2] The attribution is somewhat forced, yet a label is useful ; we may call it the *Ricardesque effect.*

A rise up the hierarchy of techniques must be associated with a rise in the wage rate. But the more capital (in terms of accumulated product) has been absorbed by increasing the amount of machinery in existence, the lower the wage rate associated with a given amount of accumulation. The more the capitalists have been able to take advantage of the Ricardesque effect, the less the workers have benefited from the Wicksell effect.

(When progress in technical knowledge is economising capital in terms of accumulation by increasing the productivity of a given amount of real capital or when opportunities for mechanising production stimulate accumulation which would not otherwise occur, the case is altered. Our equilibrium conditions tell us nothing about the effect of inventions, or the vagaries of effective demand.)

REAL WAGES

The neo-classical system is based on the postulate that, in the long run, the rate of real wages tends to be such that all available labour is employed. In spite of the atrocities that have been committed in its name there is obviously a solid core of sense in this proposition. To return to our road builders, employment per unit of output is much higher in Gamma than in Alpha, and it seems obvious that this is connected with the fact that real wages there are much lower—that the plethora of labour keeps real wages down, and so helps to get itself employed. Let us try to see what this means.

The basic data of our system are : the labour force, the amount of capital,[3] and the state of technical knowledge, expressed as the hierarchy, ranged according to degrees of mechanisation, of the possible techniques of production. In order to satisfy the neo-classical postulate of full employment, the given amount of capital must employ the given amount of labour.

[1] The numerical example on p. 89, note 2, shows that, on the stated assumptions, a very large reduction in the rate of interest, from 10 per cent to 6 per cent, reduces the supply price of a given balanced outfit of equipment, when the life of individual items is ten years, only in the ratio of 58 to 55 ; with a life as long as twenty years, a reduction in the rate of interest from 10 per cent to 5 per cent reduces supply price in the ratios of 66 to 58. This suggests that the interest effect is not very large. On the more realistic assumption that costs for repairs rise with the age of plant, so that earnings are larger in the earlier years, the effect of interest would be less than in the example. The rise in the wage rate entailed by the fall in the rate of interest must in most ordinary cases lead to a rise, on balance, in the cost of capital in terms of product, and cases in which the interest effect more than offsets the Wicksell effect seem likely to be rather peculiar.

[2] Hayek, " The Ricardo Effect," *Economica*, . May, 1942

[3] Accumulation may be in course of proceeding in an equilibrium position but, if so, it is going on slowly, relatively to the amount of capital already in existence, and it is going on in a manner which does not violate the internal equilibrium of the system, so that, at any moment, it is legitimate to postulate an existing amount of capital.

At any given wage rate, the interplay of competition between capitalists, each seeking to maximise his own profits, is assumed to ensure that the technique will be chosen that maximises the rate of profit. Thus the technique is a function of the wage rate. The outfit of productive equipment in existence is determined by the technique and the total amount of capital. A given outfit of equipment offers a given amount of employment. Thus we have the amount of employment as a function of the wage rate. We can then state the neo-classical postulate : the wage rate is assumed to be such that the technique of production is such that the given quantity of capital employs the given labour force. It is necessary to postulate that the amount of real wages (which is not the same thing as the wage bill but is governed by it) in relation to the cost of subsistence is at least sufficient to maintain the given labour force in being.

There is no difficulty in principle (though infinite complication) in removing the trick assumptions and introducing into our analysis specialised natural resources, varieties of skill of labour, and demand equations for individual commodities in terms of relative prices, total income and the distribution of income between households. (We could not, however, digest economies of scale, for that implies the existence of active entrepreneurs, as opposed to our capitalists whose only function is to own capital and use it to employ labour.) We should then be able to make use of the supply-and-demand analysis which neo-classical economics has developed with so much elaboration, and to show that the technique will be chosen, each capitalist reckoning wages in terms of his own product, that maximises the total of profits.

The equation relating the amount of employment to the wage rate is independent of the medium in which wages are paid to workers. Since it is more usual to pay in money than in peanuts, let us see how it works out when wages are paid in money, that is, some unit of generalised purchasing power.

The relation of the money prices of particular commodities to their money-wage costs is influenced by supply and demand for scarce factors, by the level of effective demand and by the price policies of imperfectly competitive rival producers. At any given money wage level, the resulting complex of prices determines the wage in terms of a composite unit of the product which is actually being produced. We have ruled out fluctuations in effective demand but we must consider the influence of competition.

Competition which is relevant to our argument is competition in the long-period sense—the readiness with which capitalists break into a market where a more than average rate of profit is ruling and bid down the price of the commodity being sold there. When a market is dominated by a monopolist, price in relation to money wages is such that the rate of profit on his capital is higher than the average for the economy as a whole. Capitalists from outside are anxious to enter this market, and in doing so they cut the price of the commodity, thus raising the wage rate everywhere in terms of this bit of the composite product, while lowering it in other lines from which capital is deflected. When competition is free and active, as we assume it to be, this process of competing away excess profits and raising sub-average profits is always completely successful, and at any moment a uniform rate of profit is ruling throughout the system.

When profits are being held above the general level in a particular line, by monopolistic restrictions, the process is impeded, for the existence of monopoly (in the long-period sense here appropriate) means precisely that an individual capitalist is able to prevent others from breaking into his preserves and competing away his profits.

In this situation a trade union may succeed in raising the money wage that its members receive, so that wages in terms of product in general for that particular group of workers are raised, while the monopolist (pursuing some long-range policy) does not necessarily counter by raising the money price of his commodity correspondingly, so that the wage of labour taken as a whole is raised. The existence of a few monopolies here and there does not necessarily depress real wages for labour as a whole ; indeed it may cause them to be higher than that which corresponds to the competitive equilibrium position, for the monopolists may be exploiting other capitalists and sharing the spoil with their own workers. But the general prevalence of monopoly must depress real wages (unless the strength of the trade unions and the complaisance of the monopolists are so great as to prevent the rate of profit from rising above the competitive level) for by keeping up money prices it reduces the wage rate, in our sense, which governs the real wage rate that interests the workers in their capacity as consumers.

When there are elements of monopoly in an economy, different rates of profit obtain in different lines of production, without much tendency to equalisation throughout the system. The argument in terms of an equilibrium rate of profit corresponding to a given ratio of capital to labour then cannot be applied without a great deal of complication. It is for this reason that the assumption of free competition (in the long-period sense) is necessary to it.

Given that competition establishes a uniform rate of profit throughout the economy, and given technical knowledge and the quantity of capital (in terms of product) there is one value of the wage rate which is compatible with full employment of any given labour force.

The neo-classical economists derived from this proposition a doctrine which cannot, in fact, be based on it. They maintained that the level of wages determines the amount of employment, and that, when unemployment occurs, workers (unless frustrated by the misguided policy of trade unions) offer themselves at a lower real wage rate than that ruling, and go on doing so till all are employed.

This doctrine was challenged by Keynes, on the ground that the wage bargain does not determine the real wage. Keynes' argument was developed to deal with short-period situations, but it applies with full force to equilibrium positions. A change in the peanut price, or the money price, of a man hour of labour alters the equilibrium price, in terms of peanuts or of money, of each commodity proportionally and leaves the equilibrium rate of profit and of wages unchanged. In short, the purchasing power, whether of money or of peanuts, over commodities in general, is governed by its purchasing power over labour, and a change in the peanut price or the money price of labour does not affect the price of labour in terms of commodities in general.

From the point of view of any one employer his wage bill in terms of his own product is by no means the same thing as the real wage which the worker gets. The wage bill to the capitalist and the wage received by the workers are the same only in a strictly one-commodity world, as Ricardo saw when he imagined a system in which corn is the only product, and the wage contract is made in terms of corn.[1] Wage bargaining is conducted in terms of what the worker gets, and the wage which enters into the wage bargain is not the same thing as the wage which determines the quantity of employment. Thus the conception of the level of employment being determined by the wage bargain cannot be expressed (outside the corn economy) in a way which has a meaning.[2]

[1] *Works and Correspondence of David Ricardo*, ed. Sraffa, Vol. I, p. xxxii.

[2] The difficulty cannot be evaded by postulating that wages are paid in kind without the use of money. To specify a "non-monetary" or "non-Keynesian" economy it is not sufficient to postulate that the society in question has not yet got round to inventing a standardised medium of exchange. We must postulate that the very idea of generalised purchasing power is unknown, so that each separate employer pays his workers in his own product, and the workers barter the products amongst themselves without any triangular dealing.

ACCUMULATION

What becomes of the neo-classical doctrine if we read it the other way round : that the rate of profit tends to be such as to permit all the capital that comes into existence to be employed ? Suppose that the wage rate has been established at a level which yields some conventional minimum real wage, and that, the technique having been chosen which maximises the rate of profit, the quantity of capital in existence does not employ all available labour, so that there is a reserve of unemployment. Accumulation can then proceed at a constant factor ratio and constant rate of profit until all available labour is employed. If population is increasing at least as fast as capital is accumulating, full employment is never attained, and the expansion of the economy can continue indefinitely (we have postulated that there is no scarcity of land, including all non-produced means of production).

So far the argument is dismally simple. What are we supposed to imagine to happen when there is full employment in the long-period sense, that is, when there is sufficient plant in existence to employ all available labour ? One line of argument is to suppose that the capitalists who are accumulating act in a blindly individualistic manner, so that a scramble for labour sets in ; the money (or peanut) wage rate is bid up, and prices rise in an indefinite spiral. (It is of no use to bring the financial mechanism into the argument, for if the supply of the medium of exchange is limited, the interest rate is driven up ; but what the situation requires is a fall in the rate of interest, to encourage the use of more mechanised techniques.)

Or we may postulate that the capitalists, while fully competitive in selling, observe a convention against bidding for labour—each confines himself to employing a certain share of the constant labour force. Then any one who wishes to increase the amount of capital that he operates shifts to a more mechanised technique. Those who first make the change may be supposed to compete for wider markets and so to reduce prices relatively to money wages. The Ricardesque effect is thus brought into play, and the switch to more mechanised techniques proceeds at a sufficient rate to absorb new capital as it accrues. Alternatively, we might imagine that an excessive number of plants of the less mechanised type are actually built, and that their redundancy, relatively to labour to man them, reduces profit margins, so that the wage rate rises and brings the Ricardesque effect into play. (Whichever line we follow the argument is necessarily highly artificial, for in reality the state of trade is the dominant influence on investment. The situation which promotes the mechanisation of production is full employment and full order books, that is to say, a scarcity of labour relatively to effective demand, but the equilibrium assumptions do not permit us to say anything about effective demand.)

Somehow or other, accumulation may be conceived to push down the rate of profit, and raise the factor ratio.

But the very notion of accumulation proceeding under equilibrium conditions at changing factor ratios bristles with difficulties. The rate at which the factor ratio rises is not governed in any simple way by the pace at which accumulation goes on—it depends upon the extent to which the rising wage rate causes capital to be absorbed

Several neo-classical economists have invented, as a prophylactic against unemployment, " commodity money " in various forms, which would produce an effect as though the wage contract were made in terms of the composite commodity representing all output. As a practical proposal, such a scheme is rather fanciful, but as a pedagogical device explaining the nature of money, it is excellent. Another way of imagining the same result is to suppose that the workers are paid in terms of a percentage of the money proceeds of the firm employing them, that is, that the workers take on the risks of entrepreneurship. This also starts an interesting train of thought.

Robinson Crusoe lived in a completely amonetary world, as Defoe makes him point out in his meditations over a parcel of coins. For this reason he is the patron saint of neo-classical economics.

by the Wicksell effect. Moreover, the effect of a given change in the factor ratio depends upon the speed at which it is made, relatively to the length of life of plant. If capital per man is rising rapidly some capitalists will still be operating Delta and Gamma plants while others have already installed Alpha plants. (In these conditions the trick assumptions become very tricky indeed.)

Even if we can find a way through these complications, there remains the formidable problem of how to treat expectations when the rate of profit is altering. An unforeseen fall in the rate of profit ruptures the conditions of equilibrium. Capitalists who are operating on borrowed funds can no longer earn the interest they have contracted to pay, and those operating their own capital find themselves in possession of a type of plant that they would not have built if they had known what the rate of profit was going to be.

On the other hand, if we postulate that accumulation goes on in the expectation of a gradually falling rate of profit, the whole basis of the analysis becomes immensely complicated. We can no longer argue in terms of a single interest rate. There is a complex of rates for loans of different lengths, the rates for shorter terms standing above the rates for longer terms. Moreover, the pace at which the rate of profit falls as the factor ratio rises is dictated by technical conditions. Over its early reaches the factor-ratio curve may be supposed to be steep, with the rate of profit falling slowly. Then it passes over a hump, with a rapid fall in the rate of profit, and flattens out again with a lower but more slowly falling rate of profit. To be correct, the expectations of the capitalists cannot merely be based on past experience but require a highly sophisticated degree of foresight.

Thus the assumptions of equilibrium become entangled in self-contradictions if they are applied to the problem of accumulation going on through time with a changing factor ratio. To discuss accumulation we must look through the eyes of the man of deeds, taking decisions about the future, while to account for what has been accumulated we must look back over the accidents of past history. The two points of view meet only in the who's who of goods in existence to-day, which is never in an equilibrium relationship with the situation that obtains to-day.

In short, the comparison between equilibrium positions with different factor ratios cannot be used to analyse changes in the factor ratio taking place through time, and it is impossible to discuss changes (as opposed to differences) in neo-classical terms.

The production function, it seems, has a very limited relevance to actual problems, and after all these labours we can add little to the platitudes with which we began : in country Gamma, where the road builders use wooden shovels, if more capital had been accumulated in the past, relatively to labour available for employment, the level of real wages would probably have been higher and the technique of production more mechanised, and, given the amount of capital accumulated, the more mechanised the technique of production, the smaller the amount of employment would have been.

A CHANGE IN TECHNICAL KNOWLEDGE

It remains to inquire whether the analysis can be applied to a change in our third basic datum, the state of technical knowledge.

In the neo-classical system, technical knowledge is altered by inventions. An " invention " is conceived not merely as an isolated innovation in a method of production or the design of equipment. It is a discovery which has a wide range of applications and which raises the productivity of labour over a wide range of factor ratios.

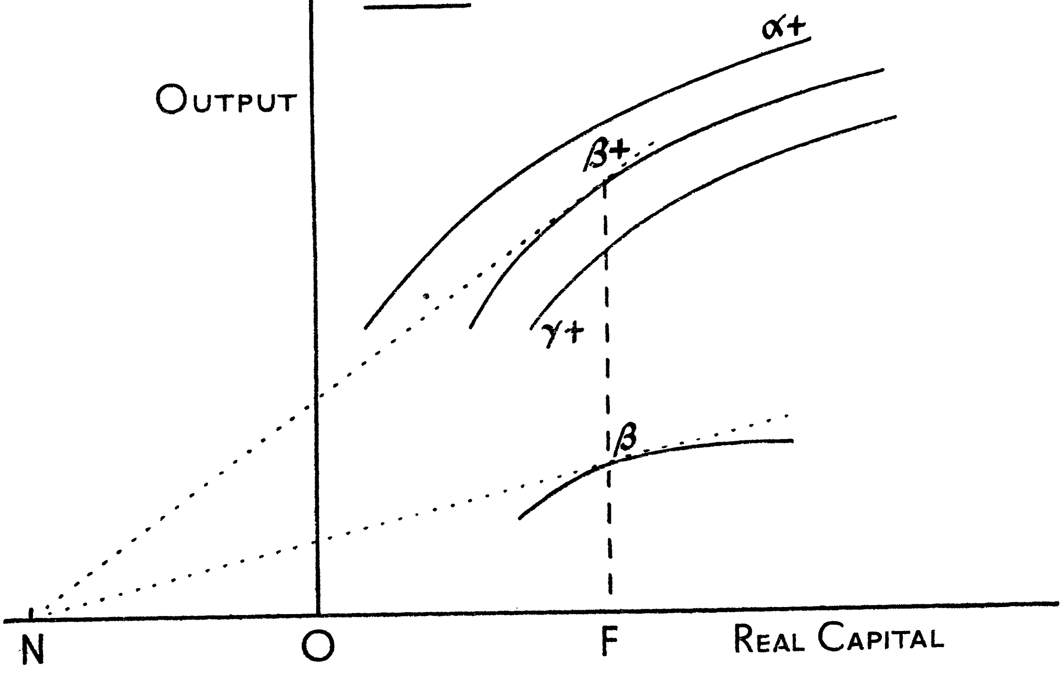
The argument is usually conducted in terms of the effect of inventions upon the output and the relative earnings of given quantities of capital and labour. Thus we are supposed to imagine an economy taking a standing jump from one stationary position to another and landing itself in equilibrium in a new state of knowledge with the same supplies of factors of production as before. This makes the contradictions involved in the neo-classical conception of equilibrium and the ambiguities involved in the conception of a constant quantity of capital more formidable than ever.

