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Heterogeneous Capital, the Production Function and the Theory of Distribution [1,2]

The notion of capital as a " factor of production ", on which the theories of production and distribution dominant since the latter part of the last century ultimately rely, has been the object of considerable discussion in recent years. As is well known, these theories had their origin in a reformulation in terms of homogeneous land and " intensive " margins, of the Malthusian theory of rent. It was shown that the rent of land—which had appeared as the residuum left after deducting the share of wages and profits calculated at the rate given by the marginal product of a dose of " labour with capital "—could also be viewed as the marginal product of land, with the sum of wages and profits as the residuum. The two factors, land and " labour with capital ", could then be put on the same footing and the theory extended to any number of factors.

The way was thus open, it was thought, to explaining in terms of marginal productivity the division of the product between labour and capital, which the Classical economists had analyzed by altogether different principles. But in order to explain the rate of profit along these new lines, capital had to be conceived, ultimately, as a single magnitude: and had accordingly to be measured as a value quantity, unlike labour or land which were physical quantities. The extension of the " law of rent " to distribution between labour and capital therefore raised the danger of circular reasoning: the value of a capital good, like that of any product, changes with those very rates of wages and interest which are to be explained by means of " quantities " of capital. Some of the originators of these theories were conscious of the difficulty, but their attempts to deal with it were limited to very special hypotheses. It is therefore not surprising that in recent literature instances should have cropped up showing that the basic propositions of the theory are in fact controvertible. Up to now, however, little or no attempt seems to have been made to see the implications of the failure of these propositions for the problem of value and distribution. An attempt to do this is the main aim of the present article.

We shall begin by discussing a defence of traditional theory put forward by Samuelson [16] and claiming that in some cases heterogeneous capital goods can be reduced to quantities of a homogeneous " capital ", the marginal product of which equals the rate of interest: this, Samuelson thought, would display the corresponding version of traditional theory as a useful " parable " giving " insights into the fundamentals of interest theory ".[3]

Sections I and II of this paper will therefore examine the relations between the wage, the rate of interest and the product per worker in the two-commodity economy—with only a consumption good and a capital good—which Samuelson used for his argument.

[1] A paper containing the material now included in Sections I-III of the present paper was submitted for publication in April 1963 and accepted for publication subject to revision shortly afterwards. The present paper is a revised version of the extended paper, including three new sections, which reached the Editors in October 1968.

[2] I wish to thank Mr Sraffa for his comments and criticisms. I am also grateful for the help derived from discussion of this paper in seminars at the Faculty of Economics in Cambridge, and at the " Consiglio Nazionale delle Ricerche " in Rome. Professor Gandolfo of the University of Siena has been of considerable help with the mathematics underlying some propositions of Section IV.

[3] Cf. Samuelson [16], p. 193.

Section III will then show that a production function giving the interest rate as the marginal product of capital is compatible with these relations *if, and only if*, the conditions of production of the capital good are always identical with those of the consumption good: an hypothesis which turns the original "heterogeneous capital model" into one where capital, besides being homogeneous and hence measurable in physical terms, is also homogeneous with the consumption good.

In Section IV, the results reached that far will be generalized from Samuelson's two-commodity economy to one where any number of commodities are produced.

In that more general setting, Sections V and VI will consider how the discussion in the earlier Sections bears on various formulations of traditional theory. Attention will be focussed on the idea that in a competitive economy wages and interest are governed by the demand and supply for "capital" and labour, the core of traditional theory in all its versions. The fact that "capital intensity" in the economy need not increase as the rate of interest falls (the wage rises), undermines, it will be argued, the explanation of distribution in those terms. In the Appendix, these negative conclusions will be illustrated by numerical examples showing how far the relations between the rate of interest, the wage, the value of capital and the physical product per worker may differ from what received theory claims.

The final pages of Section VI will then consider some of the problems which arise when that explanation of value and distribution is abandoned. In this connection, we shall refer to the different approach used by the Classical economists up to Ricardo.

I. PRODUCTION WITH A SINGLE "SYSTEM"

The economy Samuelson assumes in his article is one where production takes place in yearly cycles and where a single consumption good A exists, obtainable by a number of alternative "systems of production", α, β, γ, etc. Each "system", e.g. α, consists of two "*methods of production*": a method for the direct production of A by means of fixed quantities $l_a^{(\alpha)}$ of labour, and $c_a^{(\alpha)}$ of a capital good $C^{(\alpha)}$ specific to the method; and the method for producing $C^{(\alpha)}$ by $l_c^{(\alpha)}$ of labour and $c_c^{(\alpha)}$ of itself. The capital good $C^{(\alpha)}$ is assumed to decay according to a yearly "rate of mortality" $d^{(\alpha)}$, independent of age.[1] Constant returns to scale are assumed in both industries and, hence, for the "system of production" as a whole.

Further, the system is such that: (i) a surplus can be obtained over the pure replacement of machines (i.e. $1-d^{(\alpha)}c_c^{(\alpha)}>0$); (ii) the capital good enters the production of both commodities (i.e., both $c_c^{(\alpha)}$ and $c_a^{(\alpha)}$ differ from zero); (iii) some labour is required directly or indirectly to produce each of the two commodities (i.e., $l_c^{(\alpha)}$ differs from zero, as do either $l_a^{(\alpha)}$, or $d^{(\alpha)}$, or both).

To begin with, suppose that only one system is known. The hypotheses of a uniform rate of interest or of profits r, and of a uniform wage w, enable us to write the two price equations

$$\begin{aligned} 1 &= l_a w + c_a p_c(r+d), \\ p_c &= l_c w + c_c p_c(r+d), \end{aligned} \qquad \ldots(1)$$

where we assume w to be paid at the *end* of the production cycle, and where p_c is the price of the capital good C, expressed, like w, in terms of A. In system (1), l_a, c_a, d, l_c, c_c are all known quantities, while r, w, and p_c are the three unknowns. The two equations are then sufficient to define the relation between r and w in the economy, given by

$$w = \frac{1-c_c(r+d)}{l_a+(l_c c_a - l_a c_c)(r+d)}. \qquad \ldots(2)$$

[1] Samuelson [16], p. 197. This way of dealing with fixed capital evades the problems specific to *fixed* capital, which are problems of joint production (cf. below, p. 13). In these Sections, however, we shall retain the assumption of a "rate of mortality" d, since $d = 1$ gives the correct treatment of *circulating* capital.

Assumptions (i), (ii) and (iii) we made about the system of production ensure that for $r = 0$ there is a positive " maximum wage " which we shall indicate by W. On the same assumptions, as r rises from zero, w falls as a continuous differentiable function of r, reaching zero for a finite " maximum rate of interest " R. It can also be shown that for $0 \leqq r \leqq R$ the price p_c of the capital good is positive.[1] The curve representing function (2) in the relevant interval $0 \leqq r \leqq R$ (Fig. 1) is Samuelson's " Factor-price Frontier ": we shall call it the *wage-curve* of the given system of production.

It is now convenient to introduce the notion of " integrated consumption-good industry ". By that we shall mean the composite industry where the proportion I of the C industry to the A industry is the one which just ensures the replacement of the capital goods consumed in the composite industry. The proportion I can be easily calculated. To a unit of the A industry, there must correspond a size of the C industry sufficient to replace the quantity $c_a d$ of C consumed in the A industry *plus* the quantity $c_a Id$ of it consumed in the C industry at its size I. The proportion I is therefore given by $I = c_a d + c_c Id$, i.e.

$$I = c_a d/(1-c_c d).$$

(In what follows we shall, for short, refer to the " integrated consumption-good industry " as the *integrated industry*.)

Recourse to the integrated industry is necessary if the *net* product—the product to be divided between wages and interest—is to consist of a *physical quantity* of the consumption good. An economy with zero net accumulation merely consists of the integrated industry.

A few important quantities and relations can now be read from the wage-curve WR of Fig. 1.

(i) the segment OW—measuring the wage when interest is zero—also measures the net *physical* product per labourer obtainable from the " integrated industry " with the given system of production;

(ii) as a result, given a wage Ow_1, the segment w_1W measures the amount of the consumption good received as interest for each worker employed in the integrated industry;

(iii) the tangent of the angle w_1PW measures the value, relative to the consumption good, of physical capital per worker in the integrated industry at the wage Ow_1. This is so because:

$$\tan w_1PW = w_1W/w_1P = w_1W/Or_1 = \frac{\text{interest per worker}}{\text{rate of interest}} = \text{value of capital per worker.}$$

(For brevity we shall refer to the tangent of the angle w_1PW as the *slope* of WP);

[1] For $r=0$, function (2) gives $W = \dfrac{1-dc_c}{l_a(1-dc_c)+dl_cc_a}$ where W must be positive by assumption (i) above, and finite by assumption (ii). Similarly for $w=0$ we have $R = (1-dc_c)/c_c$, with R as the net " rate of reproduction " of the capital good, positive by assumption (i) and finite by assumptions (ii) and (iii). We now note that function (2) defines a straight line in the special case $l_cc_a - l_ac_c = 0$, and a rectangular hyperbola with asymptotes parallel to the r and w axes in the general case where $l_cc_a - l_ac_c \neq 0$. In the first case we have the *decreasing* straight line:

$$w = \frac{1-dc_c}{l_a} - \frac{c_c}{l_a} r.$$

In the case of the hyperbola, the product of the distances of each point of the curve from the asymptotes is the *positive* quantity $\dfrac{l_cc_a}{(l_cc_a-l_ac_c)^2}$, and consequently we have a *decreasing* hyperbola. The positive intercepts, W and R, then imply that the segment of the curve for $0 \leqq r \leqq R$ lies on a single portion of the hyperbola, and is therefore continuous and differentiable.

