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"Laws of Production and Laws of Algebra: The Humbug Production Function"

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I. Introduction

Aggregate neoclassical theory has most often been formulated by way of analogy with the corresponding micro-economic theory, the justification being that general equilibrium models are empirically useless unless simplified greatly. Recent capital controversies, however, have shed much light on the conditions requisite for "surrogate" production functions, these conditions being tantamount to assuming that at any instant of time the *simple labor theory of value* prevails in the economy.[1] The irony of this is inescapable; furthermore, these results imply that most economists must somehow steer a treacherous path between repudiating the assumption of simple labor value pricing while retaining its conclusion,[2] since the use of such functions in both theoretical and empirical analysis is widespread. The explanation for this popularity, it appears, is the fact that the empirical basis for aggregate production functions appears strong. Not just any functions either, for in both time series and cross-section studies, the Cobb-Douglas function seems to stand out above all others; "the sum of coefficients usually approximate closely to unity," and there is a striking "agreement between the labor exponent and the share of wages in the value of output."[3] It would seem therefore that empirical results strongly support both the constant returns to scale aggregate production functions and the aggregate marginal productivity theory of distribution, almost in spite of their theoretical deficiencies.

In a recent paper, Franklin Fisher concedes that the requirements "under which the production possi-

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[1] The original attempt to provide a theoretical justification for a "surrogate" production function is in Samuelson (1962). The strict conditions necessary for this aggregate behavior are derived in an excellent article by Garegnani (1970).

[2] Joan Robinson (1971) has repeatedly pointed out that an even more serious criticism of "surrogate" production function behavior is that *at best* it represents *alternative* positions of equilibrium. Movements along such a curve are only comparisons of the possible equilibria allowed by a given state of technology, not movements which would actually take place. To quote her, "Time, so to say, is at right angles to the blackboard on which the curve is drawn." (p. 255.)

[3] A. A. Walters (1963), p. 27.

bilities of a technically diverse economy can be represented by an aggregate production function are far too stringent to be believable."[4] He proposes therefore to investigate the puzzling uniformity of the empirical results by means of a simulation experiment: each of N industries in this simulated economy is assumed to be characterized by a microeconomic Cobb-Douglas production function relating its homogeneous output to its homogeneous labor input and its own *distinct* machine stock. The conditions for theoretical aggregation are studiously violated, and the question is, how well, and under what circumstances, does an aggregate Cobb-Douglas function represent the data generated? In such an economy, the aggregate wage share is often variable over time, so that in general an aggregate Cobb-Douglas would not be expected to give a good fit. What seems to surprise Fisher, however, is that when the wage share happens coincidentally to be roughly constant, a Cobb-Douglas production function will not only fit the data well but *also* provide a good explanation of wages, *"even though the true relationships are far from yielding an aggregate Cobb-Douglas," suggesting that "the view that the constancy of labor's share is due to the presence of an aggregate Cobb-Douglas production function is mistaken. Causation runs the other way and the apparent success of aggregate Cobb-Douglas production functions is due to the relative constancy of labor's share."* (emphasis added).[5]

It is obvious that so long as aggregate shares are roughly constant, the appropriate econometric test of aggregate neoclassical production and distribution theory requires a Cobb-Douglas function. Such a test would then apparently cast some light on the degree of returns to scale (through the sum of the coefficients), and the applicability of aggregate marginal productivity theory (through the comparison of the labor and capital exponents with the wage and profit shares, respectively). What is not obvious, however, is that so long as aggregate shares are constant, an aggregate Cobb-Douglas function having apparently "constant returns to scale" will always provide an exact fit, for any data whatsoever. *In addition, under fairly reasonable conditions, such a function will seem also to possess "marginal products equal to respective factor rewards," thus seeming to justify neoclassical aggregate distribution theory.* These propositions, it will be shown, are *mathematical* consequences of constant shares, and it will be argued that the puzzling uniformity of the empirical results is due in fact to this law of algebra and not to some mysterious law of production. In fact, in order to emphasize the independence of this result from any laws of production, an illustration is provided in the form of the rather implausible data of the "Humbug" economy, for even these data are perfectly consistent with a Cobb-Douglas function having "constant returns to scale," "neutral technical change," and satisfying "marginal productivity rules," so long as shares are constant.