However, it is possible once again to distil something from the ingredients of the argument. Technical progress may be incorporated into the conditions of equilibrium if we postulate that inventions are expected to be made at a certain rate and that, in fact, they do succeed each other in a smooth and regular manner. All capital goods are, therefore, provided with obsolescence funds, which allow equipments to be transmogrified to suit new techniques without any loss of capital in terms of product. An invention then does not give a shock to the system rupturing the conditions of equilibrium, and we can continue to rule out problems of effective demand by assuming that saving is geared to investing so that if accumulation is taking place it goes on at a steady rate without any perturbations.

We are also obliged to put a heavy weight on the trick assumptions, for it is impossible to compare the efficiency of different methods of production unless they are designed for similar outputs.

Subject to these provisos, we can (with due reservations) make use of the foregoing analysis to compare equilibrium positions of the economy at two points of time. In the first position, say, Beta technique is in use.

FIG. II



In the diagram β is the point of equilibrium on the factor-ratio curve in the first position; the factor ratio is OF; and the elasticity of the productivity curve through β is e (the productivity curve is drawn up using the Beta rate of profit as the notional interest rate : its elasticity is equal to the share of profit in output when equilibrium obtains at β).

Now draw the productivity curve for the new state of knowledge using the rate of profit which obtains at β as the notional interest rate. Let $\beta+$ be the point on this curve corresponding to the factor ratio *OF*.

If the elasticity of the curve at $\beta+$ were equal to e, the wage rate at factor ratio *OF* would have been raised by the improvement in technique in the same proportion as output. Capital per unit of output, and the rate of profit on capital would, therefore, be the same as in the first position. If the elasticity at $\beta+$ were less than e, the wage rate would be raised more than in proportion to output, and the share of profit in output and the rate of profit on capital would be reduced (readjusting for a lower notional interest rate reinforces these relations). The converse would hold if the elasticity at $\beta+$ were greater than e.

A lower elasticity of the productivity curve implies that as we move up the new hierarchy of techniques, Gamma + to Beta + to Alpha +, output per man rises proportionately less than was the case with the old hierarchy. There is, so to say, less scope in the new situation for the Ricardesque effect to work. Thus inventions which alter the shape of the productivity curve in this way (over the relevant range of factor ratios) may be called *unfavourable* to capital, those which leave the elasticity at a given factor ratio unchanged, *neutral*, and those which raise it, *favourable* to capital.

The nature of the technical change does not by itself determine a new position of equilibrium. That depends also upon how much accumulation took place while the technical change was being made. As we have seen, for the factor ratio to remain unchanged, when the inventions have been neutral, capital must have increased in the same proportion as output. With any smaller amount of accumulation the factor ratio is reduced, the system must be supposed to have moved in the direction of, say, $\gamma+$ or $\partial +$, and the share of profit in output and the rate on capital are higher than they were at β.

Thus the effect of inventions, unless they are highly unfavourable to capital in a technical sense, is to raise the rate of profit obtainable by a given amount of capital, and to increase its relative share in output.

The above analysis throws some light on the controversy between the economists and the Luddites, and supports Ricardo's argument *On Machinery*.[1] When the nature of inventions is sufficiently favourable to capital, the wage rate at the factor ratio *OF* is reduced by them. But if the wage rate ruling at β provided a real wage no greater than the subsistence minimum, it is impossible for the wage rate in a new position of equilibrium to be any lower. There is a possible position of equilibrium at a higher factor ratio, say at $\alpha+$, where the wage rate is the same as that which prevailed at β. But to provide a higher factor ratio for a constant labour force at a constant wage rate requires an increase in the quantity of capital. If the quantity of capital has not increased sufficiently, the higher factor ratio can be attained only by reducing employment. There is then a " substitution of capital for labour " in a literal and brutal sense.

To return to our road builders—if inventions made it profitable to introduce bulldozers in country Gamma the workers there would be much worse off than they were with wooden shovels, unless at the same time there were a sufficient accumulation of capital to provide the labour displaced from road building with employment in other industries.

This scheme of analysis provides the basis for the model of a capitalist system

[1] *Principles*, Third Edition, Chapter XXXI.

enjoying continuous expansion. Where technical progress is neutral and accumulation goes on at just such a rate as to keep the ratio of capital to output constant, the share of capital and labour in output and the rate of profit on capital remain constant and the model is free from " internal contradictions."[1]

In other cases the complications arising from changes in the distribution of income between classes and changing expectations of profits are too great to be digested by the assumptions of equilibrium, and it is idle to pursue the argument any further without taking account of the problem of effective demand.

CONCLUSION

The tenor of our argument has been mainly negative and the level of abstraction maintained in the analysis is very high. Nevertheless, we can draw some general conclusions from it. The conclusions sound obvious enough, but perhaps that is all to the good, as it shows that the propositions drawn from the abstract argument are not in conflict with common sense.

The rate of profit on capital will tend to be higher, and real wages lower :

(1) the more plentiful are the technical opportunities for mechanising production ;

(2) the slower is the rate of capital accumulation in relation to the growth of population ;

(3) the weaker is the force of competition (and the weaker is the bargaining power of the workers, when competition is weak).

Given the degree of competition and the rate of growth of population, the course of the rate of profit over the long run (abstracting from short-period fluctuations) depends on the interaction between technical progress and the rate of accumulation. Technical discoveries (unless extremely unfavourable to capital) are continuously tending to raise the rate of profit and accumulation is tending to depress it. Prosperous capitalist economies are those where the rate of profit is falling in spite of rapid technical progress, and miserable ones those where the rate of profit is high in spite of technical stagnation.

Cambridge. JOAN ROBINSON.

APPENDIX

THE FACTOR-RATIO CURVE AND THE CAPITAL-RATIO CURVE

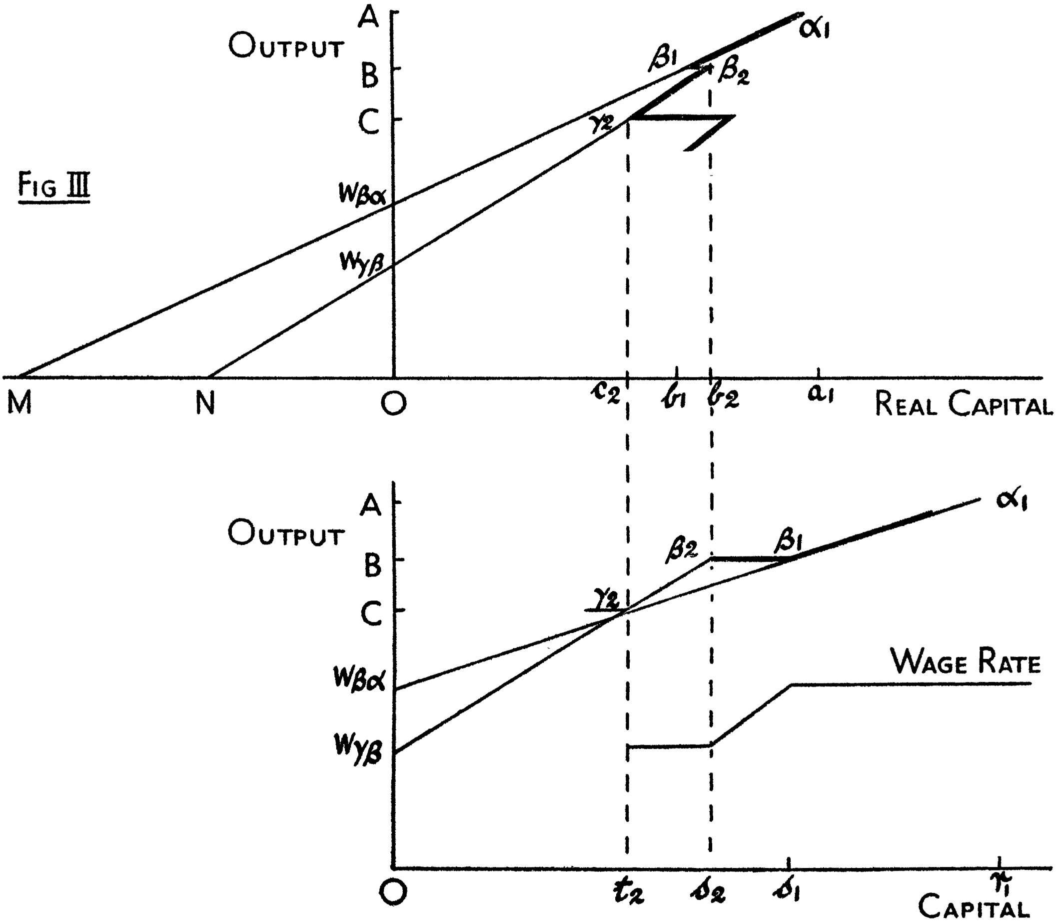
The relations between the factor ratio and the ratio of capital (in terms of product) to labour employed discussed above (p. 93) can be set out diagramatically as follows :

The upper half of Fig. III (overleaf) repeats Fig. I. The slope of the straight line, $\gamma_2 \beta_2$ is the ratio of an increment of output to the increment of real capital with which it is associated, and at a constant wage rate an increment of output is an increment of profit ; thus this slope is the ratio of the increment of profit to an increment of real capital. Therefore, by producing $\beta_2 \gamma_2$ to cut the y axis in $w_{\gamma\beta}$ we find the wage rate, $OW_{\gamma\beta}$, at which Gamma and Beta techniques are indifferent. When output is OC, $W_{\gamma\beta}C$ is profit per man employed, and when output is OB, $W_{\gamma\beta}B$ is profit per man. Similarly, $OW_{\beta a}$ (when $a_1\beta_1$ cuts the y axis in $W_{\beta a}$) is the wage rate at which Beta and Alpha techniques are equally profitable.

[1] At this point the argument joins on to that set out in *The Rate of Interest and Other Essays*, p. 90.

At any factor ratio, the amount of capital per man (that is, capital in terms of product) is equal to real capital per man multiplied by the wage rate. Thus, at β_2, capital per man is $Ob_2 \,.\, OW_{\gamma\beta}$ and the rate of profit on capital ruling over the range $\gamma_2\beta_2$ is equal to $W_{\gamma\beta}B/Ob_2 \,.\, OW_{\gamma\beta}$. Produce $\beta_2\gamma_2$ to cut the x axis in N. Then $OW_{\gamma\beta}/ON = W_{\gamma\beta}B/Ob_2$. Therefore, the rate of profit on capital is equal to $1/ON$. Similarly the rate of profit over the range $\beta_1\alpha_1$ is $1/OM$ when $\alpha_1\beta_1$ cuts the x axis in M.

FIG III

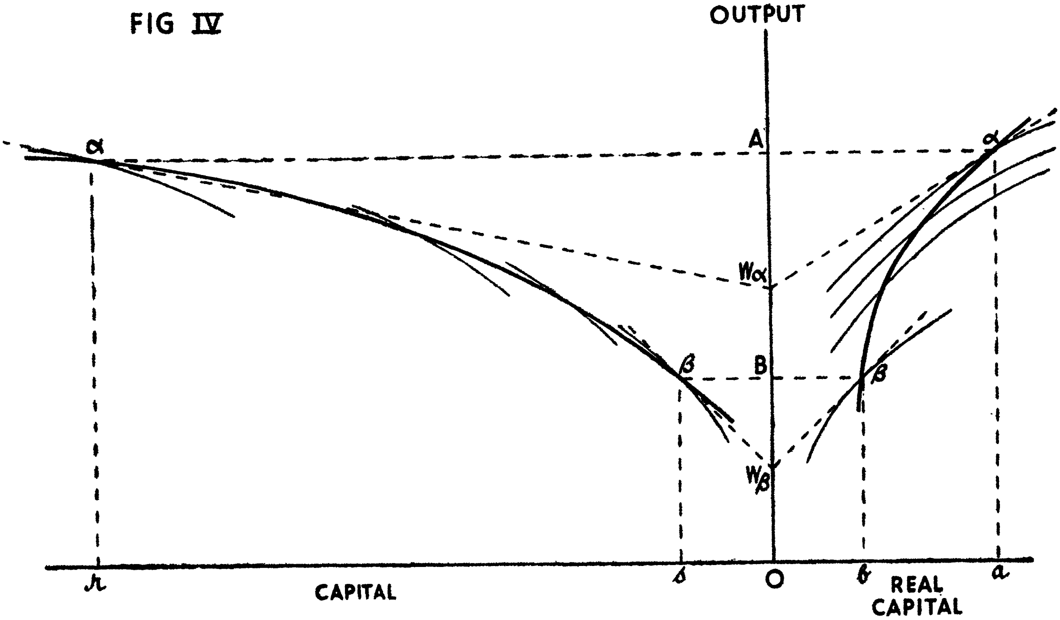


We can now draw a curve relating output to the ratio of capital to labour employed, that is, to capital per man in terms of product. In the lower half of Fig. III the y axis is identical with that of the upper half ; the x axis measures capital per man, instead of real capital per man. Physical capital, that is the who's who of capital goods corresponding to each rate of output, is unaffected by the manner in which the items composing it are valued, and points given the same names (β_1, α_2, etc.) correspond to identical technical situations. The distance Os_2 on the lower x axis is equal to the area $Ob_2OW_{\gamma\beta}$ in the upper half of the diagram. To facilitate comparison, we take the distance $OW_{\gamma\beta}$ as the unit for x on the lower axis, and make the distance Ob_2 equal to Os_2. Then the two straight lines $\gamma_2\beta_2$ are identical.

At β_2 all men are employed with Beta technique, and a rise in the factor ratio and the capital ratio requires a rise in the wage rate to $OW_{\beta\alpha}$. Capital per man then increases to Os_1, on the lower x axis, while real capital per man falls to Ob_1 on the upper x axis. (In this case the rise in the wage rate more than offsets the fall in real capital due to a lower rate of profit. The Wicksell effect outweighs the interest effect.

The slope of $\gamma_2\beta_2$ in the lower half of the diagram ($W_{\beta\gamma}B/Os_2$) is the rate of profit on capital which obtains when Gamma and Beta techniques are indifferent, and the slope of $\beta_1\alpha_1$ is the rate of profit when Beta and Alpha techniques are indifferent. The greater length of s_1r_1 on the lower x axis compared to b_1a_1 and the smaller slope of the corresponding straight line $\beta_1\alpha_1$ reflects the excess of the $\beta\alpha$ wage rate over the $\gamma\beta$ wage rate.

FIG IV



Smoothing out discontinuities in the productivity curves, we may draw a pair of diagrams for the factor-ratio curve and the capital ratio curve as in Fig. IV. We take real capital as the x axis in the right hand part of the diagram and capital in the left hand.

Corresponding to each of the family of productivity curves in the right hand is what we may call a pseudo-productivity curve on the left hand, showing what capital per man would be if the wage rate were that which is compatible with the rate of interest used in drawing the corresponding productivity curve. Each pseudo productivity curve has a meaning only in the neighbourhood of the point of equilibrium corresponding to the wage rate on the basis of which it is drawn.