Turning now to the positivity of p_c, we note that system (1) in the text gives p_c as the following function of r:

$$p_c = \frac{l_c}{l_a+(l_cc_a-l_ac_c)(r+d)}.$$

It follows that if $l_cc_a - l_ac_c \geqslant 0$, then p_c is positive for any non-negative value of r. In the remaining case, where $l_cc_a - l_ac_c < 0$, we have $p_c > 0$ for $r < \dfrac{l_a}{l_ac_c - l_cc_a} - d$; but $\dfrac{l_a}{l_ac_c-l_cc_a} - d > R$ because $R = \dfrac{1}{c_c} - d$, and we can conclude $p_c > 0$ for $0 \leqq r \leqq R$.

(iv) Since physical capital per worker in the integrated industry is a fixed physical quantity of capital good C, the change in the slope of WP, as we move along the wage-curve WR, indicates how p_c changes with changes in the division of the product between wages and interest.

It follows from (iv) that p_c is constant only when the wage-curve is a straight line. This will be the case if the proportion between physical capital and labour is the same in the two industries (i.e. $c_c/l_c = c_a/l_a$). Then, as r varies, the change in interest-costs relative to wage-costs must affect the two products equally, leaving their relative value p_c unchanged.

In general, however, the wage-curve will be either concave or convex to the origin. A concave curve, like WR in Fig. 1, shows that p_c rises with r. That will happen when, with the given system, the proportion between physical capital and labour is higher in the C industry (i.e. $c_c/l_c > c_a/l_a$), so that the rise in interest-costs affects the capital good more than the consumption good. Similar reasoning shows that the wage-curve will be convex in the remaining case, where the ratio of physical capital to labour is lower in the C industry.

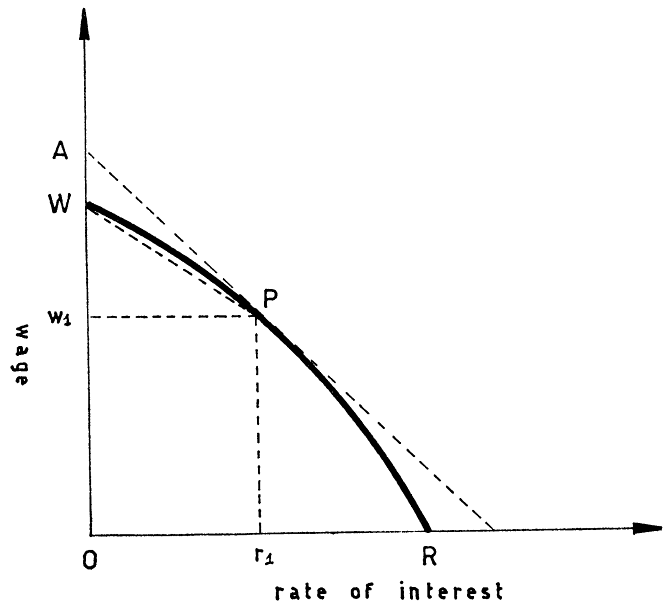

FIGURE 1

The " wage-curve ": OW is the net physical product per worker; tan. w_1PW is the value of capital per worker when the wage is Ow_1.

II. PRODUCTION WITH MANY " SYSTEMS "

We may now return to the hypothesis that several systems are available for the production of A. We shall have as many wage-curves as there are alternative systems. Since w is always measured in terms of the same commodity A, all the wage-curves can be drawn in the same diagram, as illustrated in Fig. 2 for the case of two systems.

At any level of the rate of interest, producers will choose the cheaper way of producing A. But, clearly, the costs of A produced with different systems will depend on the system which happens to be in use. We may take the example of Fig. 2. If α is in use at r_2, the wage is $w_2^{(\alpha)}$ and the prices of $C^{(\alpha)}$ and $C^{(\beta)}$ are calculated for r_2 and $w_2^{(\alpha)}$. If however β and not α is in use at r_2, the wage is $w_2^{(\beta)}$: the prices of $C^{(\alpha)}$ and $C^{(\beta)}$ are different and thus the costs of A produced with the two alternative systems will also differ from what they are when α is in use. The question is then whether the order of the two systems as to cheapness might not itself change according as system α or β is in use. If the order should so change we would have endless switching back and forth between α and β, or, alternatively, no tendency to change whichever system happens to be in use.

These possibilities, however, can be ruled out: the cheaper system will be the same at both wage rates and price systems. Moreover, the tendency of producers to switch to whichever system is cheaper in the existing price situation, will bring them to the system giving the highest w; while systems giving the same w for the same r will be indifferent and can co-exist.[1] It follows that the relation between r and w will be represented by the

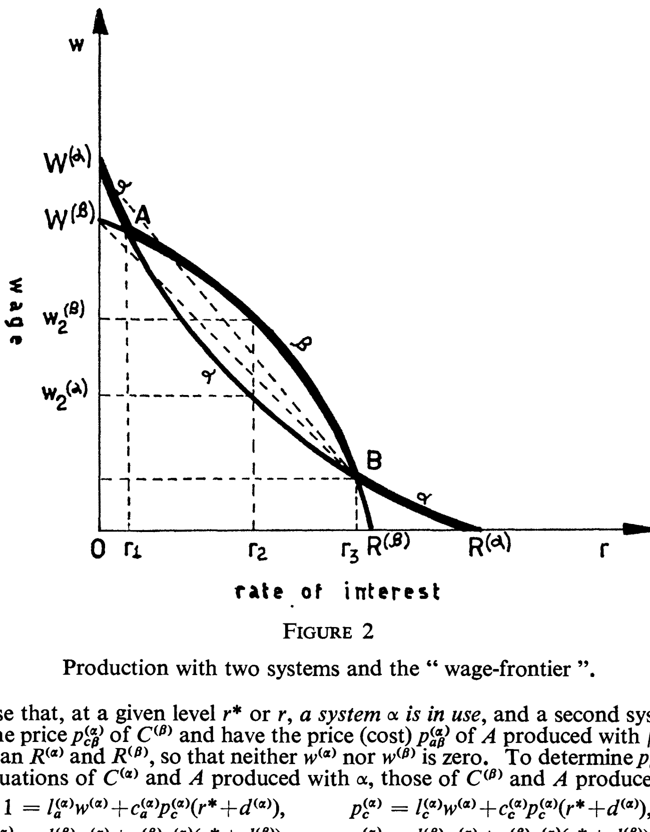

FIGURE 2

Production with two systems and the " wage-frontier ".

[1] Let us suppose that, at a given level r^* or r, *a system* α *is in use*, and a second system β is considered in order to know the price $p^{(\alpha)}_{c\beta}$ of $C^{(\beta)}$ and have the price (cost) $p^{(\alpha)}_{a\beta}$ of A produced with β. We shall assume that r^* is smaller than $R^{(\alpha)}$ and $R^{(\beta)}$, so that neither $w^{(\alpha)}$ nor $w^{(\beta)}$ is zero. To determine $p^{(\alpha)}_{c\beta}$ and $p^{(\alpha)}_{a\beta}$, we must add to the price-equations of $C^{(\alpha)}$ and A produced with α, those of $C^{(\beta)}$ and A produced with β:

$$\begin{aligned} 1 &= l^{(\alpha)}_a w^{(\alpha)} + c^{(\alpha)}_a p^{(\alpha)}_c (r^* + d^{(\alpha)}), & p^{(\alpha)}_c &= l^{(\alpha)}_c w^{(\alpha)} + c^{(\alpha)}_c p^{(\alpha)}_c (r^* + d^{(\alpha)}), \\ p^{(\alpha)}_{a\beta} &= l^{(\beta)}_a w^{(\alpha)} + c^{(\beta)}_a p^{(\alpha)}_{c\beta} (r^* + d^{(\beta)}), & p^{(\alpha)}_{c\beta} &= l^{(\beta)}_c w^{(\alpha)} + c^{(\beta)}_c p^{(\alpha)}_{c\beta} (r^* + d^{(\beta)}). \end{aligned} \qquad \text{...(i)}$$