[4] F. Fisher (1971), p. 306.

[5] Fisher (1971), p. 306.

II. Laws of Algebra

Let us begin by separating the aggregate data in any time period into output data (Q, the value of output), distribution data (W, π, wages and profits, respectively), and input data (K, L, the index numbers for capital and labor, respectively). Then we can write the following aggregate identity for any time t:

$$Q(t) \equiv W(t) + \pi(t) \tag{1}$$

Given *any* index numbers $K(t)$, $L(t)$, we can always write:

$$q(t) \equiv w(t) + r(t)\, k(t) \tag{2}$$

where $q(t)$ and $k(t)$ are the output-labor and capital-labor ratios, respectively, and $w(t) \equiv W(t)/L(t)$, $r(t) \equiv \pi(t)/K(t)$ are the wage and profit rates, respectively. The above equation is therefore the fundamental identity relating output, distribution, and input data. Defining the share of profits in output as s, and the share of wages as $1 - s$, we can differentiate identity 2 to arrive at identity 3 (time derivatives are denoted by dots, and the time index, t, is dropped to simplify notation):

$$\dot{q} = \dot{w} + \dot{r}k + r\dot{k} \equiv w\left(\frac{\dot{w}}{w}\right) + rk\left(\frac{\dot{r}}{r}\right) + rk\left(\frac{\dot{k}}{k}\right)$$

$$\frac{\dot{q}}{q} \equiv \frac{w}{q}\left(\frac{\dot{w}}{w}\right) + \frac{rk}{q}\left(\frac{\dot{r}}{r}\right) + \frac{rk}{q}\left(\frac{\dot{k}}{k}\right)$$

noting that

$$s \equiv \frac{rk}{q}, \quad 1 - s \equiv \frac{w}{q},$$

we can write

$$\frac{\dot{q}}{q} \equiv \frac{\dot{B}}{B} + s\frac{\dot{k}}{k} \tag{3}$$

where

$$\frac{\dot{B}}{B} \equiv [(1-s)\,\dot{w}/w + s\,\dot{r}/r].$$

It is important to note that all relationships so far are *always* true for *any* aggregate data at all. Suppose now we are faced with data which exhibit constant aggregate shares, so that $s = \beta$. Then we can immediately integrate the identity (3) to get[6]

$$q = B(c_0\, k^\beta),\ \text{where}\ B \equiv e^{\int(\dot{B}/B)dt},\ c_0 \equiv \text{constant of integration.} \qquad (4)$$

Equation (4) is strikingly reminiscent of a constant returns to scale aggregate Cobb-Douglas production function with a shift parameter B. But in fact, it is not a *production* function at all, but merely an algebraic relationship which always holds for *any* output-input data Q, K, L, even data which could not conceivably come from any economy, so long as the distribution data exhibits a constant ratio. Furthermore, since the $\dot{B}/B$ term in identity (3) is a weighted average of the *rates of change* of w and r, respectively, it seems empirically reasonable to expect many measures of K, L would give a capital-labor ratio k which is weakly correlated with $\dot{B}/B$. With measures for which the above is true, *$\dot{B}/B$ may be considered to be primarily a function of time,* so that B will also be solely a function of time. Then we can write

$$q = B(t)\,[c_0\, k^\beta] \qquad (5)$$

$$Q = B(t)\,[c_0\, K^\beta L^{1-\beta}]. \qquad (5')$$

The above algebraic relationship has several interesting properties. First, it is homogeneous to the first degree in K and L. Second, since $\beta = s \equiv rk/q$, the partial derivatives $\partial Q/\partial K$, $\partial Q/\partial L$ are equal to r, w, respectively. And third, the effect of time is "neutral," as incorporated in the shift parameter $B(t)$. What we have, actually, is *mathematically* identical to a constant returns to scale Cobb-Douglas production function having neutral technical change and satisfying marginal productivity "rules." And yet, as we have seen, *any production data whatsoever can be presented as being "generated" by such a function,* so long as shares are constant and the measures of capital and labor such that k is uncorrelated with $\dot{B}/B$. Therefore, precisely because (5′) *is* a mathematical relationship, holding true for large classes of data associated with constant shares, it cannot be interpreted as a production function, or any production relation at all. If anything, it is a *distributive* relation, and sheds little or no light on the underlying production relationships.[7] In fact, since the constancy of shares has been taken as an empirical datum throughout, equation (5′) does not shed much light on any theory of distribution either.