The factor-ratio curve cuts the family of productivity curves from below as it rises. The capital-ratio curve cuts the pseudo productivity curves from above.

The point β on each curve corresponds to the output OB. Draw a tangent to the productivity curve through β. By the same reasoning as above, its intercept on the y axis, OW_β, is the wage rate at which the corresponding technique will be in use. As before, we take OW_β as the unit for the capital axis, so that the tangent to the pseudo-productivity curve through β on the capital-ratio curve is drawn as identical with that to the productivity curve through β on the factor-ratio curve. The elasticity of the tangent, $W_{\beta}B/OB$, is the ratio of profit to output, or relative share of capital in product.

Similarly, draw tangents at α. The elasticity of the pair of tangents is the same, the greater distance to the left of the left hand position of α compensating for the smaller slope of the tangent. The slope of the tangent on the left ($W_{\alpha}A/A\alpha$) is the rate of profit on capital.

It is convenient for some purposes to conceive the productivity curves as continuous, each technique requiring an indefinitely small increase in capital per man compared to the last. But it is hard to picture what this would mean in reality (even in the famous wine cellar). Moreover, the essential nature of the mechanism of the relationship between the rate of wages and the choice of technique can be understood only in terms of discontinuous curves ; we must, therefore, think of a point, such as β or α in Fig. IV, as being a small straight line segment of a curve, over which two techniques are indifferent, rather than as representing the product of a single technique.

The geometry reveals a curious possibility.[1] It may happen that, over a certain range, a reduction in the rate of interest produces a larger reduction in the capital cost of the equipment for a lower than for a higher technique, so that successive wage tangents become steeper as the rate of profit falls. They may then find contact with productivity curves at successively lower points, so that a lower rate of profit (and a higher wage rate) results from the use of a *less* mechanised technique. This might occur if the plant required for less mechanised techniques had a much longer gestation period or working life, so as to be much more sensitive to the interest rate than that for more mechanised techniques. This " perverse " behaviour of the factor-ratio curve, where it occurs at all, can be only over a certain range. At very low values of the rate of interest the differential effect (as between techniques) of changes in the interest rate must be small, so that there must be an upper range, on the way to Bliss, over which the factor ratio curve rises to the right as the rate of interest falls ; and the degree of mechanisation must have reached a certain level before there is any scope for it to fall, so that there must be a lower range over which the factor ratio curve rises. In the case where the curve at first rises in a " normal " manner, then falls to the left " perversely," then rises again, there is evidently a certain range of techniques which provide possible positions of equilibrium at three different wage rates. A good deal of exploration of the possible magnitude and behaviour of the interest effect is needed before we can say whether the above is a mere theoretical rigmarole, or whether there is likely to be anything in reality corresponding to it.

[1] This was pointed out to me by Miss Ruth Cohen.

The Production Function and the Theory of Capital: A Comment

I. INTRODUCTION

In her note on the Production Function, Joan Robinson has drawn attention to the difficulties inherent in any attempt to measure the quantity of capital in a community by a single number, and of the consequent dangers in teaching pupils to regard output as a function of the amounts of labour and capital employed. In an effort to avoid in time " sloppy habits of thought," she has adopted the position that " when we consider what addition to productive resources a given amount of accumulation makes, we must measure capital in labour units " and hence determined a method of measuring the quantity of capital under the equilibrium conditions of a simplified model : this threw light on the manner in which the factor ratio (quantity of capital available per employed person) affected the choice of productive technique, the rate of interest and the real wage rate, and hence the distribution of product between capital and labour. For brevity we shall refer to the labour-units of quantity of capital, which Joan Robinson uses, as J.R. units.

The present comments will be directed towards the following points.

(*i*) If we propose to regard output as a function of the quantities of labour and capital employed, it is not very convenient to measure capital in J.R. units because, if we do,

(*a*) The same physical stock of capital equipment and working capital, producing the same flow of consumption goods, can appear under two equilibrium conditions, differing only in respect of the rate of interest and rate of real wages, as two different amounts of capital.

(*b*) The wage-rate of labour and the reward per unit of capital will, in general, differ under perfect competition from the partial derivatives of output with respect to the quantities of labour and capital employed.

(*c*) Output per head may be negatively correlated with quantity of capital per head measured in J.R. units, despite the assumption of a given state of technical knowledge. This can lead to the paradoxical result that a reduction of the capital per head (in J.R. units) is required to increase productivity.

(*ii*) If we abandon J.R. units of capital and employ instead a straightforward chain-index of quantity of capital, then we can again obtain, in principle, a production function $O = f\ (L\ C)$ with the property that the social product O is distributed into shares $L\,\frac{\partial f}{\partial L}$ for labour and $C\,\frac{\partial f}{\partial C}$ for capital.

(*iii*) A clear and rigorous deduction of these results requires a careful statement of the assumptions underlying the model and the explicit exclusion of certain exceptional cases. The exceptional cases themselves are of some interest as indicating situations where the production function is no longer single-valued and in which some of Joan Robinson's conclusions are falsified.

(*iv*) The use of the chain-index for measuring capital facilitates the discussion of the case where a " continuous spectrum " of techniques is available. The distribution of O into $L\,\frac{\partial f}{\partial L}$ and $C\,\frac{\partial f}{\partial C}$ still holds in this case.

(*v*) The remaining sections of the article are concerned with the effects of relaxing the three simplifying conventions :

(*i*) only stationary states are to be considered and compared ;
(*ii*) there are no technical advances ;
(*iii*) labour and capital are the only two factors employed.

(*vi*) An appendix giving particular arithmetical and algebraic examples throws light on the working of the general model.

II. CHOICE OF UNITS FOR MEASURING QUANTITY OF CAPITAL

In her introduction, Joan Robinson complains that the student of economic theory is taught to regard output as a function of the amounts of labour and capital, but is not taught in what units the quantity of capital is to be measured. Her own answer is that under the simplified conditions which she assumes, the quantity of capital should be measured in units of labour, and should be equated to the labour input which it costs, compounded at the ruling rate of interest : if the capital has already been used to produce output, an appropriate deduction from its cost should be made on this account.

There is nothing inconsistent in this method, but it is not the only possible method and it is inconvenient if we wish to regard output as a function of the quantities of labour and capital. Suppose, for example, that as described in Joan Robinson's section "Technique of Production," there exists a hierarchy of techniques, which may become profitable at various stages of capital development. To each such technique there will correspond a range of interest rates at which it can be fully competitive (given appropriate wage-rates) compared to all other techniques. It is thus possible to conceive two stationary states, each using one and the same technique, with identical amounts and composition of capital equipment, with identical labour input and product-output : yet, although both in equilibrium, they may have differing real wage-rates, and two different interest rates, each within the range over which the technique can be fully competitive. For purposes such that output is to be regarded as technically determined by the amounts of labour and capital employed, it would be convenient to regard the quantities of capital employed in these two stationary states as the same, because the capital stock, the labour input and the output stream are identical in the two states. Yet because the rate of interest differs in the two states, the quantities of capital as measured by J.R. units must differ also.

Conversely, it is easy to see that two equilibrium stationary states may exist, with different techniques, with the same labour input, but with different interest rates, and different outputs, which will appear, with J.R. units, to have equal amounts of capital as well as of labour. Thus, whilst on the one hand the same physical capital can be measured as two different amounts of capital, yet on the other hand, the same amounts of labour and capital may result in different outputs. The function giving output in terms of the factor inputs fails to be single-valued.

These difficulties arise from the index-number problem involved in measuring the quantity of capital. They result simply from the fact that mere difference in interest rates, without necessarily corresponding to any difference in the productive possibilities or physical characteristics of the stocks of capital available in two stationary states, can yet affect their cost measured in J.R. units. Hence, comparing the amounts of capital in a sequence of stationary states, we shall obtain a set of numbers reflecting differences of interest rates as well as differences relevant to productive potential. There is a close analogy to an attempt to compare quantities of production by an index giving their money values in a sequence of stationary states with slightly different price systems.

It may be asked whether these inconveniences disappear when we consider not a sequence of discrete equipments but a continuous spectrum, with the appropriate rate of interest altering continuously as we pass down the spectrum. The answer is that the most glaring inconveniences do disappear, but the basic weakness of the method remains : differences in interest rates which are irrelevant to the production possibilities (although not to the profit possibilities) of two sets of capital equipment still are allowed to affect the comparison of their amounts of capital when measured in J.R. units. That the distortion that this involves may be so great as to contradict common sense is suggested by the following extreme example. Suppose that there is a continuous spectrum of basic equipments E_u, with u a continuous variable. If now instead of discussing stationary states, we think of a very slow progress with constant employment, but changing type of equipment providing an increasing output, we should not go very far wrong by supposing wages and interest rates to move through the stationary state values appropriate to the various types of equipment.

Suppose that constant replacement of worn-out equipment by types providing more output per head involves the withholding of some labour from producing consumption goods, then this situation is such as is normally described as one with positive net investment, and we can legitimately require that any proposed system of measurement may show the quantity of capital to be increasing.

In Appendix II, Section II, the following simple example is discussed in some detail. Each basic equipment E_u costs the work of 100 men spread evenly over one year ; when complete, the equipment E_u needs $100u$ men to work it and produces a uniform output flow at the rate of $100\ (1 + 11u)^{\frac{1}{11}}$ units per annum : at the end of one year the equipment wears out. It is shown that if 1 per cent of the labour force is withheld from other activity and devoted to replacing worn-out equipments by a (larger) number of equipments needing less men to operate them in relation to those required to build them, then as u steadily decreases and the number of equipments increases, the rate of interest will fall, real wages will rise and the output of food will rise. If in this example we use the chain-index method described in Section III below, and in Appendix I, each equipment can be regarded as 100 units of capital, and the total quantity of capital will increase at a rate of K per cent per annum where K is the proportion that the annual net income bears to the value of all capital.

Now consider how this process appears when J.R. units are used for the measurement of capital. It is shown in the Appendix that an equipment of type E_u will at the time of its use contain approximately $1{,}000u/\log\ (1 + 10u)$ J.R. units of capital. If the total number of men employed is $100N$ then the total number of equipments of type E_u is approximately $\dfrac{N}{1 + u}$, so that the total number of units of capital is approximately $1{,}000Nu/(1 + u)\ \log\ (1 + 10u)$. Numerical calculation shows that if $u = 2.326$ initially and capital accumulation proceeds as described above, u will fall from 2.326 towards zero, and the quantity of capital measured in J.R. units will simultaneously *fall* from about $219N$ down to $100N$. Thus, in this example, a process of capital accumulation carried out by labour withheld from making consumption goods, and financed out of saving, appears when J.R. units are used as a steady *decrease* in the quantity of capital. This is an extreme example of the negative bias induced in measurement of net investment when J.R. units are used, and this bias is due to including a negative element reflecting a fall in the rate of interest.

Thus our warning against the incautious use of J.R. units is based not merely on considerations of convenience, but also on the danger that as soon as we draw approximate conclusions from a comparison of stationary states, about a process of very slow investment, their use may cause what plainly is *positive* net investment in the customary sense of these words to appear as *negative* net investment.

Another inconvenience arising from the use of J.R. units is that if the marginal productivity of labour is obtained by partial differentiation of the production function, it will in general be found to differ from the equilibrium wage of labour, when J.R. units of capital are used, despite whatever heroic assumptions of perfect competition may be adopted. This symptom again suggests that " keeping the amount of capital constant " in J.R. units does not correspond to what is usually understood by keeping the amount of capital constant.

A natural method by which to construct an index of quantity of capital in a historical sequence would be to form a chain index, increasing the index at each step by the proportion in which the cost of the capital at current wage and interest rates at the end of the step exceeded the cost of capital at the beginning of the step, calculated at the same wage and interest rates. By shortening the steps, the distortion due to choosing wage and interest rates at the *end* of each step could be made as small as we pleased.

The same method can be used to construct an index of quantity of capital in a sequence of stationary states, and provided these are arranged in an order so that the difference between one and the next is always a small step, the distortion due to the method can again be reduced to negligible proportions. The method has the advantage that changes of cost merely due to changes in the interest rate do not affect this measure of the quantity of capital.

The technique of constructing the index number is further explained in the following section and in Appendix I. It will suffice to say here, that to a statistician accustomed to the problem of sorting out quantities from price-changes, this measure of quantity of capital would probably seem the most satisfactory one (at any rate for comparison of states fairly close together in the sequence), and that the use of this measure removes the more glaring difficulties in the way of regarding aggregate output as a function of the amounts of labour and capital employed. In particular, we shall show in Section IV that the rewards per unit of the factors are once again given by the partial derivatives of the aggregate production function if stationary state conditions with perfect competition are assumed.

A development of the theory of capital using these units needs a careful statement of the simplifying assumptions underlying the model. This is attempted in the following section, which describes a " discrete " model closely similar to that discussed by Joan Robinson.

III. SIMPLIFYING ASSUMPTIONS FOR DISCONTINUOUS MODEL

In line with Joan Robinson, we shall assume :

(*i*) The output of consumption goods is homogeneous. We shall refer to these goods as food.

(*ii*) Food may be produced in a constant stream by any of a number of techniques, each of which employs a distinctive outfit of equipment, and a constant stream of labour, part of which may be devoted to maintaining or replacing the equipment.

(*iii*) Equipment is already complete at the moment when the food stream begins to flow, having been built up by a varying stream of labour during the past.

(*iv*) For each technique constant returns prevail, in the sense that the equipment outfit and the labour stream are infinitely divisible ; that when each is multiplied by any number λ, so also is the food output stream ; and that the outputs and inputs of the sum of two or more different equipment outfits is the sum of their individual outputs and inputs.

(*v*) At any level of food-wages of labour, the rate of interest will settle at the highest level which any employer can pay without making losses.

(*vi*) The conditions of the stationary state hold, in the sense that everyone believes that prices, wage-rates of labour and interest rates will remain fixed for ever and

either this is true

or we retain this as a convention in calculating our rates of interest in assumption (*v*).

It follows that at any given food-wage-rate of labour V, there will be a rate of profit R_{vs} associated with each equipment E_s, which an employer can earn on its capital-value if he builds E_s and uses it for producing food. By assumption (*v*), if the rate of food-wages of labour is V, competition will drive the rate of interest to the level $R\ (V) = \underset{s}{Max}\ R_{vs}$ of the greatest of the R_{vs}. We may call this the competitive rate of interest at V, and when food-wages are at V, only that (those) equipment(s) E_s for which $R_{vs} = R\ (V)$ will be built for use. We may call this (these) equipment(s) "competitive at V."

There may be equipments which are not competitive at any V : these we shall call ineffective equipments : equipments which for some V are competitive will be called effective equipments.

In order to avoid giving special attention to possible exceptional cases, we shall introduce the following further assumptions.

(*vii*) There exists a finite set of "basic" equipments $E_1, E_2, \ldots, E_n$, such that any effective set of equipment is composed of one or more of these basic equipments.

It follows from assumptions (*iv*) and (*vii*) that any equipment competitive at V is composed only of those basic equipments which are competitive at V.

(*viii*) There is never more than one food-wage-rate at which two given basic equipments are both competitive.

(*ix*) Every set of values of V for which a basic equipment is competitive is a closed connected set.

From among our basic equipments select those each of which are competitive at more than one food-wage-rate, and hence over a closed range of rates. It follows from assumption (*viii*) that these ranges do not overlap and from (*v*) that between them they cover the whole range of V from O to V_{max}, the level at which the competitive interest rate is zero. Hence the ranges of V fall into a natural order and we may number our selected basic equipments accordingly $E_1, E_2, \ldots, E_m$, letting E_1 be that which is competitive at zero food-wage and E_m that which is competitive at V_{max}.