Using the value of $w^{(\alpha)}$ resulting from the first two equations, we can determine $p^{(\alpha)}_{c\beta}$ and hence $p^{(\alpha)}_{a\beta}$; system β will be cheaper or dearer than α (in the price situation corresponding to α) according as $p^{(\alpha)}_{a\beta}$ is less or more than 1. If β, and not α, had been in use we would have had:

$$\begin{aligned} p^{(\beta)}_{a\alpha} &= l^{(\alpha)}_c w^{(\beta)} + c^{(\alpha)}_a p^{(\beta)}_{c\alpha} (r^* + d^{(\alpha)}), & p^{(\beta)}_{c\alpha} &= l^{(\alpha)}_c w^{(\beta)} + c^{(\alpha)}_c p^{(\beta)}_{c\alpha} (r^* + d^{(\alpha)}), \\ 1 &= l^{(\beta)}_a w^{(\beta)} + c^{(\beta)}_c p^{(\beta)}_c (r^* + d^{(\beta)}), & p^{(\beta)}_c &= l^{(\beta)}_c w^{(\beta)} + c^{(\beta)}_c p^{(\beta)}_c (r^* + d^{(\beta)}). \end{aligned} \qquad \text{...(ii)}$$

Now, w and the prices of the four commodities (A produced with α appearing as different from A produced with β) which result from equations (ii) will generally differ from those obtained from equations (i). But the difference can be seen to arise here purely from a change of the value-unit, which was A produced with α in (i), and is A produced with β in (ii). (It should be noted that this result is due purely to the fact that the two systems have no common commodity-input, and does not hold for more complex systems of production). The ratios between the prices and the wage are therefore the same in both systems of equations, and we have $\frac{w^{(\beta)}}{w^{(\alpha)}} = \frac{1}{p^{(\alpha)}_{a\beta}} = p^{(\beta)}_{a\alpha}$. Since when $p^{(\alpha)}_{a\beta} \lesseqgtr 1$, then $p^{(\beta)}_{a\alpha} \gtreqless 1$, the order of the two systems as to cheapness is the same at both systems of prices. Moreover, $p^{(\alpha)}_{a\beta} < 1$ if, and only if, $w^{(\beta)} > w^{(\alpha)}$: thus the systems giving a w higher than $w^{(\alpha)}$ are cheaper than α at the prices corresponding to α. Accordingly, whichever the system initially in use, the switch to cheaper methods will finally bring us to the system giving the highest wage. On the other hand, when $w^{(\alpha)} = w^{(\beta)}$, $p^{(\alpha)}_{a\beta} = 1$: systems giving the same w are therefore equally profitable at the given level of r, and can co-exist.

" outside " broken line generated by the intersecting wage-curves: the line which Samuelson named " north-east frontier " and which we shall call *wage-frontier*. Where the " wage-frontier " has a " corner " a switch of systems occurs, while at the " corner " itself the two systems, which we may call " adjacent ", can co-exist. Thus, in the example of Fig. 2, the " wage-frontier " is $W^{(\alpha)}ABR^{(\alpha)}$, with switches from α to β, and back from β to α occurring as r rises from zero to its maximum $R^{(\alpha)}$.

We could now examine how net physical product and value of capital per worker in the integrated industry change as r varies and producers switch from system to system. So, for example, Fig. 2 shows that, at r_3—with r rising—producers switch to system α, having a *higher* product per labourer (measured by $OW^{(\alpha)}$, larger than $OW^{(\beta)}$) and having, at r_3, a *higher* value of capital per worker (measured by the slope of $W^{(\alpha)}B$, steeper than that of $W^{(\beta)}B$). These results are both in striking contrast with received theory, as is the fact that in the highest range of r producers " switch back " to a system which had already been in use for low levels of r. It is convenient however to postpone these matters, and make some further assumptions which—while necessary for a discussion of Samuelson's surrogate production function—will allow us to meet traditional theory on the more familiar ground where systems of production " change continuously " with r.

We shall assume, accordingly, that the four coefficients l_a, d, c_c and $(l_c c_a)$[1] can change as given continuous functions of a parameter u: i.e. to each value of u included in a certain range, there will correspond a unique set of values of the coefficients and, hence, a system for the production of A. (As an example, we might imagine " wine " being produced with a continuously variable quantity of direct labour, where the " wine " produced with a given quantity of labour requires a specific *quality* of grapes, in turn produced by themselves and labour in fixed quantities.)[2]

We shall also suppose that the " family " of systems thereby defined is such that: (i) the wage-curve of each system cannot contribute segments, but only points, to the " wage-frontier "; (ii) that the " wage-frontier " no longer shows any " corner ". The " wage-frontier " then becomes a smooth " envelope " which is tangent at each point to one wage-curve, and encloses the whole family of them from above. An illustration is given in Fig. 3, where we have drawn some members of one such family of wage-curves having E as their " envelope ".[3]

On these assumptions, the system of production—i.e., the level of u defining the system in use—" changes continuously " with r. By this expression we mean that: (i) any change of r, however small, brings about a change of system; (ii) one system only is in use at each possible level of r.

Let us now consider the properties of an economy meeting these hypotheses. As the rate of interest rises, the wage must fall because the " envelope " is tangent to a decreasing curve at each point. Little however can be said about its curvature. Where the envelope

[1] Function (2) depends on $l_c c_a$, and not on l_c and c_a taken separately. This is as we should expect, since the arbitrary choice of the physical unit of C (which affects l_c and c_a, but not $l_c c_a$) cannot alter the relation between r and w.

[2] The assumption of continuity in the text is made *only* in order to show that the criticisms of traditional theory raised in this paper are independent of whether or not we assume systems of production which change by indefinitely small steps. In fact this assumption would be justified if the qualitative differences in the means of production used in different systems could be reduced to quantitative differences of the *single input* " capital ". But when differences in the kinds of input have to be taken into account, the only acceptable hypothesis is that of a *finite* number of alternative systems.

[3] In mathematical terms the position is as follows. The four coefficients are given by the functions $l_a = v_1(u)$, $l_c c_a = v_2(u)$, $c_c = v_3(u)$ and $d = v_4(u)$, defined for an interval of u in which all four functions are positive. Substituting these functions in function (2) (see p. 408 above), we obtain

$$w = \frac{1-v_3(u)\{r+v_4(u)\}}{v_1(u)+\{v_2(u)-v_1(u)v_3(u)\}\{r+v_4(u)\}}, \qquad \ldots(2.1)$$

where (2.1) is the " parametric equation " of the family of wage-curves. When the functions giving the four coefficients are appropriate, the family of curves defined by (2.1) admits a " smooth " envelope enclosing them from above. The equation of the envelope, or its points, can then be found by well-known procedures (cf. e.g. Courant [5], pp. 171-174).

is tangent to wage-curves which are either convex to the origin or straight lines, the envelope must be convex: but where the wage-curves are concave, the envelope may well be concave or a straight line. The envelope may therefore have any curvature, with convex and concave segments alternating.

As we have just seen, at any possible level of r, a single system is in use in the economy.[1]