I emphasized earlier that the theoretical basis of aggregate production function analysis was extremely weak. It would seem now that its apparent empirical strength is no strength at all, but merely a statistical reflection of an algebraic relationship.

III. Applications

It is obvious that one can apply equation (5′) in many ways. Section A) below will re-examine Solow's famous paper on measuring technical change. Section B) will present a numerical example to illustrate the generality of equation (5′) and section C) will extend the preceding analysis to cross-section studies.

A) *Technical Change and the Aggregate Production Function: Solow (1957)*

In what is considered a "seminal paper,"[8] Robert Solow introduced in 1957 a novel method for measuring the contribution of technical change to economic growth. Since that time several refinements of Solow's original calculations have been established, all aimed at providing better measures of labor and capital by taking account of education, vintages of machines, etc., but the basic approach has remained unchanged.[9]

Solow's model is a familiar one. Equation (6) represents the assumed constant returns to scale aggregate production function, with neutral technical change represented by the shift parameter $A(t)$, while equation (7) states that "factors are paid their marginal products" (rewritten below in terms of the share of profits in output).[10]

$$q = A(t)\, f(k) \qquad (6)$$

$$\frac{df}{dk}\frac{k}{f} = s \equiv \text{share of profits in output.} \qquad (7)$$

Solow's main purpose was to isolate what I will call the "underlying" production function $f(k)$ by distinguishing between shifts of the production function (due to "technical" change) and movements along it (due to changes in the capital-labor ratio k). To do this, he differentiates (6), and using (7) to substitute s for $(\dot{f}/f)/(\dot{k}/k)$, he arrives at

[6] In $q = \int \dot{B}/B\ dt + \beta \ln k + c_0$, which gives us $q = [e^{\int \dot{B}/B\,dt}][c_0\, k^\beta]$.

[7] I thank Professor Luigi Pasinetti for having pointed this out in his comments on an earlier version of this paper.

[8] Solow (1957).

[9] For a summary of subsequent refinements, see Nelson (1964).

[10] Note that $f(k) = q/A(t)$. The marginal product of capital is $dq/dk = A(t)[df/dk] = [q/f][df/dk]$. Setting this equal to r and rearranging gives (7), since $s \equiv (rk)/q$.

$$\frac{\dot{q}}{q} = \frac{\dot{A}}{A} + s\frac{\dot{k}}{k}. \quad (9)$$

Equation (8) is derived from the *assumptions* of a constant returns to scale aggregate production function, with distribution determined by marginal productivity rules. Equation (3), derived earlier from an *identity* and therefore always true for any production and distribution behavior, is mathematically identical to (8) above. It follows therefore that $\dot{A}/A = \dot{B}/B \equiv [(1-s)\dot{w}/w + (s)\dot{r}/r]$; that is, Solow's measure of technical change is merely a weighted average of the growth rates of the wage w and rate of profit r.

Solow's data provide him with a series for q, k, and s for the United States from 1909–1949. From this he calculates $\dot{q}/q$ and $\dot{k}/k$, and uses them in (8) to derive a series for $\dot{A}/A$; since $\dot{A}/A$ appears uncorrelated with k, he concludes that technical change is essentially neutral, and by setting $A(0) \equiv 1$, he is able to arrive at a series for $A(t)$, the neutral technical change shift parameter. Finally, since (6) relates q, $A(t)$, and the hitherto unspecified "underlying" production function $f(k)$, he can use the series for $A(t)$ to derive one for $f(k)$. Plotting $f(k)$ versus k, he gets a diagram with a noticeable curvature, and concludes that the data "give a distinct impression of diminishing returns."[11] In fact, Solow finds that this "underlying" production function is extremely well represented (with a correlation coefficient $R = .9996$) by a Cobb-Douglas function:

$$\ln \hat{f}(k) = -0.729 + .353 \ln k. \quad (9)$$

Given our preceding analysis in section II, it is not difficult to see why Solow's results turn out so nicely. We know for instance that his data exhibit roughly constant shares, and that his measures for K, L yield a residual $\dot{A}/A$ uncorrelated with k. From purely algebraic considerations, therefore, one would expect the data to be well-represented by the functional form in (3), a form which is mathematically identical to a constant returns to scale Cobb-Douglas function, with neutral technical change and "marginal products equal to factor rewards." In fact, the algebra indicates that Solow's "underlying" production function should be of the form

$$f(k) = c_0 k^{\beta} \quad (10)$$

β is of course the (roughly) constant share, and c_0 is a constant of integration which depends only on the initial points q_0, k_0 of the data.[12] Thus, on purely algebraic considerations one would expect the "underlying" production function to be characterized by:

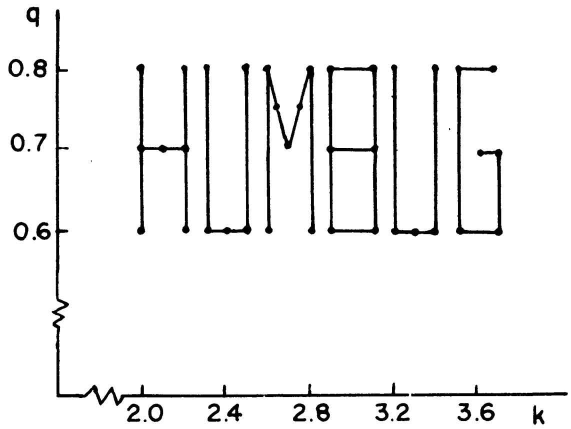
$$\ln f(k) = -0.725 + .35 \ln (k). \quad (11)$$

This, of course, is virtually identical to Solow's regression (equation (11)) as it should be, for it is a law of algebra, not a law of production!

B) *The Humbug Production Function*

It is possible to illustrate the generality of the preceding analysis by means of a numerical example. Consider, for example, an economy with the output-input data illustrated in figure 1 below, and

FIGURE 1. — THE HUMBUG ECONOMY



the same profit share s as the U. S. (Solow's data, (1957)). Using this data, for q, k, and s, one can calculate $\dot{q}/q$ and $\dot{k}/k$, and use them to calculate $\dot{A}/A$. Plotting $\dot{A}/A$ on k gives a scatter diagram with no apparent correlation (left out here for reasons of space; calculations appear in the appendix). Following Solow, one can set $A(0) \equiv 1$, and thus arrive at a series for $A(t)$, which is illustrated in figure 2. Finally, one can use the $A(t)$ to derive the "underlying" production function $f(k) = q/A(t)$, which when plotted versus k in figure 3 also gives a distinct impression of "diminishing returns." In fact, a regression of $f(k)$ on k gives $\ln \hat{f}(k) = -0.453 + .34 \ln (k)$, $R = 0.9964$. Since the profit share for the years involved is roughly constant around 0.34, one has the remarkable conclusion that even the Humbug data can be extremely well represented by a Cobb-Douglas production function having constant returns to scale, neutral technical progress, and marginal products equal to factor rewards.

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[11] Solow, ibid, p. 320.

[12] From (3), $q = B(t)[c_0 k^{\beta}]$. Solow identifies $B(t) = A(t)$, and since $A_0 = 1$, $q_0 = c_0 k_0^{\beta}$. Solow uses the years 1909–1942 in his regressions, and for these years $s \cong \beta = .35$. Also from table 1, p. 315, $q_0 = .623$, $k_0 = 2.06$, which gives $\ln c_0 \cong -0.725$.