We may denote by V_1 that V at which both E_1 and E_2 are both competitive, and in general by V_s that rate at which E_s and E_{s+1} are both competitive. We may denote by R_s the competitive rate of interest at V_s, namely $R\ (V_s)$.

We may describe any adjacent pair of equipments E_s and E_{s+1} as consecutive equipments.

At this point we part company with Joan Robinson and introduce a definition about quantity of capital which conflicts with any use of J.R. units.

DEFINITION. The ratio of the quantities of capital in any two equipments which are both competitive at the same rate of interest (and food-wage-rate)[1] is equal to the ratio of their costs calculated at that rate of interest (and food-wage-rate).

[1] The reference to the food-wage-rate is not essential to the definition, but is included in order to facilitate the extension of the definition to the case with many factors employed.

Since this definition applies to every pair of consecutive equipments, it determines the amount of capital in every one of the selected basic equipments, save for an arbitrary multiplying constant. It does this without contradiction, since the assumptions ensure that none of them is competitive at the same rate of interest as is any other except the two adjacent to it in the sequence : moreover, any consecutive pair E_s and E_{s+1} compete at a unique rate of interest R_s. The definition also covers without ambiguity those basic equipments which are competitive at only one interest rate. To extend the measure to mixed equipments composed of more than one basic equipment we adopt the following definition :

DEFINITION. The quantity of capital in any mixed equipment is the total of the quantities in the basic equipments of which it is composed.

We refer to this method of comparing the amounts of capital in different effective equipments as the chain index method, because of the obvious analogy with a chain index of quantities. The extension of the definition to the case where basic equipments form a continuous spectrum, instead of a sequence, is discussed in Appendix I.

IV. THE EQUALITY OF MARGINAL PRODUCT AND REWARD

Let E and E' be any two equipments both competitive at the rate of interest $R(V)$. Let employer A employ quantities Y of E and Y' of E', but employer B use quantities $Y + y$ of E and $Y' - y$ of E'. Then the cost at food-wage-rate V and interest rate $R(V)$, of the total equipment of each employer is the same. Hence the interest paid by each employer is the same. Hence, interest rates being at the competitive level $R(V)$ proper to V, the difference in the two wage-bills must equal the difference between the values of the two product-flows, since under competition profits of each employer are zero. It follows that the extra product of the employer employing the more labour is just sufficient to pay the wages of that labour at the competitive rate, or in technical language the competitive wage of labour equals the marginal product of labour, the quantity of capital being held constant.

This may be expressed algebraically as :

$$w_x = \frac{\partial}{\partial x} f(x, z) \quad \text{(4.1)}$$

where w_x is the food-wage of labour,
x is the amount of labour employed,
z is the quantity of capital,
$f(x, z)$ is the flow of product from these quantities of factors.

Now by our assumption (*iv*) of constant returns $f(\lambda x, \lambda z) = \lambda f(x, z)$ for all real λ and hence :

$$x \frac{\partial f}{\partial x} + z \frac{\partial f}{\partial z} \equiv f \quad \text{(4.2)}$$

Also under competition, by assumption (*v*) :

$$x\, w_x + z\, w_z = f \quad \text{(4.3)}$$

where w_z is the food-reward under competition of each unit of capital.

Hence $z w_z \equiv f - x\, w_x \equiv f - x \dfrac{\partial f}{\partial x} \equiv z \dfrac{\partial f}{\partial z}$

Hence $w_z \equiv \dfrac{\partial f}{\partial z}$ ....(4.4)

or in other words, the reward of each unit of capital is equal in value to its marginal social product.

Our method has thus the added convenience that it provides a means of expressing capital as a quantity and yet enabling us still to say that under perfect competition the two factors, labour and capital, are each paid according to their marginal productivity to society.

V. POSSIBLE ANOMALIES IN THE TWO--FACTOR MODEL

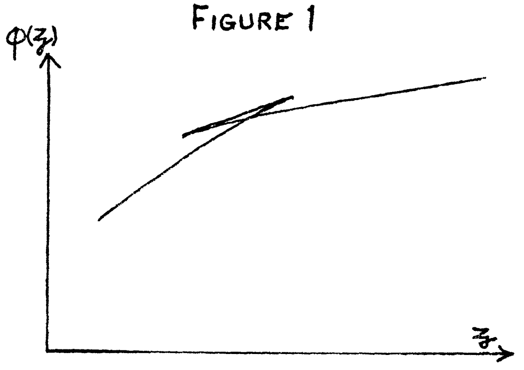
It may seem intuitively obvious that the function $f(x\ z)$ expressing output as a function of labour and capital must be single-valued. But our assumptions are not sufficient to ensure this.

Let $f(1, z) = \phi(z)$ then by our assumption of constant returns $f(x, z) = x\ \phi\left(\frac{z}{x}\right)$ so that a knowledge of $\phi(z)$ is sufficient for a knowledge of $f(x, z)$.

Contrary to intuitive expectation, our assumptions do not ensure that a graph of $\phi(z)$ is a single-valued curve sloping upwards to the right. For example, a graph of the form shown in Fig. 1 is quite possible.

The further assumption that is needed in order to eliminate this possibility is that of two equipments E_{s-1} and E_s (both competitive at R_{s-1}), E_s (that competitive at the lower range of interest) will have the higher productivity, i.e. the higher ratio of food output to labour input. Under this assumption a gradual fall in the rate of interest would entail increases both in productivity and in the quantity of capital per head. But although this may fit in well with our preconceived notions, there is no logical justification for the assumption. It is logically possible that over certain ranges of the rate of interest, a fall in interest rates and rise in food-wages will be accompanied by a *fall* in output per head and a *fall* in the quantity of capital per head.

FIGURE 1



Suppose now, that instead of comparing stationary states, we are considering a sequence of states in time. If we conceive of the rise in food-wages and the accompanying fall in the rate of interest as being caused by a steady process of net investment, with all labour employed, it is interesting to consider what would happen next when a further rise in real wages and fall in the rate of interest would make competitive only equipment with lower productivity and employing *more* men per unit quantity, and thus requiring negative net investment. Presumably, the only way that investment could remain positive without a prolonged interval of disinvestment would be for food-wages to leap up and the rate of interest to leap down to levels where capital equipment even more productive than that in existence became competitive.

The fall in the rate of interest would be sufficient despite offsetting factors to cause a sudden increase in the (demand) value of existing capital equipment bringing windfall gains to its owners : on the other hand, the rise in wages would be sufficient, despite offsetting factors, to raise the replacement cost of existing equipment even more than its (demand) value, so that no more of it would be produced. During the switch to the new type of equipment it would no longer be true that the factor-rewards were equal to the values of their marginal products. A fuller discussion of this case is given in Appendix II.

A related inadequacy of our model arises in connection with our assumption (*ix*). This rules out the possibility that an equipment may be competitive over *two* ranges of the rate of interest, although not competitive over an intermediate range. This assumption is necessary in order to get neat results, and intuition suggests that the excluded case is unrealistic, but it is shown in the Appendix by simple numerical examples that there is no logical justification for the assumption: it is as easy to imagine a world featuring the excluded case as one free of it. If we drop assumption (*ix*) we admit again the possibility of two stationary states each using the same items of equipment and labour force, yet being shown as using different quantities of capital, merely on account of having different rates of interest and of food-wages.

VI. ACCUMULATION AND TECHNICAL PROGRESS

The model which we have so far discussed suffers from three serious limitations.

(*i*) It is confined to stationary states.
(*ii*) A given state of technical knowledge is assumed.
(*iii*) Labour and capital are the only two factors of production.

The concluding sections of the article will be concerned with the extension of the theory to the case where more than two factors are employed, but before advancing to this, it is worth while to consider what interest our results may have in spite of the two other limitations (*i*) and (*ii*).

Joan Robinson has pointed out that a rigorous discussion of the theory under conditions of steady increase in capital per head would be excessively complicated. However, the interest of a comparison of a sequence of stationary states is due to the presumption that this will give us a first approximation to a comparison of successive positions in a slow process of steady accumulation. This presumption is far stronger when we are considering a spectrum of basic equipments E_u, with u a continuous variable, than when the basic equipments form a discrete series E_s with s 1, 2, 3, . . . n.

Provided that R, the rate of interest, is now regarded as the short-term rate, and employment is assumed constant, it is reasonable to expect that where the rate of net investment is of the first order of small quantities, then by using the stationary state analysis to provide snapshots of stages in a process of growth we shall incur errors only of the second order of small quantities. The result suggested by the above theory is that we may then regard output as determined by a function of the amounts of labour and capital employed, where capital itself is increasing at a rate equal to the net rate of saving measured in our units of capital. The rewards of the two factors at any stage may be found by the usual marginal rule: in particular, the investment will increase or decrease the relative share of capital according as the elasticity of substitution of capital for labour is greater or less than unity. On the other hand, it is worth noting that the *mere* knowledge of the production function, although it enables us to obtain the reward per unit of capital at each point, cannot of itself enable us to calculate the rate of interest at any point without further information.

It is reassuring to find that the orthodox analysis fits in so well with the new presentation, once a convenient method has been found for measuring the quantity of capital. But this does not mean that the new presentation adds nothing to the old. It shows that the form of the production function cannot be properly known until the whole history of the advancing economy is known : for it is only then that the quantity of capital can be appropriately measured. This capital must be a balanced outfit as regards age-distribution, and must include capital equipment under construction sufficient to enable the balanced outfit to be maintained. Similarly, the labour employed must include labour engaged in replacing and maintaining capital equipment. The appropriate definition of output is net output, i.e. output excluding maintenance and replacement of capital equipment : similarly, the appropriate concepts of saving and investment are net concepts, giving the excess of net output over consumption output, so that investment is equal in value to the rate of increase in the quantity of capital, as well as to the rate of saving. There is room for argument whether this concept of saving is the one that may most reasonably be regarded as a function of income, but this argument is quite distinct from any of the topics discussed in this article. It is, however, relevant to the question of the rate at which the accumulation of capital is likely to continue in any given model : this question we shall not pursue here.

In real life, the introduction of more productive capital equipment takes place most often because of advances in technical knowledge. Such investment lies outside the scope of the model so far discussed. The production function itself depends upon the state of technical knowledge, and the results we have obtained depend on the assumption that nobody expects technical advances to be made. New technical discoveries would involve a change in the whole production function and an entirely new theory is required to investigate the effects of this.

The difficulty of tracing the effects of technical advance in a model like ours lies in the need to decide which capital equipment after the change can be provided without further saving, and with constant employment, to replace the existing equipment as it wears out. The difficulty of this decision is due to the difference in the times required to build up different types of equipment.

These difficulties may be side-tracked by assuming the simple conditions of the model discussed above in Section II and again in Appendix II, Section II. Here each equipment costs the labour of 100 men spread over one year : it needs $100u$ men to operate it and produces $100\phi(u)$ tons of food over one year, at the end of which it wears out. It is easily shown that in this case the wage of labour is the slope of the curve relating $\phi(u)$ to u, and the relative share of labour is $\left(1 + \frac{1}{u}\right)$ times the slope of the curve relating $\log \phi(u)$ to $\log u$. Retaining the simplifying assumptions, we may represent a technical advance by a transference from the curve $\phi(u)$ to some higher curve $\psi(u)$: moreover, in virtue of our special assumptions, the effect of the advance will be a move from a point on the $\phi(u)$ curve to a point on the $\psi(u)$ with u unchanged. The new wage will be given by the slope of the $\psi(u)$ curve at this point and the new relative share of labour will be $\left(1 + \frac{1}{u}\right)$ times the slope at the corresponding point of the curve relating $\log \psi(u)$ to $\log u$.

Roughly speaking, we may regard capital-saving inventions as having the effect :

(a) $$\psi(\theta u) \equiv \theta\phi(u) \qquad \theta > 1 ;$$

and inventions which economise labour in the using of equipment as having the effect :

(b) $$\psi(\theta u) \equiv \phi(u) \qquad \theta < 1 ;$$

and inventions which economise labour equally in the building of equipment and in the use of equipment as having the effect :

(c) $$\psi(u) \equiv \theta\phi(u) \qquad \theta > 1.$$

If for further simplicity we supposed $\phi(u)$ to have been such that the relative share of labour was insensitive to changes in the factor-ratio, this would be supposing that the slope of the curve relating $\log \phi(u)$ to $\log u$, was approximately proportionate to $\left(1 + \frac{1}{u}\right)$ It could then be shown that the effects of the three types of technical advance would be:

(*a*) to increase wages but decrease labour's relative share;
(*b*) to increase wages and increase labour's relative share;
(*c*) to increase wages but leave the relative shares unchanged.

These results could most easily be obtained by considering the effects at given u, on the slope of the curve relating $\log \phi(u)$ to $\log u$, of a uniform shift of the curve:

(*a*) at 45 degrees upwards and to the right;
(*b*) to the left;
(*c*) upwards.

These results, whilst being suggestive and of some interest in their context, depend on the extra set of simplifying assumptions built into the model for this purpose: these assumptions exclude any differences in the construction-periods of different types of equipment. We shall not attempt to analyse the effects of technical advance when this assumption is relaxed.

The remainder of the article will be concerned with the removal of our other simplifying assumption that only two factors, labour and capital, are employed.

VII. EXTENSION OF THEORY TO CASE WHERE THREE FACTORS ARE EMPLOYED

It is possible to extend the above theory to the case where several homogeneous factors are employed with equipment, if we limit attention to the production function for the economy as a whole, and if the amounts employed of the homogeneous factors in the economy as a whole are fixed or in fixed proportions. In this case, we may define a composite factor, composed of the homogeneous factors, combined in the given fixed proportions, and regard output in the economy as a whole as a function of the amounts employed of the composite factor and of capital. It is possible to measure capital in such a manner that in any stationary state equilibrium, units of capital and units of the composite factor will be rewarded according to their marginal productivity.

The extension of the earlier theory is not quite so straightforward as might be supposed owing to the fact that the relative cost of two outfits of equipment will no longer depend only on the rate of interest, but also on the relative wage-rates of the various homogeneous factors. This complication is sufficiently serious to wreck any attempt to regard output as a function of the quantity of capital and the amounts of the homogeneous factors, *each* homogeneous factor being paid according to its marginal product.

The possibilities and limitations of the extended theory may be adequately illustrated by a consideration of the case where there are two homogeneous factors, labour and land.

We first amend our assumptions of Section II (*i*) to (*vi*) by inserting the words "and land" after "labour" whenever it occurs.

We now note, as a consequence of our assumptions, that at any pair (V, W) of food-wage-rates of labour and land there will be a rate of interest $R_{\lambda, VW}$ which an employer can just afford on its capital cost if he builds E_λ and uses it for producing food. By assumption (*v*) if the food-wages of labour and land are (V, W) competition will drive the rate of interest to the level

$$R(V, W) = \underset{\lambda}{Max}\, R_{\lambda,\, VW} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.1)$$

of the greatest of the $R_{\lambda,\, VW}$. We may call this the competitive rate of interest at (V, W). When food-wages of labour and land are at (V, W) only that (those) equipment(s) E_{λ} will be built for which $R_{\lambda,\, VW} = R\,(V, W)$ we may call these equipments "competitive at (V, W)."

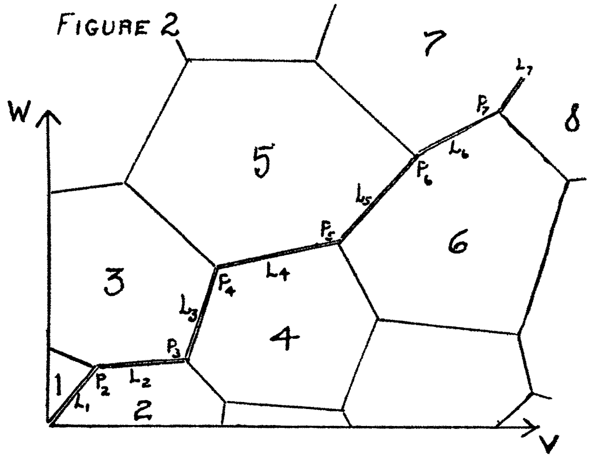
Equipments competitive at some (V, W) will be called effective, the others will be called ineffective.