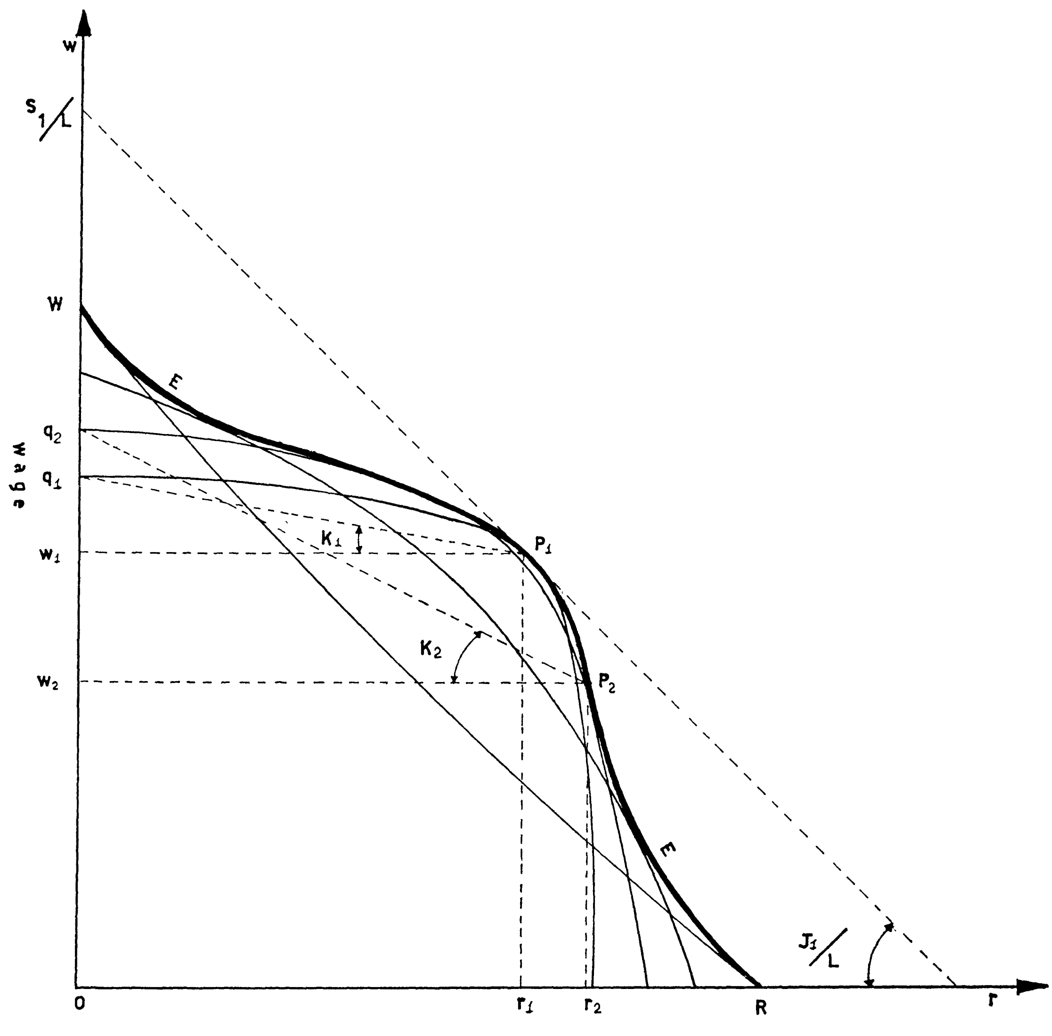

FIGURE 3

E is the " envelope " of the wage-curves.

We shall therefore find in the " integrated industry " (p. 409) a determinate physical product and a definite physical capital per worker. We shall indicate this net product by q, using k

[1] The hypothesis of a " smooth " envelope has ruled out the co-existence of " adjacent " systems, shown by " corners " of the wage-frontier. A second kind of co-existence is however conceivable: that in which two or more wage-curves are *tangent* to each other. Indeed, we might conceive families of wage-curves where two or more of them are tangent to each point of the " envelope ". These possibilities have been ignored in the text. (Cf., however, note 2, p. 432 in the Appendix.)

for the value of the physical capital per worker, expressed in terms of A and calculated at the given level of r. The way in which q and k vary with r can easily be seen from the diagram of the envelope. We take the wage-curves tangent to the envelope at given levels of r: q and k can then be read in the way we saw in the preceding Section. So, in the example of Fig. 3, as r rises from r_1 to r_2, q rises from q_1 to q_2, and k rises from k_1 to k_2.

In the economy we thus have definite relations between w, q, k and r. In what follows, these relations will be indicated by the three single-valued functions $w = e(r)$, $q = q(r)$, $k = k(r)$.

III. THE "SURROGATE PRODUCTION FUNCTION"

Thus equipped we may turn to Samuelson's " surrogate production function ". We have a " real " economy with a family of alternative systems giving the " envelope " $w = e(r)$ as the relation between r and w. We assume zero net accumulation so that the function $q = q(r)$ gives the net physical product per worker in the " real " economy. Samuelson's problem is whether these two relations might hold in an " imaginary " economy where the consumption good A is produced by labour and a capital homogeneous with A, and w and r are therefore determined by the marginal products of the two factors.

The problem can be restated as follows. We are asked to define a function $S = S(J, L)$ homogeneous of the first degree—with S as the quantity of (net) product, J that of capital, and L that of labour—satisfying the following two conditions:

(i) $\partial S/\partial L = e(\partial S/\partial J)$, where $w = e(r)$ is the relation between w and r in the " real " economy;

(ii) $S/L = q(\partial S/\partial J)$, where $q = q(r)$ is the relation between net product per worker and r in the same " real " economy.

The function $S(J, L)$, if it existed, would be a " surrogate production function " for the " real " economy in the sense that it would determine the relations between r, w and q once J—the quantity of " surrogate capital "—has been appropriately defined. This would in fact show that heterogeneous capital goods can be expressed as quantities of an appropriately defined homogeneous capital, in accordance with what Samuelson calls the " Clark-Ramsey parable ".

To see whether the " surrogate function " can be defined, we begin by using Euler's theorem to write $S(J, L)$ in the form

$$S/L = (\partial S/\partial L)+(\partial S/\partial J)(J/L). \qquad ...(3)$$

In the "imaginary" economy, the equilibrium rate of interest would always be equal to $\partial S/\partial J$. Condition (i) then permits us to write function (3) in the form

$$S/L = e(r)+r(J/L). \qquad ...(3.1)$$

Differentiation of function (3) with respect to $\partial S/\partial J$ gives [1]

$$J/L = -\frac{d(\partial S/\partial L)}{d(\partial S/\partial J)}, \qquad ...(4)$$

and, using condition (i) again,

$$J/L = -e'(r), \qquad ...(4.1)$$

where $e'(r)$ is the derivative of the envelope-equation for the " real " economy.

[1] We first obtain:

$$\frac{d(S/L)}{d(J/L)}\frac{d(J/L)}{d(\partial S/\partial J)} = \frac{d(\partial S/\partial L)}{d(\partial S/\partial J)} + \frac{J}{L} + \frac{\partial S}{\partial J}\frac{d(J/L)}{d(\partial S/\partial L)},$$

and then, since $d(S/L)/d(J/L) = \partial S/\partial J$, we have

$$\frac{J}{L} = -\frac{d(\partial S/\partial L)}{d(\partial S/\partial J)}.$$

Result (4.1) is of some interest and, by itself, sufficient to rule out the " surrogate function " in a first class of cases. It shows that, *if* $S(J, L)$ is to give the relation between r and w in the " real " economy, *then* to each level of $\partial S/\partial J$ there must correspond a ratio J/L equal to the " slope " of the envelope at the point where $r = \partial S/\partial J$. We saw, however, that the " real " economy may give an " envelope " which—in parts or throughout—is *concave* to the origin. Then J/L would *rise* with $\partial S/\partial J$, and the function $S(J, L)$ could not be a production function: equilibrium in the " imaginary " economy of the " Clark-Ramsey parable ", requires that the marginal product of " capital " should *not* rise when the ratio of " capital " to labour J/L rises. A straight-line envelope, which we saw was also conceivable, would be in even more striking contrast with the " parable ": we should then have to admit that the " marginal products " change, when the ratio of capital to labour does not.[1] But even the convexity of the relation between r and w does not ensure the existence of the " surrogate function ": the conditions for that are stricter.

To find them, let us return to function (3.1) and, using result (4.1), re-write it in the form

$$S/L = e(r)+r\{-e'(r)\}. \qquad ...(3.2)$$

It is now clear that condition (i) above is *sufficient* to define the function $S(J, L)$. No freedom is left for adapting $S(J, L)$ so as to satisfy condition (ii): we can only ascertain whether S/L as defined by (3.2) is identical with $q = q(r)$. That this is in general *not* so can be easily seen from the graph of the envelope (cp. Fig. 3). Function (3.2) gives S/L as the intercept on the w axis of the *straight-line* tangent to the " envelope " at the corresponding level of r. On the other hand, q is the intercept on the w axis of the *wage-curve* touching the envelope at the same point. Function (3.2) therefore over-estimates q at all levels of r where the system in use gives a wage-curve concave to the origin (see Fig. 3 at the point where $r = r_1$). It underestimates q when the wage-curve is convex. It only gives the correct q when the wage-curve is a straight line. Consequently, function (3.2) satisfies condition (ii) as well as condition (i) and a " surrogate production function " exists, *if and only if* all wage-curves are straight lines (see Fig. 4).

The implications of this condition must now be traced. We know from Section I that the wage-curve is a straight line when, in a given system, the proportion of capital goods to labour is the same in the A and C industries, so that the relative value of the two commodities is constant as the division of the product between wages and interest changes. We may now go further and note that, but for the arbitrary choice of the capital-good unit, the input-coefficients of the two industries are identical. The system is therefore *indistinguishable* from one where A is produced by itself and labour.[2] Indeed, since " heterogeneity " of commodities can here be properly defined only as a difference in their conditions of production, a straight-line wage-curve *means* that A is produced by itself and labour.

When this is true for the whole family of systems, we have that A is produced with varying proportions of itself to labour: the proportions indicated by the " slopes " of the wage-curves. It is then no surprise that the relations between r, w and q are compatible with the " Clark-Ramsey parable ". The assumption of equal proportions of inputs has turned the " real " economy with " heterogeneous capital-goods " into the " imaginary " economy of the " Clark-Ramsey parable ", where, in Samuelson's own words,

> " labour and homogeneous capital . . . produce a flow of *homogeneous net national product* which can consist of consumption goods or of net capital . . . formation *the two*

[1] In fact, in the case of a straight-line envelope, no differentiable homogeneous function can satisfy condition (i) above.