FIGURE 2. — TECHNICAL "REGRESS"

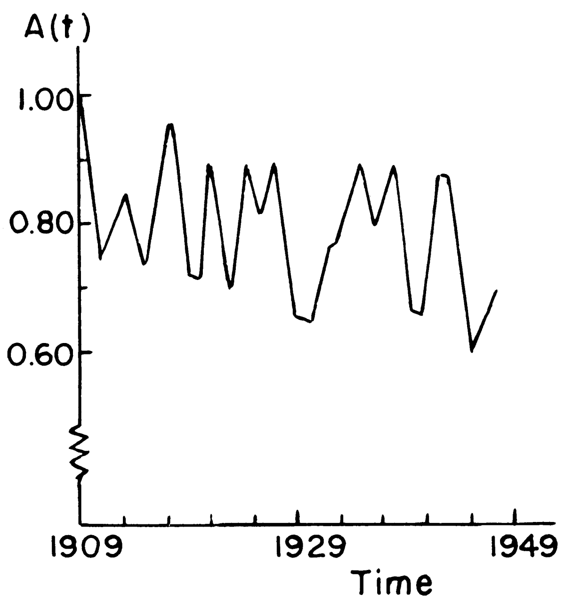
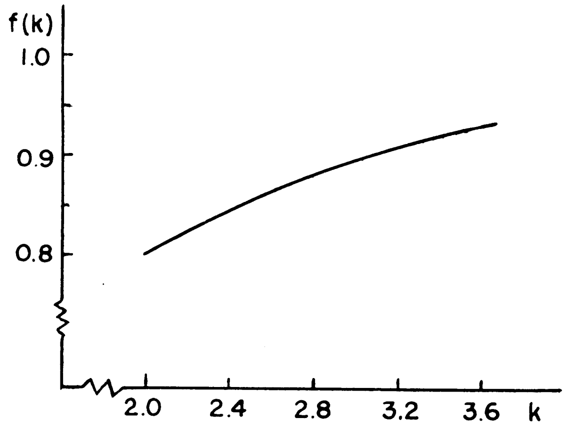


FIGURE 3. — UNDERLYING HUMBUG PRODUCTION FUNCTION



C) *Cross-section Aggregate Production Functions*

The direct analogy to constant shares in time series is the case of uniform profit margins (profits per dollar sales) in cross-section data. Using the subscript i for the i^{th} industry (or firm), and defining $\beta = s_i = r_i k_i / q_i$ as the uniform profit margin, we can rewrite equation (3) as

$$\frac{dq_i}{q_i} = \left[(1-\beta)\frac{dw_i}{w_i} + \beta\frac{dr_i}{r_i} \right] + \beta\frac{dk_i}{k_i}. \tag{12}$$

Then, so long as the term in brackets is uncorrelated with dk_i/k_i, the above equation is algebraically similar to a simple linear regression model $y_i = bx_i + u_i$, with the term in brackets playing the part of the disturbance term u_i. Obviously, for any data in which the bracketed term is small and uncorrelated with the dependent variable dk_i/k_i, the "best" fit will be a cross-section Cobb-Douglas production function with constant returns and factors paid their marginal products.

There are still other ways in which one may explain the apparent success of a Cobb-Douglas in cross-section studies, the best single reference being Phelps Brown's (1957) critique. In a subsequent note, Simon and Levy (1963) show that any data having uniform wage and profit rates across the cross section can be closely approximated by the ubiquitous Cobb-Douglas function having "correct" coefficients, even though the data reflect only mobility of labor and capital, not any specific production conditions. Once again, it would seem that the apparent empirical success of the Cobb-Douglas function having "correct" coefficients is perfectly consistent with wide varieties of data, and cannot be interpreted as supporting aggregate neoclassical production and distribution theory.

IV. Summary and Conclusions

It has been conceded in recent literature that the theoretical basis of aggregate production functions is, at best, weak. At the same time, the empirical basis has generally been presented as being strong. In particular, the striking dominance of the constant returns to scale Cobb-Douglas function, with estimated marginal products in close agreement with "factor rewards," has often been taken as providing a good measure of support for production function analysis, in spite of the theory. The main purpose of this paper has been to show that the empirical results do not in fact have much to do with production conditions at all. Instead, it is demonstrated that when the distribution data (wages and profits) exhibit constant shares, there exist broad classes of production data (output, capital, and labor) which can always be related to each other through a functional form which is mathematically identical to a Cobb-Douglas with constant "returns to scale," "neutral technical change," and "marginal products equal to factor rewards." Since the above is a mathematical consequence of constant shares, true even for very implausible production data (such as the Humbug economy of section III, B), it is argued that the so-called empirical strength of production function analysis is in reality nothing more than a statistical reflection of the (unexplained) constancy of income shares.