We now introduce assumption (*vii*) unmodified. It follows from this assumption that any equipment competitive at VW is composed of basic equipments effective at VW.

We modify assumption (*viii*) to the form,

(*viii*) There is no closed region of finite area, of the plane of (V, W) throughout which two basic equipments are both competitive.

(*ix*) Every set of the values of (V, W) for which a basic equipment is competitive is a closed connected set.



It is helpful at this stage to consider Fig. 2, showing which basic equipments are competitive at various wage-rates V, W.

Each cell of the diagram represents a region of (V, W) in which one particular basic equipment is competitive (the cells have for simplicity been drawn as hexagonal with straight sides, but, in general, the sides will be curved and the number of vertices may vary from cell to cell). If we imagine R rate of interest to be measured along a third axis, then we may regard the diagram as showing a lot of intersecting surfaces, only that with the highest R, appearing above each point (V, W). Where rents are high and wages low, the competitive equipment is likely to employ a high proportion of labour to land and vice versa : hence, if we divide the basic equipments into those with more men per acre, and those with less men per acre than the density laid down for the economy as a whole, the two types are likely to be separated on our diagram by a critical boundary such as that shown by the double line, those employing many men per acre lying above it.

It is clear that the only pairs of wage-rates which will allow a stationary state employing the required number of men per acre will be those corresponding to points on the critical boundary which we have just described. For only when (V, W) is any such point will it be possible to combine a basic equipment employing less than the required number of men per acre with one employing more than it, and thus to employ the correct number of men per acre in the economy as a whole.

Having found the critical boundary we may number off the basic equipments whose cells have edges along the boundary, odd numbers lying to one side, and even numbers to the other side of it. To each consecutive pair will correspond cells with a common edge along the critical boundary, and from such a pair we may construct a composite equipment employing the correct number of men per acre. Let the composite equipment composed of basic equipments E_s and E_{s+1} be called F_s, then the points (V, W) at which F_s is competitive form that segment of the critical boundary which joins the cells of E_s and E_{s+1}: call this segment L_s. Finally, let the point where L_{s-1} meets L_s be called P_s: then at the wage-rates (V, W) represented by P_s, both composite equipments F_{s-1} and F_s are competitive, and so is any combination of them.

From this point, the theory proceeds as in the case of two factors only. The composite equipments $F_1 \ldots F_{m-1}$ now take the place of the basic equipments $E_1, E_2 \ldots E_m$ of the two-factor theory. Analogously to that theory, we compare the quantities of capital in two consecutive composite equipments, by costing them at the factor-wages (VW) at which both are competitive, and at the rate of interest which is competitive at that rate. The composite equipment all employs the same number of men per acre, so we may measure men and land in terms of a composite factor embodying men and land in that proportion. When this is done output may be expressed as a function of quantity of capital and quantity of composite factor, and it follows by the same argument as before that in each stationary state, units of capital and units of the composite factor will each be paid according to their marginal productivity.

This extended theory is, however, limited by the fact that quantity of capital has been defined only for composite equipments and not for basic equipments. In any stationary state which does not employ land and labour in the specified proportions, the quantity of capital remains undefined. Hence, we cannot speak of the marginal productivity of labour, the quantities of land and capital being kept constant. Any attempt to define quantity of capital to cover these situations, and to develop a function relating output $f(x, y, z)$ to the quantities of labour, land and capital will fail to satisfy the hoped-for equations:

$$w_x \equiv \frac{\partial f}{\partial x} \qquad w_y \equiv \frac{\partial f}{\partial y} \qquad w_z \equiv \frac{\partial f}{\partial z} \quad \ldots\ldots\ldots\ldots (7.2)$$

where w_x, w_z, w_z denote the wage-rates of the three factors.

This failure springs from the fact that the ratio of the costs of two consecutive basic equipments E_s and E_{s+1} will *not* in general be the same at the wage-rates and competitive interest rates corresponding to the two points P_s and P_{s+1}, at *both* of which the two basic equipments are competitive. This divergence is associated with changes in the relative wage-rates of labour and land and to the fact that the two basic equipments embody different proportions of land and labour.

It is a matter of interest that the quantities of capital can still be compared as between the *odd*-numbered basic equipments—namely those employing *more* than the required number of men per acre. To compare the quantities of capital in basic equipment E_{2s-1} and $E_{2s}+1$ we simply compare their costs in the only situation in which both are competitive, namely at the wage-rates and interest rates proper to P_{2s} : similarly, we may compare the quantities of capital in even-numbered basic equipments (those using *less* than the required number of men per acre). But we cannot satisfactorily compare the amount of capital in basic equipments using more than the required number of men per acre with that in those using less. To loosen the style let us call these two types of capital labour-using and land-using.

We can regard output as a function of four variables, namely the amounts of labour x, land y, labour-using capital z_1 and land-using capital z_2, provided we confine attention to those stationary states in which wage-rates and interest rates are at such levels that it would be *possible* to produce without loss, using the required number of men per acre. This may still allow considerable variation of the number of men employed per acre, as it includes the possibility of using either only land-using equipment or only labour-using equipment. Although there are four factors, there are from a technical point of view only three degrees of freedom in varying them, so that only a three-dimensional subset in the region (x, y, z_1, z_2) represent combinations of factors which are technically possible. When, therefore, we represent output as a function $O \equiv f(x, y, z_1, z_2)$ of technically possible combinations of the four factors, we cannot in general give meaning to the partial derivatives, and so we cannot say that each factor is rewarded according to its marginal product.

But an analogous proposition can be established, namely, that if $(\Delta x, \Delta y, \Delta z_1, \Delta z_2)$ represents any technically possible small variation in the amounts of the four factors, then :

$$\Delta f \equiv W_x \Delta x + W_y \Delta y + W_{z_1} \Delta z_1 + W_{z_2} \Delta z_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7{\cdot}3)$$

where Wx, Wy, Wz_1, Wz_2 are the wage-rates of the four factors before the change. This proposition asserts that any combination of factors which has a productivity at the margin receives a wage-rate equal to that marginal product. This proposition will apply for example to appropriate combinations of any pair of the four factors.

The need for measuring capital by the two variables z_1 and z_2 arises from the fact that the wage-rates of labour-using capital and land-using capital will be in different ratio to one another in different stationary states.

VIII. EXTENSION OF THEORY TO THE CASE WHERE SEVERAL FACTORS ARE EMPLOYED

Even in the case where only land, labour and equipment were employed, a rigorous statement of the simplifying assumptions needed for clear-cut results has not been achieved. Many of our assumptions were packed into the drawing of Fig. 2, where we placed the numbered cells neatly along a corridor without thorough supporting discussion.

This topological device does not spring so readily to hand when we extend the discussion to the case where several factors are employed, and one can only suggest by analogy what results should be obtainable from a proper enlargement of the simplifying assumptions to that case.

If the quantities employed (or their proportions) of the homogeneous factors are fixed for the economy as a whole, we should still be able to express output as a function of the quantities employed of :

(*i*) a single composite factor,
(*ii*) capital,

and in such a way that both the composite factor and capital would, in each permissible stationary state, be rewarded according to its marginal product.

But if we wish to construct a function of output to cover cases where the quantities of homogeneous factors employed are *not* in the required proportions, we should have to represent capital by as many variables as there are homogeneous factors. In this case, marginal adjustments to the quantities of factors would in general involve combinations of them. It should be possible to prove that in each of certain permissible stationary states any such combination of factors would receive a wage-rate equal to its marginal productivity.

IX. ACKNOWLEDGMENT

I have been greatly helped in the writing of this article by criticism from Mrs. Robinson, Professor Kahn and Mr. Johnson, and by instruction from Mr. Kaldor. I need only add the customary rider that none of them bears any responsibility for the short-comings of the article.

APPENDIX I

THE CHAIN INDEX METHOD OF MEASURING CAPITAL

In Section IV a definition was given which determined the amounts of capital in each of a discrete series of selected basic equipments. This definition may be reformulated as follows: The unit of capital is so defined that where $C(s, R, V)$ denotes the cost per unit of capital, of equipment type E_s at interest rate R and wage-rate V, then for all s, V:

$$C(s, R_s V) = C(s + 1, R_s, V)$$

where R_s denotes that rate of interest at which both the consecutive equipments E_s and $E_s + 1$ are competitive.

When we consider a continuous spectrum of equipments E_u with u a continuous variable, we may adapt this definition as follows: Let $C(u, R, V)$ denote the cost per unit of capital of equipment of type E_u at rate of interest R and wage-rate V, the units of capital must be such that for all u and V,

$$\left\{ \frac{\partial}{\partial u} C(u, R, V) \right\}_{R = R_u} = 0$$

where R_u is the rate of interest at which E_u is competitive.

APPENDIX II

NUMERICAL EXAMPLES

I. SIMPLIFIED TWO-FACTOR MODEL WITH A FINITE SEQUENCE OF BASIC EQUIPMENTS

Suppose that there are N basic equipments $E_1, E_2 \ldots E_N$. Let x_s be the number of men required to operate and maintain E_s and let O_s be the annual food output produced by E_s. Let the cost of E_s consist of the expenditure of X_s man-years of labour during a short interval of time at T_s years before the food flow begins. If we define date of completion of the machine to be that at which the food flow begins, then the food cost of E_s at the date of completion is given by:

$$C_s = VX_s\ e^{RT_s} \qquad \text{(1.1)}$$

where R is the rate of interest and V the food-wage-rate.

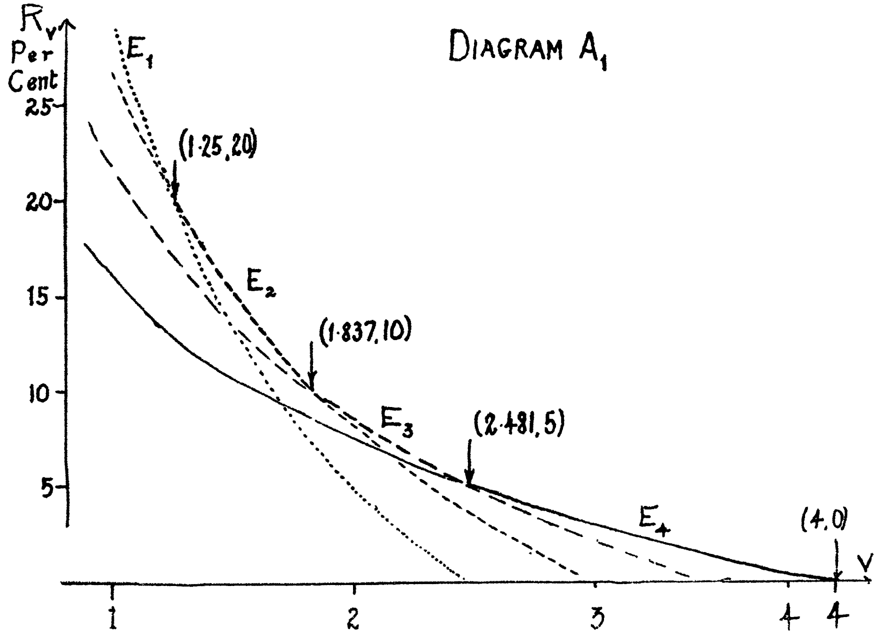
Suppose that each E_s can be maintained permanently. The highest interest rate R_{vs} which can be afforded with E_s at wage-rate V is given by :

$$O_s = V\{x_s + R_{vs}X_s e^{R_{vs}T_s}\} \qquad \text{(1.2)}$$

(i) Numerical example involving no anomalies

Suppose that there are four basic equipments and that their technical coefficients x_s O_s T_s X_s are those given in the following table.

s	x_s	O_s	T_s	X_s
1	4	10	0	20
2	3	9	1	17.193
3	2	7	2	14.920
4	1	4	4	10.054



The values of R_{vs} calculated for formula (2) are shown for each equipment, for food wages in the range one to four tons of food per annum, in Diagram A_1. The competitive interest rate at each food-wage is shown by the envelope of the four curves. It can be seen from the diagram that for food-wages up to and including 1.25 tons of food per annum, equipment 1 is competitive : in the range 1.25 to 1.837 tons of food per annum, inclusive, equipment 2 is competitive : in the range 1.837 to 2.481 tons of food per annum, inclusive, equipment 3 is competitive : and in the range 2.481 to 4 tons of food per annum, inclusive, equipment 4 is competitive. At food-wages exceeding 4 tons per annum, the wage-bill would exceed the maximum national income technically possible.

We may distinguish two possible kinds of stationary state :

(*i*) Pure—employing only one basic type of equipment. The food-wage-rate may be at any level at which that basic equipment is competitive, and the rate of interest will be at the corresponding competitive level.

(*ii*) Mixed—employing some combination of a consecutive pair of equipments at the food-wage level at which both are competitive, and at the competitive level of the rate of interest.

In our model, just three mixed types of stationary state are possible:

(*i*) Employing a combination of E_1 and E_2 at a wage of 1.25 tons of food per annum and interest rate 20 per cent per annum.

(*ii*) Employing a combination of E_2 and E_3 at a wage of 1.837 tons of food per annum and interest rate 10 per cent per annum.

(*iii*) Employing a combination of E_3 and E_4 at a wage of 2.481 tons of food per annum and interest rate 5 per cent per annum.

To establish what quantity of capital is embodied in a unit of each of the four basic equipments we merely compare the costs in each consecutive pair at the rate of interest at which they compete. We find that at a rate of interest of 20 per cent the cost of items of equipments 1 and 2 are in ratio 20 : 21. At a rate of interest of 10 per cent, items of equipments of types 2 and 3 have costs in ratio 34.89 : 33.26. At a rate of interest of 5 per cent items of equipments of types 3 and 4 have costs in ratio 40.76 : 30.38. Accordingly, we take units of the four basic types of equipment to represent quantities of capital 20, 21, 20.16 and 14.96.

Food output may now be uniquely expressed in terms of the amounts x z of labour and capital employed: we find it correctly given (in tons of food per annum) by

$$f\,(x\ z) \equiv 1.250x + 0.25z \text{ where } 5x \leqslant z \leqslant 7x$$

$$f\,(x\ z) \equiv 1.837x + 0.1661z \text{ where } 7x \leqslant z \leqslant 10.08x$$

$$f\,(x\ z) \equiv 2.481x + 0.1011z \text{ where } 10.08x \leqslant z \leqslant 14.96x \quad \ldots\ldots\ldots\ldots (1.3)$$

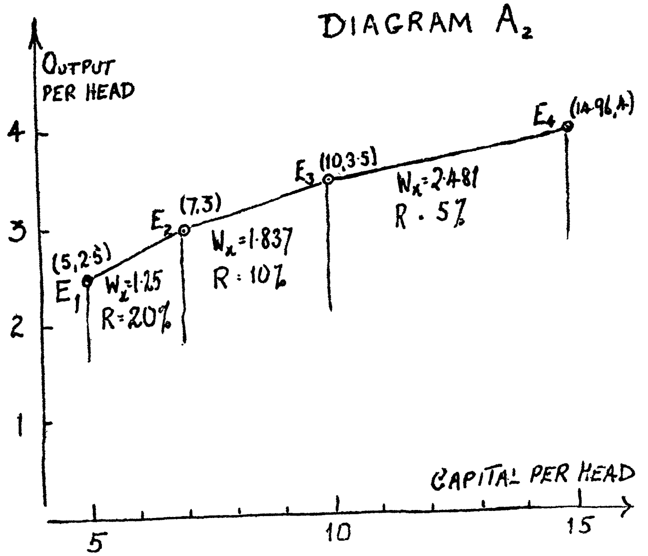
The three forms of the equation correspond to the three mixed types of stationary state and they overlap at the two pure types with basic equipments 2 and 3.