[2] Samuelson writes: " under our postulations, one can rigorously estimate (surrogate capital) by

$$J = V = P_\alpha K_\alpha + P_\beta K_\beta + \cdots$$

where the equilibrium market (numeraire) prices of the heterogeneous physical capitals are weights which *most definitely do change* as the real wage and interest rate are higher along the factor price frontier " ([16], p. 201, our italics). But in fact, under Samuelson's hypothesis of " equal proportions of inputs ", the prices of the " heterogeneous " physical capitals are " weights " which do *not* change as r and w change.

being infinitely substitutable (in the long run, or, possibly even in the short run) *on a one-for-one basis* ".[1]

In fact " surrogate capital " and " capital " are one and the same thing: at any given level of r, J/L, the " slope " of the envelope, is also the " slope " of the wage-curve tangent to the

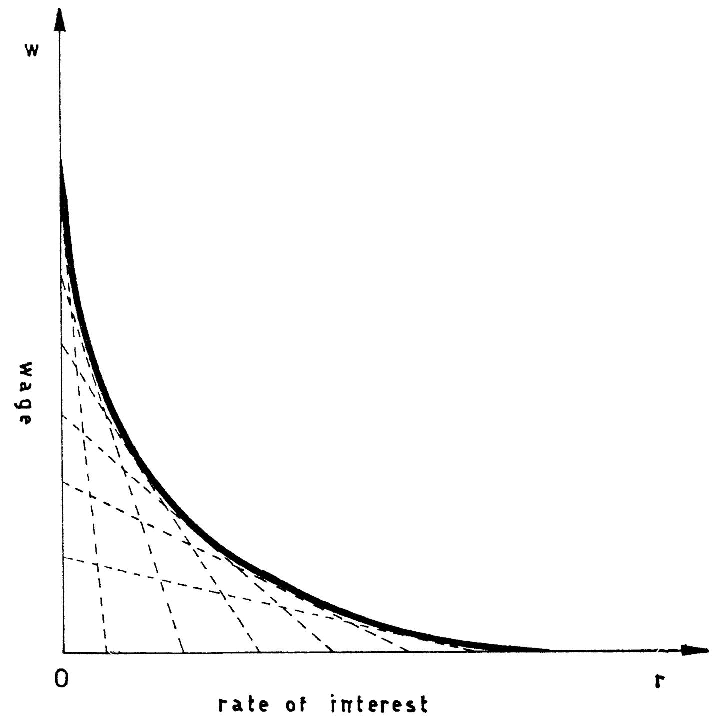

FIGURE 4

The " envelope " when the consumption good is homogeneous with the capital good.

envelope at that rate of interest, and measures the proportion of A to labour required to produce A with the system in use at that level of r. Samuelson's " surrogate production function " is thus nothing more than the production function, whose existence in such an economy no critic has ever doubted.[2]

[1] [16], p. 200; our italics. It should be noted how Samuelson himself emphasizes that when the price of " consumption " is constant relative to " net capital formation " (i.e. in his words: " the two [are] infinitely substitutable . . . on a one-for-one basis "), the two commodities are *homogeneous* (i.e., they constitute " a flow of homogeneous net national product ").

[2] We have supposed a stationary economy (above, p. 414). This is in fact the assumption we must make in order to discuss the " surrogate function " in the case where A and C are heterogeneous (if A and C are homogeneous, the level of accumulation is irrelevant). For, with some net accumulation, the net product would consist of A and of capital goods different at different levels of r. Then, as r changes, the net product per worker (a product measurable only in value terms) reflects also the changes in the *composition* of output—a circumstance quite distinct from the changes in the system of production, which alone the production function is meant to express. So, for example, should the economy grow in scale year by year, at a proportional rate equal to the rate of interest, the value relative to A of net product per worker would be equal to the S/L of equation (3.2) (cf. Bhaduri [1], p. 288 and, for a more general discussion of the ratio of capital to labour in conditions of accumulation, Spaventa [18]): but it would be incorrect to describe this case as one where the traditional production function is valid. By doing so we would have to admit, whenever the envelope is concave, equilibrium with rising " marginal products ": or even, should the envelope be a straight line, " marginal products " changing without any change in the proportions of the factors (p. 415 above).

IV. A GENERALIZATION

Suppose now, for a moment, that a second consumption good was produced in the economy besides A, and let it be a " luxury good ", a good not consumed by workers. The introduction of this good would not have affected the relation between r and w. Each " system " for the production of A—i.e. a " method of production " of A, and the " method of production " of the corresponding capital good—would still give the two equations (1); and these would determine the relation between r and w *independently* of the new price-equations for the " luxury good " and any means of production specific to it. The part these additional equations would play is only that of determining the prices of the " luxury good " and its means of production (once r, w and p_c are known).

What we have here is the application of a principle which Ricardo first perceived:[1] in any economy, the relation between the wage and the rate of interest depends exclusively on the methods of production of the commodities that are either wage-goods *or* means for their direct or indirect production. The way is now open for extending the inquiry of the preceding Sections to an economy with any number of commodities.

Let us consider an economy where commodities A_1, A_2, ..., are produced in yearly cycles, each in a distinct industry. By assuming an industry for each commodity we rule out joint production, and thus the possibility of a satisfactory treatment of fixed capital. We shall therefore complete the step and assume that the means of production are entirely consumed in each yearly cycle of production. As before, we suppose that no scarce natural resources exist, and that each industry has constant returns to scale. To begin with, we shall also assume that only one " method of production " is known for each industry. Thus, the production of a unit of commodity A_1 requires fixed quantities l_1 of labour and a_{11}, a_{21}, ..., of commodities A_1, A_2, etc.: these quantities define the " method of production " of A_1. Similarly the " methods of production " of A_2, A_3, etc., are defined, respectively, by the sets of input-coefficients $l_2, a_{12}, a_{22}, \ldots$; $l_3, a_{13}, a_{23}, \ldots$; etc.

The wage w is paid at the end of the yearly production cycle, and is a quantity of the *wage-commodity G* consisting of the *h wage-goods* A_1, A_2, ..., A_h taken in the quantities $\lambda_1, \lambda_2, \ldots, \lambda_h$, respectively.[2] We can now sort out from among all the commodities those

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The problem of the " surrogate function " has been associated by Samuelson with the measurement of relative shares of labour and capital by means of the elasticity of the wage-curve (cf. [16], pp. 199-200). Now, if in figure 1 (p. 410) we draw the tangent AP to the " wage-curve at the point P, the elasticity of the curve at that point is Ow_1/w_1A. On the other hand, relative shares in the stationary economy are Ow_1/w_1W at P (cf. proposition (ii) on p. 409). Consequently, the elasticity of the wage-curve does *not* measure relative shares in the stationary economy unless the wage-curve is a straight line. (In the latter case, A and C being homogeneous, the elasticity of the wage-curve measures relative shares, whatever the level of accumulation.) When A and C are heterogeneous, the elasticity measures relative shares only when the proportional rate of accumulation equals r.

[1] Both in letters of 1814 and in his *Essay on the effect of the low price of corn on the profits of stock* (1815), Ricardo maintained that " it is the profits of the farmer that regulate the profits of all other trades " ([13], vol. VI, p. 104; cf. also vol. IV, p. 23). As Mr Sraffa pointed out, that principle is founded on the assumption that " corn " is the only constituent of the wage, and requires only labour and itself to be produced ([13], vol. I, p. XXXI): " corn " is then the only commodity " entering the wage ", and the conditions of its production determine the rate of profits in the economy, once the wage is given. The principle was then taken up in a more general form by L. von Bortkiewicz ([2], English tr. p. 206; [3], English tr., p. 21; cf. also Pasinetti [12], p. 85. For the principle and comments on its formulation in Bortkiewicz, see, by the author, [7], pp. 31-33, 54n, 202).

[2] Changes in the wage level are generally associated with changes in the kinds and proportions of the commodities consumed by workers. It may then appear that a changing wage, measured in terms of an unchanging set of " wage-goods ", becomes an abstract value-quantity on the same footing as the wage in terms of any other commodity. However, certain commodities always play a primary role in workers' consumption and therefore provide a more significant measure of the wage. In addition, when, as in most concrete problems, the concern is for non-drastic changes in distribution around a given situation, the goods consumed by workers in that situation can be legitimately singled out as " wage-goods ".

which are means of production for the h wage-goods, or means of production for those means of production, etc. We shall indicate by $m(m \geqq h)$, the number of commodities which either are wage-goods or enter (directly or indirectly) the production of the wage-goods. For short, we shall call these m commodities, " commodities entering directly or indirectly the wage ", or simply *commodities entering the wage*. We shall indicate them by $A_1, A_2, ..., A_m$. The methods of the m " commodities entering the wage " constitute the *system of production* of the " wage commodity " G.