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APPENDIX

Humbug Data

Year	Actual Share of Property Income	"Humbug" Output Per Worker	"Humbug" Capital Per Worker					
	s	$q(t)$	$k(t)$	$\dot{q}/q$	$\dot{k}/k$	$\dot{A}/A$	$A(t)$	$f(k)$
1909	0.335	0.80	2.00	−0.125	0.000	−0.125	1.000	0.800
1910	0.330	0.70	2.00	−0.143	0.000	−0.143	0.875	0.800
1911	0.335	0.60	2.00	+0.167	0.000	+0.167	0.750	0.800
1912	0.330	0.70	2.00	0.000	+0.050	−0.017	0.875	0.800
1913	0.334	0.70	2.10	0.000	0.048	−0.016	0.860	0.814
1914	0.325	0.70	2.20	−0.143	0.000	−0.143	0.846	0.826
1915	0.344	0.60	2.20	+0.333	0.000	+0.333	0.725	0.828
1916	0.358	0.80	2.20	0.000	0.045	−0.016	0.965	0.830
1917	0.370	0.80	2.30	−0.250	0.000	−0.250	0.948	0.843
1918	0.342	0.60	2.30	0.000	0.044	−0.015	0.710	0.845
1919	0.354	0.60	2.40	0.000	0.042	−0.015	0.700	0.857
1920	0.319	0.60	2.50	+0.167	0.000	+0.167	0.690	0.870
1921	0.369	0.70	2.50	+0.143	0.000	+0.143	0.805	0.870
1922	0.339	0.80	2.50	−0.250	0.040	−0.264	0.921	0.869
1923	0.337	0.60	2.60	+0.333	0.000	+0.333	0.678	0.885
1924	0.330	0.80	2.60	−0.063	0.019	−0.069	0.902	0.887
1925	0.336	0.75	2.65	−0.067	0.019	−0.073	0.840	0.893
1926	0.327	0.70	2.70	+0.071	0.019	+0.065	0.780	0.897
1927	0.323	0.75	2.75	+0.067	0.018	+0.061	0.830	0.903
1928	0.338	0.80	2.80	−0.250	0.000	−0.250	0.880	0.908
1929	0.332	0.60	2.80	0.000	0.036	−0.012	0.660	0.908
1930	0.347	0.60	2.90	0.000	0.052	−0.018	0.652	0.920
1931	0.325	0.60	3.05	+0.167	0.000	+0.167	0.641	0.935
1932	0.397	0.70	3.05	0.000	−0.049	+0.019	0.748	0.935
1933	0.362	0.70	2.90	+0.143	0.000	+0.143	0.764	0.916
1934	0.355	0.80	2.90	0.000	0.052	−0.018	0.874	0.916
1935	0.351	0.80	3.05	−0.125	0.000	−0.125	0.860	0.930
1936	0.357	0.70	3.05	0.143	0.033	+0.132	0.752	0.930
1937	0.340	0.80	3.15	0.250	0.000	−0.250	0.852	0.940
1938	0.331	0.60	3.15	0.000	0.032	−0.011	0.638	0.940
1939	0.347	0.60	3.25	0.000	0.031	−0.011	0.633	0.948
1940	0.357	0.60	3.35	+0.333	0.000	+0.333	0.626	0.960
1941	0.377	0.80	3.35	0.000	0.070	−0.026	0.843	0.950
1942	0.356	0.80	3.60	0.000	−0.042	+0.015	0.820	0.975
1943	0.342	0.80	3.45	−0.250	0.000	−0.250	0.832	0.964
1944	0.332	0.60	3.45	0.000	0.044	−0.015	0.624	0.964
1945	0.314	0.60	3.60	+0.167	0.000	+0.167	0.614	0.978
1946	0.312	0.70	3.60	0.000	−0.014	+0.004	0.717	0.975
1947	0.327	0.70	3.55	—	—	—	0.721	0.970