In every stationary state of mixed type labour is paid a wage equal to $\frac{\partial f}{\partial x}$, the coefficient of x in the production function: and each unit quantity of capital earns $\frac{\partial f}{\partial z}$, the coefficient of z. In the pure types with basic equipments 2 and 3 the appropriate equation to use in calculating marginal productivity differs according to whether an increase or decrease is considered. In these pure types the factors are rewarded at rates within the closed ranges terminating in their two marginal productivities. In the pure state, with only equipment type 1, the marginal productivities give an upper limit to wage-rates and a lower limit to the reward of each unit of capital. In that with only type 4 they give a lower limit for wage-rates and an upper limit for the reward of each unit of capital.

Diagram A_2 (overleaf), which shows $\frac{f\,(x\ z)}{x}$ plotted against $\frac{z}{x}$ further illustrates the form of the production function in this example.

From our knowledge of $f\,(x\ z)$ we can calculate how labour's relative share of output varies with the quantity of capital employed per head. This brings out clearly the phenomena which Joan Robinson describes in connection with the Wicksell effect and the Ricardo effect.

Increases in labour's share only "take place" on the three occasions when the conditions are those of a pure stationary state: all increases of capital per head from one stationary state to another within the same mixed class involves an increase in output which accrues wholly to the owners of capital.



(*ii*) *Exceptional case*

To illustrate the anomalies which may arise when food-wages and productivity vary inversely, consider the model represented by the following table, which superficially represents the normal example we have just discussed.

s	x_s	O_s	T_s	X_s
1	4	12	0	20
2	2	9	0	20
3	4	16	4	20.854
4	2	10	4	21.221

When we calculate the interest rates R_{vs} at which the four equipments can compete at various food-wage-rates, we find E_1 competitive above 20 per cent, E_2 in the range 10 to 20 per cent, E_3in the range 5 to 10 per cent and E_4 at 5 per cent. Following our earlier procedure, we should argue that three mixed types of stationary state are possible, E_1 and E_2 ; E_2 and E_3 ; and E_3 and E_4 ; the three appropriate pairs of values of the food-wage and interest rate being (1.5, 20 per cent), (2.25, 10 per cent) and (3.034, 5 per cent).

In order to consider the production function, let us now assign a quantitative measure of capital as before. E_1 and E_2 may each count as 20 units ; E_2 and E_3 compete at 10 per cent and the ratio of their costs is then 20 : 31.11, so E_3 must be taken as 31.11 units. E_4 and E_3 at all rates of interest have their costs in the ratio 21.221 : 20.854 ; applying this ratio to 31.11, we find that E_4 must represent 31.657 units of capital.

We may now draw up a table showing for each of the four equipments, O_s the food output, x_z the number of persons employed, z_s the quantity of capital, and hence capital per head z_s/x_s, and output per head O_s/z_s.

We could construct a diagram[1] showing the output per head as a function of capital per head in the various possible stationary states ; the points for mixed types

[1] This diagram is not shown, but its main features would be the same as those of Fig. 1 above.

	Output	*Employment*	*Quantity of capital*	*Capital per head*	*Output per head*
S					
1	12	4	20	5	3
2	9	2	20	10	4.5
3	16	4	31.11	7.777	4
4	10	2	31.657	15.828	5

again being obtained by drawing straight lines to join the pair of points relating to the consecutive pair of equipments which are combined. We may also write down the production function $f\ (x\ z)$ as :

$$f\ (x, z) \equiv 1.5x + 0.3z \qquad 5x \leqslant z \leqslant 10x$$

$$f\ (x, z) \equiv 2.25x + 0.225z \qquad 7.777x \leqslant z \leqslant 10x$$

$$f\ (x, z) \equiv 3.034x + 0.124z \qquad 7.777x \leqslant z \leqslant 15.828x \quad \ldots\ldots\ldots\ldots\ldots (1.4)$$

These three forms corresponding to the three mixed types of stationary state, (E_1, E_2), (E_2, E_3) and (E_3, E_4).

This production function is triple-valued for those values of x and z such that $7{\cdot}777x < y < 10x$. This is no paradox, since the function merely tells about various possible stationary states. The function preserves the property that each factor is paid a wage equal to its marginal product : thus $W_x = \frac{\partial f}{\partial x}$ and $W_z = \frac{\partial f}{\partial z}$ in all the mixed-type stationary states.

This triple-valued feature of the production function cannot be attributed to our method of measuring the quantities of capital per head in E_2 and E_3 : for any plausible method would ascribe more capital per head to E_2 than to E_3 : for example, Joan Robinson's method would give as large or larger a margin in the factor ratio.

It may again be objected that no employer in his senses would use E_3 and E_4 in the combination with the same factor ratio as E_2, since he could obtain higher output per head by using E_2 : the answer is that if wage-rates were at 3,034 tons of food per annum, then E_3 and E_4 could compete at 5 per cent, but E_2 could not. *At this interest rate of* 5 *per cent,* unit quantity of E_2 would cost considerably more than unit quantity of E_3 or E_4, and so much so as more than to offset the advantage in productivity.

Although the production function is quite satisfactory for describing possible stationary states it is in this case definitely inconvenient for illustrating a time-sequence. Suppose an economy, at constant employment, slowly, out of its saving, to have converted from E_1 to E_2 equipment. Wages have stood at 1.5 tons of food per annum and interest rates at 20 per cent. If, after the conversion, wages rise just above 2.25 tons of food and interest rates fall just below 10 per cent, E_3 will become the only competitive equipment, and E_2 will still be more nearly competitive than E_4. But any attempt to use E_3 must *lower* the national income and *decrease* the quantity of capital employed, on account of its lower productivity and lower factor ratio than E_2. The only way positive net investment and the rise in productivity can continue uninterrupted is for real wages to jump up to above 3.034 and interest rates fall below 5 per cent. If this is done, there will ensue a period of conversion from E_2 to E_4, and the economy will for a time be a mixture of E_2 and E_4. But such a mixture is certainly not possible in a stationary state, since there is no real wage-rate and rate of interest at which E_2 and E_4 are both competitive.

In our particular example we have assumed that the equipments never need replacement, but the same features of a multi-valued production can arise without this simplifying device. Suppose for the moment that the equipment needs renewal after a certain life, then if the rate of interest had been pushed down to the level at which E_4 is competitive, E_2 would fail by a considerable margin to be competitive. *All* E_2 would be replaced by E_4 as it fell due, and unless the life of E_2 was long, this would represent a tremendous demand on the investment industries—so that the capital per head could increase from 10 to 15.828 during one life-time of an E_2 equipment. This would take us far from the conditions of slow and gradual accumulation, to which our model, with its expectation of constant conditions, had a limited relevance.

The outcome of this discussion is that, although our method of measuring quantity of capital provides us with a production function satisfactory for describing the family of stationary states, formidable difficulties arise when we consider a sequence of states in time in a developing economy, unless we rule out cases in which a lowering of interest rates can cause the introduction of techniques with a *lower* productivity than those used up till then. A numerical example has shown that these cases cannot be ruled out merely on logical grounds.

To illustrate the case where assumption (*ix*) breaks down so that an equipment is competitive over each of two separated intervals of V, consider the position when, of our four equipments, only E_2 and E_3 have been invented. The condition that both E_2 and E_3 should be competitive at the interest rate R may be expressed :

$$\frac{x_2 + Re^{T_2 R}X_2}{O_2} = \frac{x_3 + Re^{T_3 R}X_3}{O_3} \quad \text{(1.5)}$$

so that substituting numerical values :

$$\frac{2 + 20R}{9} = \frac{4 + 20\ 854Re^{4R}}{16}$$

Therefore $$R\{320 - 187.686e^{4R}\} = 4 \quad \text{(1.6)}$$

It may be verified that there are two positive solutions for R, namely $R = 0.3$ and $R = 0.0402$.

The corresponding food-wage-rates are 2.25 and 3.21. We find that for wage-rates below 2.25 E_2 is competitive, for wage-rates between 2.25 and 3.21 E_3 is competitive, and for wage-rates from 3.21 to 4.5 E_2 is competitive.

Moreover, since E_2 and E_3 are both competitive at $(V, R) = (2.25$, 10 per cent) and also at (3.21, 4.02 per cent), there are two possible bases for comparing the quantities of capital in E_2 and E_3. Counting E_3 as 31.11 units as before, the two possible methods ascribe to E_2 either 20 units or 25.4 units.

The formal solution is to regard E_2 as 20 units if it is used at a rate of interest of 10 per cent or over, and as 25.4 units if it is used at a rate of 4.02 per cent or less. This procedure has, however, little to recommend it, apart from its enabling one still to regard the factors as being rewarded according to their marginal productivity.

One final and somewhat fanciful remark may be made with reference to this example. Two mixed types of stationary state using E_2 and E_3 are possible, one at $(V\ R) = (2.25$, 10 per cent) and one at (3.21, 4.02 per cent). Both use the same equipment, but the question of which $(V\ R)$, and hence what income-distribution between labour and capital is fixed, is left in this model for political forces to decide. It is interesting to speculate whether more complex situations retaining this feature are ever found in the real world.

II. SIMPLIFIED TWO-FACTOR MODEL WITH CONTINUOUS SEQUENCE OF EQUIPMENTS

Consider a sequence of basic equipments E_u with u as a continuous variable. Suppose that to build any equipment E_u requires the work of 100 men spread over one year. Suppose that equipment E_u when completed requires $100u$ men to operate it and produces a flow of $100g(u)$ tons of food per annum, where $g(u)$ is a function of u. Suppose that the working life of each equipment E_u is one year, at the end of which it has no value.

At any rate of interest R and wage V the cost of any equipment E_u is:

$$K_R = 100 \frac{e^R - 1}{R} V \quad \text{.......... (2.1)}$$

when new. By the formula for balanced equipment in the integrated case given in Champernowne and Kahn, it may be shown that the expression (2.1) also gives correctly the cost of the balanced set of equipment in equilibrium. Now this expression is the same for all u, and it follows (see Appendix I) that we can regard each equipment E_u as containing 100 units of capital, consistently with the chain-index method.

We are now in a position to write down the production function, for we know that the balanced equipment E_u represents 100 units of capital, employs $100u$ men for production and 100 men for replacement and produces an output stream of $100\,g(u)$. Hence the production function must satisfy:

$$f\,(100\,(1+u),\,100) = 100\,g(u) \quad \text{.......... (2.2)}$$

and since it must be homogeneous of degree one, it must be given in terms of the function $g(u)$ by:

$$f\,(x, y) = y\,g\left(\frac{x-y}{y}\right) \quad \text{.......... (2.3)}$$

for those values of x and y which give to $u = \frac{x-y}{y}$ a value such that E_u is effective.

By the marginal principle, the wages of labour and capital are:

$$\left.\begin{aligned} w_x &= g'(u), \text{ the derivative of } g(u) \text{ where } u = \frac{x-y}{y} \\ w_y &= g(u) - (u+1)\;g'(u) \end{aligned}\right\} \quad \text{.......... (2.4)}$$

Finally, the rate of interest R_u at which E_u is competitive is given, in virtue of (2.1), by:

$$g'(u)\{\,u + e^{R_u}\,\} = g(u) \;\text{ whence }\; R_u = \log\left\{\frac{g\,(u)}{g'(u)} - u\right\} \quad \text{.......... (2.5)}$$

Although each E_u contains the same amount of capital as measured by the chain index, they do not contain equal amounts of capital measured in J.R. units. The amount Q_u of capital in J.R. units in E_u is given by the condition (2.1) as

$$Q_u = 100 \frac{e^{R_u} - 1}{R_u}$$

which in virtue of (2.5) may also be written:

$$Q_u = 100 \frac{\frac{g\,(u)}{g'(u)} - u - 1}{\log_e \left\{\frac{g\,(u)}{g'(u)} - u\right\}} \quad \text{.......... (2.6)}$$

It is possible for $Q_u/1 + u$ to decrease with decreases in u, even in cases where a decrease in u clearly involves an increase in productivity and in an increase in the proportion of the labour force devoted to replacing rather than operating equipment. That is to say, that a decrease in u involving what would ordinarily be understood as deepening or increased capital per head, will be shown as a reduction of the quantity per head of capital measured in J.R. units. As an example of this, consider the case where

$$g(u) = (1 + 11u)^{\frac{1}{11}} \qquad (2.7)$$

then production per head is

$$\frac{(1 + 11u)^{\frac{1}{11}}}{1 + u} \qquad (2.8)$$

which increases as u decreases, and the proportion $\frac{1}{1 + u}$ of labour devoted to replacement increases. But the quantity of capital per head in J.R. units is by (2.6) :

$$\frac{Q(u)}{100\,(1 + u)} = \frac{10u}{(1 + u)\log(1 + 10u)} \qquad (2.9)$$

and numerical calculation shows that this *decreases* from 2.19 to 1.00 as u decreases in the range 2.326 to 0, and the rate of interest meanwhile falls from 3.19 to 0. But production per head *increases* from 0.405 to 1.000 and capital per head, measured by the chain index, from 0.307 to 1.000.

It is evident that a variety of other forms for the function $g(u)$ could be chosen so as to demonstrate similar paradoxical results of measuring capital in J.R. units.

III. MODEL INVOLVING THREE FACTORS

In this model we shall allow five basic equipments E_1, E_2, E_3, E_4 and E_5. E_s will be supposed to cost at interest R, $\frac{X_s}{0.6 - R}$ units of labour, and $\frac{Y_s}{0.6 - R}$ units of land, as would happen, for example, if it had been built up by using $e^{-0.6t}\,X_s$ units of labour and $e^{-0.6t}\,Y_s$ units of land from the distant past $t = \infty$ up till $t = 0$. E_s will be supposed to employ x_s units of labour and y_s units of land permanently to produce a flow of O_s tons of food per annum.

The numerical values of these parameters are given in the following table :

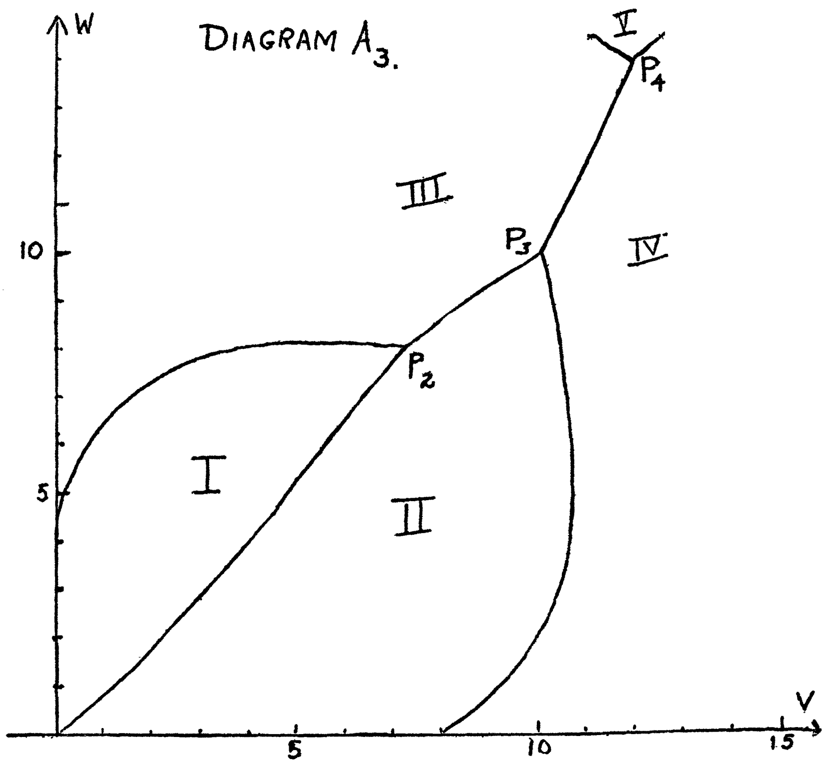
s	x_s	y_s	O_s	X_s	Y_s
1	5	4	109.6	6	5
2	3	4	88	4	5
3	3	2	64	4	3
4	1	2	40	2	3
5	2	1	38	3	3

We shall suppose that in the economy as a whole the numbers of " men " and " acres of land " are equal. It will be seen that equipments 1, 3 and 5 are labour-using, whereas equipments 2 and 4 are land-using.