The assumption of a uniform wage and rate of interest allows us to write the following $m+1$ equations, the last of which defines G as the value unit:

$$\begin{aligned} p_1 &= l_1 w + (a_{11}p_1 + a_{21}p_2 + \ldots a_{m1}p_m)(1+r), \\ p_2 &= l_2 w + (a_{12}p_1 + a_{22}p_2 + \ldots a_{m2}p_m)(1+r), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_m &= l_m w + (a_{1m}p + a_{2m}p_m + \ldots + a_{mm}p_m)(1+r), \\ 1 &= \lambda_1 p_1 + \lambda_2 p_2 + \ldots \lambda_h p_h, \end{aligned} \qquad \ldots(5)$$

where $l_1, a_{11}, ..., a_{m1}$; $l_2, a_{12}, ..., a_{m2}$; $..., l_m, a_{1m}, ..., a_{mm}$; are all known quantities, as are $\lambda_1, \lambda_2, ..., \lambda_h$. There are $m+2$ unknowns: r, w and the prices $p_1, p_2, ..., p_m$.

Having $m+1$ independent equations for $m+2$ unknowns, system (5) has one degree of freedom, which we may use to obtain w and the prices as functions of r.

To analyse the relation between r and w, it is first necessary to distinguish those commodities, if any, which are used either directly or indirectly as means of production of *all* the m commodities. These commodities we shall call commodities (or products) *basic to the system*.[1] (E.g., in the systems discussed in the preceding sections, C, the capital good, was " basic " while A, the consumption good, was not.) We shall now make the following assumptions about the " system " for producing the wage-commodity G:

(i) the system is viable, i.e. is capable of yielding a surplus over the pure replacement of the means of production;
(ii) at least one of the m commodities is basic to the system;
(iii) the direct production of at least one of its basic products requires some labour (so that labour enters directly or indirectly into the production of all m commodities);
(iv) where a group of non-basics exists such that each of them enters directly or indirectly the production of all commodities in the group, the group's " own net rate of reproduction " is greater than the " net rate of reproduction " of the basic products.[2]

Assumptions (i), (ii) and (iii) are the generalization of the parallel assumptions (on p. 2) for our earlier two-commodity systems. (There, assumption (iv) was unnecessary because consumption good A, the non-basic product, did not enter its own production.)

We can now return to the relation between r and w as determined by system (5). Assumptions (i), (ii), (iii) and (iv) ensure that, for $r = 0$, we shall have a positive " maximum " wage W, and that, as r rises from zero, w falls as a continuous differentiable function of r, reaching zero for a finite " maximum " rate of interest R. The same assumptions imply

[1] Cp. Sraffa, [20], pp. 7-8. Mr Sraffa defines as basic commodities those which enter directly or indirectly the production of *all commodities in the economy*. This stricter definition seems inconvenient here since the relation between r and w depends only on the conditions of production of the commodities " entering the wage ".

[2] The " net rate of reproduction " of the basic products is Mr Sraffa's " Standard ratio " ([20], p. 21). The " own net rate of reproduction " of a group of connected non-basics is the analogous concept obtained by considering those non-basics as if their production did not require as means of production any commodity outside the group.

that, for $0 \leqq r \leqq R$, the prices of the m commodities are positive and finite.[1] The system for the production of G therefore defines a ' wage-curve " (Fig. 5) as with Samuelson's two-commodity systems, except that, under the present more general assumptions, concave and convex segments may alternate along the wage-curve.[2]

Let us now take the m industries in the unique set of proportions which ensures a net physical product consisting entirely of G,[3] and call the resulting composite industry the " integrated wage-commodity industry " or, for short, the *integrated industry*. Then, for the reasons we saw in Section I (p. 409), the vertical intercept OW of the wage-curve (Fig. 5) measures the net physical product per worker in the " integrated industry "; while the " slope " of the straight line WP measures the value—relative to the wage-commodity—of physical capital per worker in the same industry, at point P.

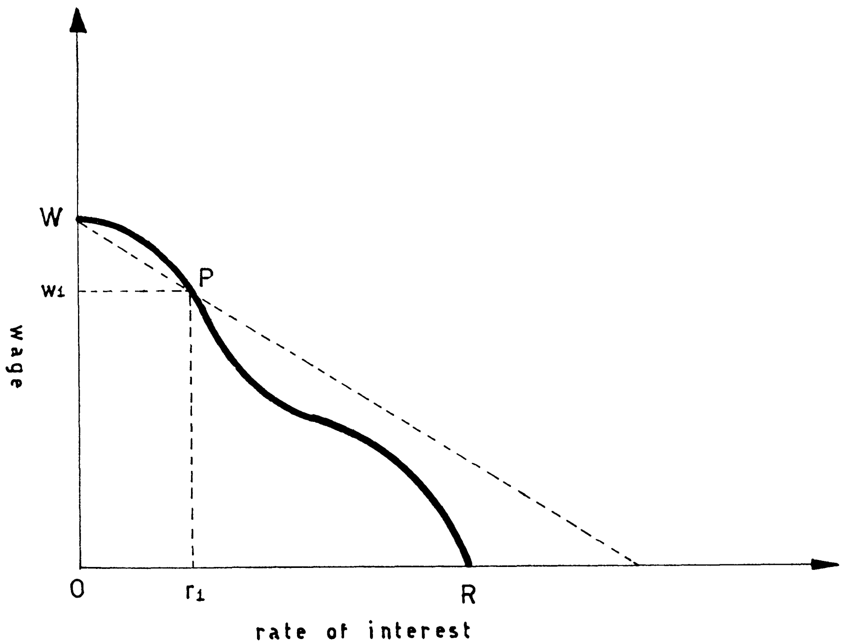

FIGURE 5

The wage-curve for systems with many commodities.

A straight-line wage-curve therefore indicates that the value of the physical capital used in the integrated industry does not change relative to the wage-commodity G as the division of the product between wages and interest changes. In that case, the equations of system (5) can be reduced to the following single price-equation for the wage-commodity:

$$1 = lw + a(1+r)$$

where l (given by $l_1\lambda_1 + l_2\lambda_2 + \dots l_h\lambda_h$) is the quantity of labour necessary for the direct production of a unit of G, and a, a constant, results from the commodity-input coefficients of the m industries. This single price-equation shows that the conditions of production of G are

1 It should be noted that assumptions (i), (ii), (iii) and (iv) do not ensure, for $0 \leqq r \leqq R$, positive prices for the " luxury goods " or their specific means of production: zero or negative prices for some of these goods would mean that they cannot be produced at the given rate of interest.

2 The wage-curve is a ratio between a polynomial of the mth degree and one of the $(m-1)$th degree in r. By a known property such rational functions admit up to $(3m-6)$ points of inflexion. (Cf., e.g. Enriques [6]). Further inquiry would be needed to find whether that maximum number can be reached in the relevant interval $0 < r < R$.

3 Cf. the notion of a " sub-system " in Sraffa [20], p. 89.

identical to those of its means of production: G and its means of production are therefore homogeneous (p. 415 above).

In all other cases, the value (in terms of G) of the capital employed in the integrated industry changes with r, and—as is shown by the alternation of convex and concave segments along the curve—the direction of change will itself change as r rises from zero to R.

So far, we have assumed that only one method is known for producing each of the commodities " entering the wage ". Let us now assume alternative methods for some or all the m commodities. Each *combination of m methods, one for each commodity*, constitutes a system for the production of the wage-commodity G. (If $i_1, i_2, \ldots, i_m$ are the number of alternative methods of production of $A_1, A_2, \ldots, A_m$ respectively, we have $i_1 \times i_2 \times \ldots i_m$ alternative systems.)[1]

We can go further and admit that the alternative methods of production of some commodity may require different means of production. Changing a method may then entail discarding some commodities and introducing new ones, each with its own method of production (or one of their alternative methods). In comparing any two systems, we shall therefore have to distinguish the commodities into two classes: those which " enter the wage " with *both* systems and are " common " to them (among these commodities we shall always find the h wage-goods), and those which are " specific " to one of the systems. (E.g. the two-commodity systems of the earlier Sections had " in common " only commodity A, the wage-good.)

After combining the alternative methods in order to form all possible systems of production of G, we may draw the corresponding wage-curves in one diagram. The problem now is to find which system will be in use at any specified level of r, given the tendency, in each industry, to switch to whichever method is cheaper in the existing price situation.

Here, as in Section II (p. 411), the question is complicated by the fact that the price situation, on the basis of which the methods are chosen, itself depends on the system which happens to be in use. It can be proved however that, whichever the system initially in use, the switch to cheaper methods will eventually bring into use the system giving the highest w. When two (or more) systems give the same w for the given r, and that w is higher than those of all other possible systems, the two (or more) systems give the same prices for all commodities they have in common, and can therefore co-exist.

The more general hypotheses of this Section do not therefore affect the main conclusion in Section II about the relation between r and w in an economy with many systems for producing the wage-commodity. That relation is represented by the 'wage-frontier " generated by the intersecting " wage-curves ": methods, and hence systems of production, change at the " corners " of the frontier, at which points two " adjacent " systems (p. 412) can co-exist. The same wage-curve however may now contribute more than two separate segments to the wage-frontier since any two wage-curves may intersect more than twice.

If we were prepared to make the necessary assumptions,[2] we could imagine the " wage-frontier " becoming a " smooth " envelope enclosing the family of wage-curves from above. The problem of a " surrogate production function " could then be approached in the same way as in Section III.

Let $w = e(r)$ be the equation of the envelope, and $q = q(r)$ be the relation between r and the net product per worker in the integrated wage-commodity industry. The problem is whether there is a function $S = S(J, L)$, homogeneous of the first degree, such that

[1] It should be noted that a change in the method of production of a commodity may entail a change in the list of basic products. It is also conceivable that certain sets of available methods may give rise to systems having no basic products. This last possibility will be ignored here, in accordance with assumption (ii) of p. 418 above.

[2] The number of commodities " entering the wage " should not change from system to system, and the $m(m+1)$ input coefficients should be appropriate functions of a single parameter u (cf. above p. 412 n. 3).

$\partial S/\partial L = e(\partial S/\partial J)$ and $S/L = q(\partial S/\partial J)$. For the reasons given above (p. 415), the function exists if, and only if, all the wage-curves are straight lines. We have already noted that a system giving a straight-line wage-curve implies that G is produced by itself and labour. It follows that $S(J, L)$ exists only when G is produced by variable proportions of itself and labour. Then J is the capital used *in the production of* G, and $S(J, L)$ is the production function of G.

But, in contrast to what we saw in Section III, the condition that all wage-curves are straight lines is now insufficient to ensure the existence of a " surrogate production function ", i.e. a function giving not only the relation between r and w, but also the relation between r and the net product *in the economy as a whole*. This is so because " luxury goods " can be produced in addition to G. The value relative to G of the net product per worker in the economy will then depend on the relative size of the various industries and will differ from $S/L = q$.[1] This, however, only confirms the conclusion we reached in Section III: a " surrogate production function " exists only for an economy where a single commodity is produced by itself and labour.

V. HETEROGENEOUS CAPITAL AND THE PREMISES OF THE TRADITIONAL THEORY OF DISTRIBUTION

Our inquiry into the properties of economies with heterogeneous capital goods now offers a convenient basis for examining the validity of traditional theory from an angle less special than that of Samuelson's " surrogate production function ". This we shall do by discussing three questions which, as we shall see, correspond to three different versions of the theory. The discussion will be conducted on the basis of the economy considered in the last Section, with constant returns to scale, no joint production and no scarcity of natural resources.

The first question is whether there exists an *aggregate* production function in which quantities of labour and " capital " explain both the level of the national product and, by means of the " marginal products " of the two factors, its distribution.[2] A second question is whether a similar production function can be conceived for any *single* commodity. The third and most important question concerns the basic premise of the traditional theory of distribution in all its formulations: the notion that a fall of r will cheapen the more capital-intensive processes of production.

Our discussion of Samuelson's " surrogate function " makes it easy to dismiss the first question. Either a single commodity is produced in the economy, and no problem of aggregation arises, or different commodities are produced, and no " aggregate production function " exists. But, despite the fashion for " aggregate production functions " in recent economic literature, they are only of secondary interest. In traditional theory, distinct production functions were generally attributed to each consumption good, and consumer demand was brought in to determine the proportions in which these goods are produced.

We therefore get nearer to the core of traditional theory when we turn to the notion that the technical conditions of production for any commodity A_i can be represented by a production function with " capital " and labour as the factors. In these versions of traditional theory it is claimed that, in any equilibrium situation, the ratio of " capital " to labour in the production of A_i would be that for which the marginal product of " capital " is equal to the ruling rate of interest. To that ratio, there would correspond a " marginal product " of labour, giving the wage in terms of A_i, and a determinate physical product to be divided between wages and interest.

[1] A similar difference between S/L and the value of the net product per worker in the economy could arise if the wage-goods were produced in proportions other than those in G. However, the assumption of a multiplicity of wage-goods can hardly be maintained where all wage-curves are straight lines: it would indeed be a fluke if G, consisting of different commodities in given proportion, required as means of production *those* commodities in *those* proportions in *all* alternative systems.

[2] Cf. Samuelson [16], pp. 193-4.

To discuss this view of production we must refer to the notions of " system of production " and " integrated industry ", applying them to the commodity A_i.[1] In parallel with the assumptions of continuity inherent in the notion of " marginal products ", we must assume that the system for the production of A_i " changes continuously " with r. What we saw about the relations between r, w, and q, and their representation by means of the " envelope " of wage-curves, can then be applied to the relations between r, the wage in terms of A_i, and the net physical product per worker in the production of A_i. The question we must now ask is whether these relations could result from a production function, with labour and an appropriately defined " capital " as the two factors.

This is simply Samuelson's problem of the existence of a " surrogate production function ", re-stated for the production of a single commodity A_i, and not for aggregate production. And, for exactly the reasons we saw above (p. 415), the function exists if, and only if, A_i is produced with varying proportions of itself and labour.[2]

But expressing the conditions of production of a commodity in terms of a production function with " capital " as a factor is a feature of only *some* versions of the traditional theory of distribution. They are the versions stemming from authors like Marshall or J. B. Clark, who thought that the principle of substitution, drawn from the reformulation of the Malthusian theory of rent in terms of homogeneous land and " intensive " margins, could be applied *without modification* to labour and " capital ". In production with unassisted labour and land, variable proportions of the factors can be shown to imply equality between the marginal products of the two factors and the rates of wages and rents in terms of the product. It was therefore thought that, in production with " capital " and labour, a similar equality would hold between the rates of interest and wages and the marginal products of these factors.

But this analogy between "capital" and labour or land was misleading. To give a marginal product equal to the *rate of interest*, "capital" must be conceived as a magnitude homogeneous with the product and must therefore be measured as the *value* of the means of production and not in physical units as is the case with labour or land. This value, however, like that of the other products, changes as r and w change. Consequently, the " quantities " of capital per worker corresponding to each system—and with it, the " production function " where those quantities appear—cannot be known independently of distribution. Every conclusion reached by postulating the contrary cannot be defended on *that* ground. And we have just seen that *one* of those conclusions is invalid: no definition of "capital" allows us to say that its marginal product is equal to the rate of interest.

The trap of drawing such a close analogy between capital and labour or land has been avoided in other versions of traditional theory. As we shall see, traditional theory —reduced to its core as the explanation of distribution in terms of demand and supply— rests in fact on a single premise. This premise is that any change of system brought about by a fall of r must increase the ratio of " capital " to labour in the production of the commodity: " capital " being the value of physical capital in terms of some unit of consumption goods, a value which is thought to measure the consumption given up or postponed in order to bring that physical capital into existence. Now, this proposition about "capital-intensity" was one of the conclusions reached by postulating a production function in which "capital" is included as a factor: the principle of decreasing marginal products would ensure that, as r falls, the

[1] So far, we have used these notions only for the wage-commodity G, but they can be applied to any commodity A_i. A " system of production " for A_i is a set of methods of production; one method for A_i, and one for each of the commodities which enter directly or indirectly the production of A_i. Appropriate proportions of these industries will give the " integrated industry " of A_i.

[2] The hypothesis that each commodity in the economy is produced by itself and labour would involve logical difficulties, unless we supposed that G, the wage-commodity, consisted of a single wage-good. This is so because each wage-good would have a distinct net " rate of reproduction ". Consequently, as r reaches the minimum of those " rates ", say that for A_1, the price of all wage-goods other than A_1 would be zero in terms of A_1. Thus, at that level of r, where the wage is zero in terms of G (but still positive in terms of wage-goods other than A_1), the economic system would become unworkable. (This difficulty is due to the fact that the systems of production of G would have no " basic " product.)

ratio of capital to labour rises, causing the marginal product of capital to fall in step with r. Some authors however thought that the proposition could be defended on other, more consistent grounds.

This was the position assumed by Boehm-Bawerk and Wicksell in what are, perhaps, the most careful formulations of the traditional theory. In their production functions, the capital goods appear in the form of a magnitude or set of magnitudes (Boehm-Bawerk's " average period of production " or Wicksell's " dated quantities of labour ") which are independent of distribution: it is to these magnitudes that the marginal productivity conditions are applied. " Capital ", the value magnitude, comes in at a second stage; when, to lay the basis for explaining interest and wages in terms of supply and demand, it is argued that a fall of r will result in a relative cheapening of the systems of production requiring capital goods of a smaller value per worker.[1]

We must therefore turn our attention to the proposition that systems with a higher " capital-intensity " become profitable at lower levels of r, and thus to the third of the questions outlined above (p. 421). That proposition is strictly associated with a second one, which claims that a fall of r lowers the relative price of the consumption goods whose production requires a higher proportion of capital to labour. Both propositions are in fact expressions of a single principle, according to which a fall of r cheapens the more capital-intensive processes of production. Granted this principle, the way is open for explaining distribution in terms of demand and supply. As r fell, both the change in the system of production for each consumption good, and consumer substitution in favour of the more capital-intensive goods, would raise the ratio of " capital " to labour in the economy. If we then assume that the quantity of labour employed remains equal to its supply (and the supply rises or, in any case, shows no drastic fall as w rises with the fall of r) it will follow that the amount of capital employed in the economy *increases* as r falls. This relation between r and the amount of capital employed could then be viewed as a demand function for capital; and competition in the capital market could be thought of as ensuring the absorption of " net saving " through appropriate falls of r.