The values of $R_{s,\,vw}$ are given by such equations as the following :

$$\frac{R_{1vw}}{0.6 - R_{1vw}}\,\{6V + 5W\} + 5V + 4W = 109.6 \qquad (3.1)$$

By calculating various loci in the $V - W$ plane of the type $R_{svw} = R_{s'vw}$ and by making a few auxiliary calculations we may construct the following diagram, which shows in which regions of the $V - W$ plane (i.e. for what factor-wage combinations) the various basic equipments are competitive.



All equipment employing one man per acre can be obtained by the appropriate blend of a labour-using (odd-numbered) equipment, with a land-using (even-numbered) equipment.

The only combinations of factor-rewards VW for which this is possible are at points along the zig-zag corridor in the diagram which runs between consecutive pairs of the five equipments.

In the example only four of the pairs of equipments (E_1, E_2), (E_2, E_3), (E_3, E_4) and (E_4, E_5) can be suitably married: let us call these married couples F_1, F_2, F_3 and F_4.

Then we can construct a table for F_1, F_2, F_3 and F_4 similar to the table above.

s	x_s	y_s	O_s	X_s	Y_s
1	8	8	197.6	10	10
2	6	6	152	8	8
3	4	4	104	6	6
4	3	3	78	5	6

The equipments F_1 and F_2 will be competitive only at the point where equipments E_1, E_2 and E_3 are all competitive: this point is the point P_2 in the diagram, namely

(7.2, 8) and the competitive rate of interest is 20 per cent. Similarly, F_2 and F_3 are both competitive at P_3, namely (10, 10) with competitive interest rate 10 per cent. Finally, F_3 and F_4 are both competitive at P_4, namely (12, 14) with competitive interest zero.

At P_2 and rate of interest 20 per cent, F_1 and F_2 have costs of 380 and 304 tons of food.

At P_3 and rate of interest 10 per cent, F_2 and F_3 have costs of 320 and 240 tons of food.

At P_4 and zero rate of interest F_3 and F_4 have costs of 260 and 240 tons of food.

Hence, counting F_1 as 38 units of capital,
we must count F_2 as 30.4 units of capital,
F_3 as 22.8 units of capital,
and F_4 as 21.05 units of capital.

Counting one man with one acre as one unit of composite factor, and letting η and ζ represent the amounts employed of composite factor and capital, we can obtain the following production function for stationary states employing F_1, F_2, F_3, F_4 or pairs of them under competitive conditions.

$$f(\eta\zeta) = 15.2\eta + 2\zeta \quad \text{for } 4.25\eta \leqslant \zeta \leqslant 5.07\eta$$
$$f(\eta\zeta) = 20\eta + 1.05\zeta \quad \text{for } 5.07\eta \leqslant \zeta \leqslant 5.72\eta$$
$$f(\eta\zeta) = 26\eta \quad \text{for } 5.72\eta \leqslant \zeta \leqslant 7.1\eta$$

and in the mixed-type stationary states the composite factor and capital will be paid at rates equal to the marginal productivities as calculated from the appropriate one of the above three equations.

Returning now to the problem of evaluating the quantities of capital in each of the five basic equipments, we note that E_1 and E_3 only compete at P_2 with $R = 20$ per cent : under these conditions their costs are 208 and 132 and E_2 and E_4 only compete at P_3 with $R = 10$ per cent : under these conditions their costs are 130 and 100 and E_3 and E_5 only compete at P_4 with $R = O$: under these conditions their costs are 150 and 130.

Hence, arbitrarily choosing E_1 and E_2 to represent 20 and 18 units of capital respectively, we find E_3, E_4 and E^1 represent 12.7, 10 and 11 units of capital respectively.

We may construct the following table :

s	x	y	z_1	z_2	O	y/x	z_1/x	z_2/x	O/x
2	5	4	20	0	109.6	0.8	4	0	21.92
1	3	4	0	18	88	1.33	0	6	29.33
3	3	2	12.7	0	64	0.67	4.23	0	21.33
4	1	2	0	10	40	2.0	0	10	40
5	2	1	11	0	38	0.5	5.5	0	19

It is now possible to express output as a function of three variables, x, z_1, z_2, representing the amounts used of labour, labour-using capital and land-using capital, it being assumed that the technically necessary amount of land is provided. The function is :

$$\left.\begin{array}{ll} f(x, z_1, z_2) = 32.08x - 2.54z_1 - 0.46z_2 & 24 - 4z_2/x \leqslant 6z_1/x \leqslant 24.8 - 4.13z_2/x \\ f(x, z_1, z_2) = 13.333x + 1.891z_1 + 2.667z_2 & 24.8 - \frac{6z_1}{x} \leqslant 4.13\frac{z_2}{x} \leqslant 41.7 - 10\frac{z_1}{x} \\ f(x, z_1, z_2) = 28.2x - 1.67z_1 + 1.18z_2 & 41.3 - 4.13\frac{z_2}{x} \leqslant 10\frac{z_1}{x} \leqslant 58.7 - 5.87\frac{z_2}{x} \end{array}\right\} \quad (3.2)$$

Consider, for example, the second equation. This relates to mixed stationary states using basic equipments 2, 3 and 4. The partial derivatives are 13.333, 1.891 and 2.667.

13.333 is the marginal productivity not merely of labour, but of labour plus the extra land needed to increase labour without altering the amount of either labour-using capital or land-using capital. It is, in fact, the marginal product of " one man and one-third of an acre." Similarly, 1.079 is the marginal product of " one unit of labour-using capital and 0.891 acres," whereas 2.667 is the marginal product of " one unit of land-using capital and one-sixth of an acre." Under competition, the wages paid for these three combinations of factors will in these stationary states be paid at rates equal to 13.333, 1.891 and 2.667.

The reason that the marginal productivities of some of these factors can be negative is that the adoption of such factors involves using less land, so that it is worth while employing them although it lowers product flow, in order to save rent. Thus the factor combination with negative productivity is always one involving a negative amount of land.

It can be verified that in our example the combined wage-rate of each factor combination is, in fact, equal to its marginal productivity as given by the appropriate regression coefficient in equations (3.2).

Oxford. D. G. CHAMPERNOWNE.

The Production Function and the Theory of Capital

INTRODUCTION

In her paper with the above title[1] Mrs. Robinson was annoyed at many of the practices of academic economists. We have reason to be grateful for her annoyance, for she seems to have written her article the way an oyster makes pearls—out of sheer irritation. Perhaps her main target is the custom of regarding output as a function of inputs of labor and " capital ". She tells us that the student—the English student, at least—is told that output is a kind of index-number, labor is a quantity of homogeneous man-hours, and then he is hustled away from the scene of the crime before he thinks to ask in what units " capital " is measured.

In the spirit of natural history I would like to record that the routine in at least some American classrooms is slightly different, if no more enlightening. If I write $Q = f(L,C)$, I simply assume that there exists only one kind of physically homogeneous capital good, and C, like L, is measured in unambiguous physical units. Of course, it's not true that only one kind of capital good exists, but then there's also more than one kind of labor, and anyhow this is neither the student's first nor last course in economic theory. All that matters is that it be made clear what part of the theory holds true in the general case.[2] I would go one step further. For many purposes it is remarkably useful to assume that there exists only one physical commodity which can either be consumed or used as capital in the production of more of itself. Then Q and C are measured in the *same* units except that Q is a flow and C is a stock. This simple but fruitful model of capital accumulation was one of the legacies to economics left by Frank Ramsey.[3] But this is by the way.

After having remarked on the degree of fakery involved in the notion of " capital " in general (for example : the composition of the capital stock changes as more is accumulated), Mrs. Robinson goes on to suggest a way of measuring " capital " by the cost of its component machines, buildings, etc., in labor units including interest accrued during the period of construction. This has a faintly archaic flavor. It doesn't seem to bother her much that on this definition two physically identical outfits of capital equipment can represent different amounts of " capital ". It wouldn't bother me either except that from the point of view of *production* two identical plants represent two identical plants.

It does bother Mr. Champernowne, and he recognizes the matter for what it is—a kind of index-number problem.[4] He gives a straightforward solution in this spirit. However, even Champernowne's chain-index can't dispel all the difficulties, as he points out.[5]

This leads me to ask : why does there have to be a useful concept of capital-in-general at all ? True, one feels that if God had meant there to be more than two factors of production, He would have made it easier for us to draw three-dimensional diagrams. But apart from this, as Mrs. Robinson remarks : " When an event has occurred we are thrown back upon the who's who of goods in existence, and the ' quantity of capital ' ceases to

[1] *Review of Economic Studies*, No. 55, p. 81.

[2] Sometimes it can be subtly suggested that the general case is really quite easy—this is known as Subscriptmanship. Sometimes it is even true.

[3] In his " A Mathematical Theory of Saving," *Economic Journal*, 1927.

[4] *Review of Economic Studies*, No. 55, p. 112, where some other paradoxes are mentioned.

[5] *Ibid.*, pp. 118, 121, 123.

have any other meaning."[1] Perhaps we should never have left the " who's who of goods in existence " in the first place. I do not contend that dispensing with the notion of the " quantity of capital " will make the theory of capital any easier. In fact it will make it harder. But the real difficulty of the subject comes not from the physical diversity of capital goods. It comes from the intertwining of past, present and future, from the fact that while there is something foolish about a theory of capital built on the assumption of perfect foresight, we have no equally precise and definite assumption to take its place. (It is clear from the context that I am not here concerned with capital as abstract purchasing power uncommitted to specific form).

In this paper I want to tackle the question from a slightly different angle : under what conditions can a consistent meaning be given to the quantity of capital ? Suppose we have a production function which relates the output of a single commodity to inputs of labor (assumed homogeneous) and the services of several kinds of capital goods. When if ever can the various capital inputs be summed up in a single index-figure, so that the production function can be " collapsed " to give output as a function of inputs of labor and " capital-in-general "? It will be seen that this is sometimes possible, but only in a very narrow class of cases.

CONDITIONS FOR COLLAPSING THE PRODUCTION FUNCTION

Formally, suppose we have a production function $Q = F(L, C_1, C_2)$ where Q is a single output, L an input of a single grade of labor, and C_1 and C_2 are inputs of the services of two distinct kinds of capital equipment (there could be more types of capital involved, but the argument would be the same). The question is : when can we write, identically :

$$(1) \qquad \begin{aligned} Q &= F(L, C_1, C_2) \equiv H(L, K) \\ K &\equiv \phi(C_1, C_2) \end{aligned}$$

That is to say, when can we collapse the production function from one having three variables to one having only two ? If this can be done, we would seem to have a right to call K an index of the quantity of capital. For the purposes of production any pattern of inputs C_1 and C_2 are equivalent so long as they yield the same value of the index K.

At first glance it may seem as if a wide class of production functions can be treated in this way. One consequence of equation (1) can be drawn immediately. Calculate the marginal rate of substitution of C_1 for C_2, i.e., the ratio of their marginal physical productivities. We find :

$$(2) \qquad \frac{MPP_1}{MPP_2} = \frac{\partial F/\partial C_1}{\partial F/\partial C_2} \equiv \frac{\frac{\partial H}{\partial K} \cdot \frac{\partial \phi}{\partial C_1}}{\frac{\partial H}{\partial K} \cdot \frac{\partial \phi}{\partial C_2}} \equiv \frac{\partial \phi/\partial C_1,}{\partial \phi/\partial C_2} \text{ which is to be independent of } L;$$

for the last ratio depends only on C_1 and C_2, and hence it and the first ratio must be *independent of L*. We have in (2) a *necessary* condition for the collapsibility of the production function (1) : the marginal rate of substitution of one kind of capital good for another must be independent of the amount of labor in use.

Here is one implication of the existence of an index of the quantity of capital. Are there any others ? At this point we can appeal to a neat theorem of Leontief's[2] which,

[1] The next sentence reads : " Then only that part of the theory of value which treats of the short period, in which the physical stock of capital equipment is given, has any application." This I doubt, but there is no time to go into it now.

[2] *Econometrica*, Vol. 15, No. 4, 1947, p. 364, Proposition I. Leontief's results would enable us to handle the case where there are also several grades of labor and the problem is simultaneously to find indices of the quantity of labor and the quantity of capital. But there seems to be no point in complicating the exposition.

applied to this situation, asserts that the answer is No. The necessary condition just stated is also *sufficient*. The invariance of the intra-capital substitution possibilities against changes in the labor input is equivalent to the possibility of finding an index of the quantity of capital.

There are two things to be noted about this condition. It is natural, and it is stringent. If it is to be possible to reduce the two (or more) capital dimensions to one, it must be true that what happens in those dimensions does not depend on where we are along other axes, such as labor. If a little C_1 could replace a lot of C_2 when we use a little labor and vice versa when we use a lot of labor, then clearly no single " average " of the amount of C_1 and C_2 would contain all the information we need. There would then be no possibility of defining *universally* equivalent bundles of C_1 and C_2.

Secondly, condition (2) will not often be even approximately satisfied in the real world. The examples which come to mind where it will hold often turn out to be cases where the types of capital equipment are homogeneous in all but name : brick buildings and wooden buildings, aluminium fixtures and steel fixtures. But note that it is not satisfied for one-ton trucks and two-ton trucks ; technical substitution possibilities will depend on the number of drivers available. And there is no special reason at all why the condition should hold for totally different species of equipment like bulldozers and trucks. In such cases no quantity of capital-in-general can be consistently defined.

There is, however, a whole class of situations in which the condition may be expected to hold and this possibility throws a new light on the meaning of the condition itself. It could be that the process of production described by F should have two stages such that *first* something called K is literally manufactured out of C_1 and C_2 alone, and *then* this substance K is combined with labor to manufacture the final output Q. In this case the index function ϕ is actually a production function itself. Obviously the inputs of C_1 and C_2 play no special role themselves ; only their yield of K matters ultimately. For example, we can imagine C_1 and C_2 to be two kinds of equipment for generating electricity which is then used in further production. Even though electric power is not itself a stock, generating capacity would be an index of the capital inputs.

In this interpretation it is useful to know whether the index-function ϕ and the collapsed function H have the characteristics we usually associate with production functions. In the next section it will be shown that they do.

PROPERTIES OF THE INDEX FUNCTIONS

Theorem : Suppose that the underlying production function F exhibits constant returns to scale with respect to L, C_1 and C_2, and obeys the generalized law of diminishing returns to variable proportions, i.e., has properly convex equal-output surfaces. Then exactly the same properties will characterize the index-function ϕ and the collapsed production function H.

This means that in (1) it is always possible to regard the index of the quantity of capital as the " output " of a production process which uses capital goods to produce capital-in-general. Moreover the final output will be a well-behaved function of other inputs and the input of capital-in-general.

To begin with we can differentiate (1) to yield :

$$(3) \qquad \frac{\partial F}{\partial L} = \frac{\partial H}{\partial L}\,; \quad \frac{\partial F}{\partial C_1} = \frac{\partial H}{\partial K}\frac{\partial \phi}{\partial C_1}\,; \quad \frac{\partial F}{\partial C_2} = \frac{\partial H}{\partial K}\frac{\partial \phi}{\partial C_2}$$

Since the marginal productivities calculated from F are positive (or at least not negative), we deduce that the marginal productivities calculated from ϕ have the same sign as the marginal productivity of K in the collapsed function H. These can all be chosen to be positive.

Next I want to show that ϕ can be taken to be homogeneous of first degree. From (2) the identity can be extracted :

$$\frac{\partial F/\partial C_1}{\partial F/\partial C_2} = \frac{\partial \phi/\partial C_1}{\partial \phi/\partial C_2}$$

The right-hand side is, except for sign, the slope of equal-K contours of ϕ. The left-hand side, a ratio of marginal productivities of a constant-returns-to-scale function, depends only on ratios of all inputs, say C_2/C_1 and L/C_1. But this ratio is independent of L, and hence depends *only* on C_2/C_1. This means that the isoquants of ϕ all have the same slope along any ray from the origin. The numbering of such " homothetic " isoquants can then always be chosen so as to make ϕ homogeneous of degree one.

Once we know F and ϕ to be homogeneous of degree one it is an easy matter to prove the same of H :

(4) $$tH(L, K) = tF(L, C_1, C_2) = F(tL, tC_1, tC_2) = H(tL, tK)$$

It remains only to show that ϕ and H have correctly-shaped isoquants. They do, but I relegate the proof to a footnote.[1]

A CORRESPONDING PRICE-INDEX

Imagine that the production function is such that a consistent index of the quantity of capital (services) can be defined. It is natural to wonder whether one can speak of the " price " of capital-in-general. This price index should have the following properties. First, it should depend only on the prices of the various capital goods themselves. Second, under constant returns to scale one would expect the price index of capital-in-general to equal the " cost of production " of the capital index :

(5) $$p_K K = p_1 C_1 + p_2 C_2$$

where C_1, C_2 and K are related by the index-production function ϕ. Finally, one can think of the cost-minimization process as broken down into stages. In the first stage the prices p_L and p_K are quoted and cost-minimization, subject to the production constraint H, leads to a preferred capital-labor ratio K/L. In the second stage the prices p_1 and p_2 are quoted and cost-minimization, subject to the production constraint ϕ, leads to a preferred ratio C_1/C_2. One would like this two-stage process to lead to the same result as straightforward cost minimization, given prices p_L, p_1, p_2, and subject to the production constraint F. What this amounts to is that one might expect the price-index of K to be such that it makes no difference whether or not the " production " of K is vertically integrated with the production of Q.

[1] Suppose $K = \phi(C_1, C_2) = \phi(C_1', C_2') = K'$. Then since F has convex isoquants $H(L, K) + H(L, K') = 2H(L,K) = F(L, C_1, C_2) + F(L, C_1', C_2') \leqq F(2L, C_1 + C_1', C_2 + C_2') = 2F(L, (C_1 + C_1')/2, (C_2 + C_2')/2) = 2H(L,\phi(C_1 + C_1')/2,(C_2 + C_2')/2))$. Hence $\phi\left(\frac{C_1 + C_1'}{2}, \frac{C_2 + C_2'}{2}\right) \geqq \phi(C_1, C_2) = \phi(C_1', C_2')$, q.e.d.

My original proof of the convexity of the $H(L, K)$—isoquants was heavily infected with second derivatives. My colleague Paul Samuelson has shown me the following direct proof. We have to show that if $H(L, K) = H(L', K')$, then $H(L, K) \leqq H(L + L')/2, (K + K')/2)$. If $K = \phi(C_1, C_2)$, choose $C_i' = mC_i$, so that $K' = mK = \phi(C_1', C_2') = m\phi(C_1, C_2)$. Then, using the convexity of F, $H(L, K) = H(L', mK) = F(L, C_1, C_2) = F(L', mC_1, mC_2) \leqq F\left(\frac{L+L'}{2}, \frac{1+m}{2}C_1, \frac{1+m}{2}C_2\right) = H\left(\frac{L+L'}{2}, \frac{1+m}{2}K\right) = H\left(\frac{L+L'}{2}, \frac{K+K'}{2}\right)$

All this can easily be accomplished. There are several ways of getting at the desired price-index. One possibility is to use (5) directly for $K = 1$ after expressing C_1 and C_2 as functions of p_1 and p_2. Another way is as follows. The minimum-cost conditions for F and for H can be written out :

$$(6) \qquad \frac{1}{p_L}\frac{\partial F}{\partial L} = \frac{1}{p_L}\frac{\partial H}{\partial L} = \frac{1}{p_1}\frac{\partial F}{\partial C_1} \equiv \frac{1}{p_1}\frac{\partial H}{\partial K}\frac{\partial \phi}{\partial C_1} = \frac{1}{p_2}\frac{\partial F}{\partial C_2} \equiv \frac{1}{p_2}\frac{\partial H}{\partial K}\frac{\partial \phi}{\partial C_2} = \frac{1}{p_K}\frac{\partial H}{\partial K}.$$

From this it is apparent that we must have

$$(7) \qquad p_K = p_1 \div \frac{\partial \phi}{\partial C_1} = p_2 \div \frac{\partial \phi}{\partial C_2}$$

Both $\partial\phi/\partial C_1$ and $\partial\phi/\partial C_2$ are monotonic functions of C_2/C_1 alone ; hence we can eliminate C_2/C_1 between these two equations and what is left is p_K as a function of p_1 and p_2 alone. It is easily verified, from Euler's Theorem, that the condition (5) will hold. And the way (7) was derived from (6) tells us that two-stage and straightforward cost-minimization must lead to identical results. A competitive entrepreneur " buying " K at the price p_K would in effect make all the same decisions as one buying C_1 and C_2 at prices p_1 and p_2.

This result, together with the earlier-proved convexity of $H(L, K)$, justifies the statement : under the strong assumptions required, there is a perfectly definite and consistent sense in which it can be said that the relative factor-price ratio p_K/p_L is a decreasing function of the ratio of capital to labor K/L.

EXAMPLES

Take first the Cobb-Douglas production function :

$$Q = L^u C_1^v C_2^w \qquad u + v + w = 1$$

It is obviously collapsible since we can rewrite it :

$$Q = L^u K^{v+w}$$

$$K = C_1^{\frac{v}{v+w}} C_2^{\frac{w}{v+w}}$$

It is easily verifiable that the marginal rate of substitution of C_1 for C_2 is (v/w) (C_2/C_1) which is indeed independent of L. Equally clearly, Q is a constant-returns-to-scale function of L and K, and similarly for K as a function of C_1 and C_2. Both H and ϕ are Cobb-Douglas functions themselves and so their isoquants have the right convexity. K, the index of the quantity of capital, is a kind of weighted geometric mean of the two capital inputs, but this is because the Cobb-Douglas function has just that kind of structure.

As for the price-index, equations (7) yield in this case, after some manipulation :

$$p_K = (v + w)\left(\frac{p_1}{v}\right)^{\frac{v}{v+w}}\left(\frac{p_2}{w}\right)^{\frac{w}{v+w}}$$

which has all the required properties.

For a second example consider the production function :[1]

$$Q = (\sqrt{L} + a\sqrt{C_1} + b\sqrt{C_2})^2$$

[1] Similar remarks apply to production functions of the general class of " means " $Q = f^{-1}\,[f(L) + f(C_1) + f(C_2)]$, further restricted to be homogeneous of first degree.

It can then be collapsed into :

$$Q = (\sqrt{L} + \sqrt{K})^2$$

$$K = (a\sqrt{C_1} + b\sqrt{C_2})^2$$

The marginal rate of substitution of C_1 for C_2 is $(a/b)\ \sqrt{C_2/C_1}$, which does not involve L. Again H and K have all the desired properties of homogeneity and convexity. The price index calculation leads to :

$$p_K = \frac{p_1 p_2}{a^2 p_2 + b^2 p_1} = \frac{1}{\frac{a^2}{p_1} + \frac{b_2}{p_2}}$$

a kind of weighted harmonic mean of the individual capital-goods prices.

This is perhaps the place to mention a curious duality relation which seems to have no apparent economic interpretation. Notice that in these two examples, and in fact in general, the price-index p_K comes out as a homogeneous function of degree one of p_1 and p_2. (This, by the way, is natural : double the component prices and you double the index.) Moreover it has all the same convexity properties as a production function. Now suppose we replace p_K by K, p_1 by C_1, and p_2 by C_2. Then the price index is transformed into a quantity-of-capital-index like the ϕ we have been talking about. Now find the price-index that corresponds to this new ϕ-index. It turns out to be the old ϕ-index ! So price and quantity indices come in pairs : if the function A as a quantity index has the function B for a price index, then B as a quantity index has A as a price index. In my second example, if we find the price index that corresponds to :

$$K = \frac{C_1 C_2}{a^2 C_2 + b^2 C_1}$$

it turns out to be :

$$p_K = (a\sqrt{p_1} + b\sqrt{p_2})^2$$

Apart from the comforting thought that at this late date no one should be surprised to find price-quantity dualities, I have no explanation to offer.

A LINEAR PROGRAMMING MODEL

Both Mrs. Robinson and Mr. Champernowne carry on their discussion in terms of discrete " activities " or processes. Everyone who invents linear programming these days seems to be charmed by it. I have used old fashioned production functions simply because the problem seemed more manageable in those terms. I have not proved similar theorems for the discrete case, but I have little doubt they are true. In any case, I conclude by showing how the parallel problem can be formulated.

Consider activities A, B, C, D, . . ., such that activity A produces a unit of output with inputs a_0 of labor, a_1 of C_1, and a_2 of C_2. Activities B, C, D, . . . are similar. If x_a units of output are produced with activity A, x_b with B, etc., then total output Q will be $x_a + x_b + x_c + x_d, \ldots$, and the total inputs of L, C_1, and C_2 respectively will be :

$$a_0 x_a + b_0 x_b + c_0 x_c + d_0 x_d \ \ldots \ = L$$

$$a_1 x_a + b_1 x_b + c_1 x_c + d_1 x_d \ \ldots \ = C_1$$

$$a_2 x_a + b_2 x_b + c_2 x_c + d_2 x_d \ \ldots \ = C_2$$

Now look at it the other way round. Given total inputs L, C_1, and C_2 how should output be allocated among the activities to yield the maximum output ? This is a garden variety linear programming problem, namely :

$$\text{Maximize } x_a + x_b + x_c + x_d + \dots$$

subject to the constraints :

$$\begin{aligned} a_0x_a + b_0x_b + c_0x_c + d_0x_d + \dots &\leqq L \\ a_1x_a + b_1x_b + c_1x_c + d_1x_d + \dots &\leqq C_1 \\ a_2x_a + b_2x_b + c_2x_c + d_2x_d + \dots &\leqq C_2 \end{aligned} \tag{8}$$

Methods of solving problems like this are now well-known. By whatever method of solution, we must finally wind up with a maximal total output Q. Varying the total inputs we repeat the process and in this way trace out a production function :

$$Q = G(L, C_1, C_2)$$

This production function is not very different from the smooth neo-classical type. It will exhibit constant returns to scale and the usual non-increasing returns as proportions vary. The main difference is that its equal-output surfaces will consist of planar pieces, joined at edges and vertices. This possibility of corners means that marginal rates of substitution are not always well-defined.

One can still ask whether this production function can be collapsed into one involving only two independent variables. The answer is, in general, certainly not. There are some obvious sufficient conditions. For instance, if the only efficient activities are some which use up only C_1 and C_2, and others which use up only L, then trivially the production function can be decomposed. The constraints (8) will then appear in partitioned-matrix form as :

$$\begin{pmatrix} A & O \\ O & C \end{pmatrix} \begin{pmatrix} X_1 \\ -- \\ X_{11} \end{pmatrix} \leqq \begin{pmatrix} L \\ -- \\ C_1 \\ C_2 \end{pmatrix}$$

where A has one row, C has two, and each has as many columns as there are activities in its group. The output of the second group of activities will serve as an index of the inputs C_1 and C_2.

Rather more generally, suppose the constraints can be written :

$$\begin{pmatrix} A & O \\ O & C \\ B & -D \end{pmatrix} \begin{pmatrix} X_1 \\ -- \\ X_{11} \end{pmatrix} \leqq \begin{pmatrix} L \\ -- \\ C_1 \\ C_2 \\ -- \\ O \end{pmatrix} \tag{9}$$

where B and D have only a single row. Now the activities of the second group do not produce final output directly at all. Instead they use up C_1 and C_2 to " produce " a fictitious output—previously called K—which is fed into the first group of activities, along with L, to produce the final output Q. In this set-up Q is equal to the sum of the activity levels of first-group activities only. In this set up it is also possible to summarize C_1 and C_2 in a single number, namely the output of the fictitious intermediate K. Linear programming problem (9) breaks down into two problems.

First, maximize $K = DX_{11}$ subject to

$$CX_{11} \leqq \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Then, maximize $Q = \Sigma X_1$ subject to :

$$\begin{pmatrix} A \\ B \end{pmatrix} (x_1) \leqq \begin{pmatrix} L \\ K \end{pmatrix}$$

The set-up (9) may very likely also be *necessary* for the collapsibility of the production function, in which case it plays the role of Leontief's theorem. In the theory of linear programming the concept of the " dual variable " plays the part of a marginal productivity. A marginal rate of substitution or price ratio would correspond to a ratio of dual variables. The dual problem to (9) is to minimize $Lu_0 + C_1u_1 + C_2u_2 + 0 \cdot u_3$ subject to :

$$\begin{pmatrix} A' & O' & B' \\ O' & C' & -D' \end{pmatrix} \begin{pmatrix} u_0 \\ \hline u_1 \\ u_2 \\ \hline u_3 \end{pmatrix} \geqq \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \hline 0 \\ 0 \\ \vdots \end{pmatrix}$$

The second group of constraints require that u_1 and u_2 bear a fixed ratio to u_3 and hence to each other. Thus the marginal rate of substitution between C_1 and C_2 is determined independently of u_0 and hence of L. Thus the possibility of rewriting the production conditions in the very special form (9) would appear to be equivalent to the possibility of defining a single index of capital inputs.

I conclude that discreteness is unlikely to help matters. Only in very special cases will it be possible to define a consistent measure of capital-in-general. Some comfort may be gleaned from the reflection that when capital-labor ratios differ widely we hardly need a subtle index to tell us so, and when differences are slight we are unlikely to believe what any particular index says.

Cambridge, Mass. ROBERT M. SOLOW.

The Production Function and the Theory of Capital—A Reply[1]

As far as I can understand Professor Solow's note, it does not touch upon the problem of capital, but is concerned rather with how to treat non-homogeneous natural resources, for instance, varieties of Marshall's meteoric stones.[2] His C_1 and C_2 are two kinds of equipment, but nothing is said about the time which it takes to produce them (gestation period) or the period over which they are expected to be useful (service life). It is no argument to dismiss a problem as antiquated until it has either been satisfactorily solved or shown to be uninteresting.

In my opinion Professor Solow is barking up the wrong tree in seeking either the problem or the solution amongst index numbers. Let us postulate a world in which there is only one kind of machine, and let us abstract from working capital. Then the problem with which Professor Solow is concerned evaporates. Now compare two machines. If one is older than the other and has a shorter prospective remaining service life, its value is less, and the difference in value varies with the rate of interest.

Next, compare two machines exactly alike in all respects (including age) except that one is drawn from an economy with a higher product-wage rate. The value of the two machines is different, and the investment required to create them is different. A difference in value remains if we deflate them by the wage rate (obtaining what I have called " real capital " or capital in terms of labour time), for in two economies with different product-wage rates the rate of profit and therefore the rate of interest are different (under the assumptions on which the argument is set out). I am sorry to have given the impression that " it does not seem to bother " me that two exactly similar pieces of equipment may represent different quantities of accumulation. I was trying precisely to show what there is to be bothered about.

Again, if the wage rate has altered during the life time of a machine, its historic cost and its reproduction cost differ.

Finally, two machines, though physically alike may require different quantities of labour to produce or to operate them (because of technical progress).

None of these questions can be dealt with in terms of an index of physical equipment, and each of them is important for the analysis of distribution and of capital accumulation.

Cambridge. JOAN ROBINSON.

[1] *Review of Economic Studies*, Vol. XXIII (2), No. 61.

[2] *Principles* (7th ed.) p. 415.