On the other hand, the assumption of a persisting equality between the employment of labour and its supply would be justified by a parallel mechanism at work in the labour market. A regular demand function for labour would exist for any given amount of capital employed in production, and competition could be thought of as bringing the wage to the level where all labour finds employment.

(After Keynes, this alluring picture of a tidy interplay between demand and supply in the labour- and capital-markets would of course be qualified as applying " in the absence of risk and uncertainty ",[2] or where monetary authorities offer a visible beneficent hand should the invisible one fail.)

This elaborate theory of distribution therefore rests on the principle that a fall of r cheapens the more capital-intensive processes of production relative to the others. But this principle is no more valid than that of the equality between the rate of interest and the marginal product of capital: in just the same way, it is invalidated by the dependence of the value of capital goods on distribution.[3]

That a fall of r may cheapen the *less* capital intensive systems of a commodity A_i can be seen from the examples given for the consumption good in Samuelson's two-commodity

[1] On the theories of Boehm-Bawerk and Wicksell, and the reasons for the failure of their arguments supporting the principle about " capital-intensity ", cf. the author's [8], pp. 563-4, and the references given there.

[2] Cf., e.g., R. Solow, [17], p. 81.

[3] If the relative value of the two sets of capital goods required by any two systems of production remained constant as r changes, a fall of r could not fail to cheapen the more " capital-intensive " of the two systems. But the fall of r can *reduce* the relative value of the set of capital-goods used in the *less* capital-intensive system: and this reduction may well make that system cheaper than the more capital-intensive one.

systems. So, at point B of Fig. 2 (p. 411), the switch with falling r is in favour of system β, whose capital goods have, at switch-point prices, the lower value per worker. And the position would not be altered if we assumed that the system of production of the commodity " changes continuously " with r. So, in the family of systems of Fig. 3 above, a fall of r can lower, as well as raise, the value of physical capital per worker in terms of the product (a value which we shall here indicate by k_i). Fig. 6 shows the relation between r and k for that family of systems. Other examples of that relation are shown in Fig. 9 of the Appendix, relating to families of systems which we shall examine there: as some of these examples show, k_i and r may fall together even in the lowest range of r.[1]

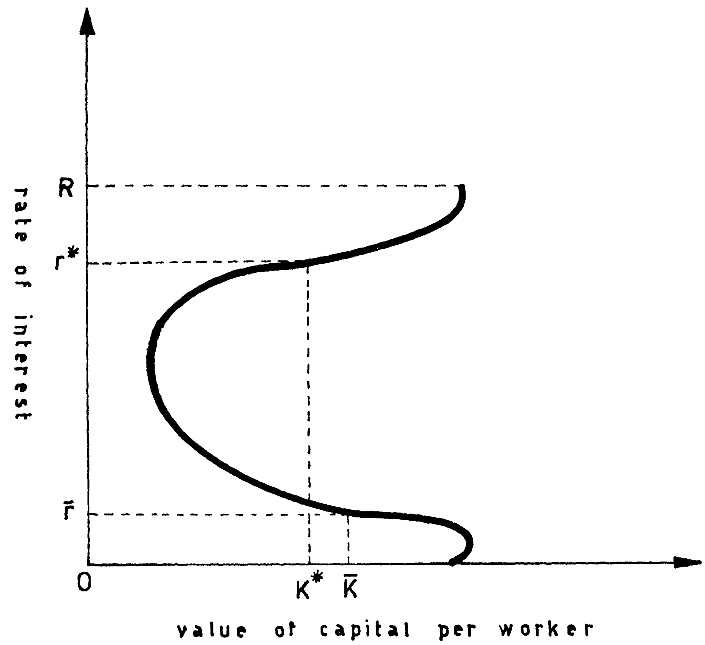

FIGURE 6

The relation between rate of interest and value of capital per worker with the family of systems of fig. 3 (p. 413).

We have seen that a fall of r may cheapen the less " capital intensive " of two systems for the production of a commodity. The same applies if the two systems relate to distinct commodities. Then by the traditional analysis of consumer choice we should conclude that, as r falls, substitution among consumption goods may lower, as well as raise, the ratio of capital to labour in the economy.

VI. HETEROGENEOUS CAPITAL AND THE THEORY OF VALUE AND DISTRIBUTION

Particular theoretical examples have forced the admission, in recent economic literature, that the switch of systems might operate in a direction contrary to the one traditionally assumed.[2] The tendency however has been to label those cases as " exceptions ": as if the

[1] In the examples referred to in the text, k_i is measured in terms of the commodity whose production we are considering. But the conclusion that k_i and r may fall together over any range of r is not affected if any other commodity is chosen as value unit. It should be noted however that the direction in which k_i changes with r may be different with different value-units since the relative value of such units will itself change with r.

[2] Cf. J. Robinson [14], p. 106; also [15], pp. 109-10, 418; D. G. Champernowne [4], pp. 118-19, 128-9; M. Morishima [11], p. 126; J. R. Hicks [9], p. 154. Cf. also D. Levhari [10], and the ensuing Symposium [22].

principle about capital-intensity had resulted from observed regularities, always liable to exception, and was not a pure deduction from postulates (like Boehm-Bawerk's " average period of production ") now generally admitted to be invalid.

Instead, it must be recognized that the traditional principle, drawn from incorrect premises, is itself incorrect. Moreover, the examples of the Appendix do not seem to indicate that the conditions in which a fall of r results in a relative cheapening of the less capital-intensive productive processes are any less plausible than those in which the opposite would be true. This appears to undermine the ground on which rests the explanation of distribution in terms of demand and supply for capital and labour.

To see why that is so, we may begin from the relation between r and the value, in the chosen unit, of the physical capital employed in the economy. This value we shall indicate by K. The relation between r and K—the traditional " demand function " for capital (saving)—was based on two assumptions: (*a*) that in the situation defined by each level of r, the labour employed is equal to the supply of it at the corresponding level of w; (*b*) that the composition of consumption output is that dictated by consumer demand at the prices and incomes [1] defined by the level of r. We shall now grant these assumptions, but we shall restrict the choice of the consumers by supposing, at first, zero net savings (i.e., in each situation, the capital goods are consumed and reproduced in unchanging quantities year by year). From these assumptions, and from what we saw about changes in the systems of production and the relative prices of consumption goods, it follows that K may fall or rise, as r falls.

To clear the ground, we must now grant traditional theory two further assumptions in addition to (*a*) and (*b*): namely that (*c*) a tendency to net saving (i.e. a fall in consumption) appearing in the situation defined by a given level of r, brings about a fall of r; (*d*) as r and w change, with systems of production and relative outputs changing accordingly, net savings realized in the economy can still be meaningfully defined, and can be measured—however broadly—by the difference between the K of the final and that of the initial situation.[2]

Let us now imagine that the economy is initially in the situation defined by the level r^* of the rate of interest, with K^* as the amount of capital.[3] Then a tendency to positive net savings appears (i.e. consumption is reduced). We assume that, after a time, the tendency to net saving disappears so that, if a new equilibrium is ever reached, the level of consumption will become that of the situation which corresponds to the new lower equilibrium value of r.

We must now ask whether—as r falls from r^* to some level $\bar{r}$ because of the initial tendency to net saving—a new situation can always be found with an additional quantity of

[1] In order to determine, simultaneously with the wage and prices, the incomes of consumers and hence the quantities of goods demanded and produced, some hypothesis is necessary regarding the distribution of the ownership of the capital-goods in the situation defined by each level of r.

[2] These assumptions are themselves highly questionable. It is beyond the scope of this article to discuss fully assumption (*d*). It should however be noted that, in order to justify this traditional assumption, we should once more refer to the economy of Samuelson's " parable ", where a single commodity is produced by itself and labour. In that economy, physical capital and K would be one and the same thing. No change of relative outputs could arise there, and changes in the systems of production would not require any qualitative change of the existing physical capital. Then, once we admit, with traditional theory, a tendency to the full utilization of resources, any change of K could be seen as resulting from an equal opposite change in consumption. But in an economy with heterogeneous capital goods, none of the conditions listed above is verified. The changes in systems of production or in relative outputs will affect the capital stock by changing the *kind* of capital goods, or by increasing the quantity of some capital goods and decreasing that of others. The possibility of referring to *physical increments* of the capital stock will fail, and with that will fail the possibility of any meaningful notion of " net saving ", not to mention " net saving " in terms of K. Then, even if we could grant traditional theory the existence of a tendency to the full utilization of resources, we would have to admit that the changes in total consumption imposed by given changes in the physical capital stock would depend on the *kind* of changes in the stock, and on the speed with which they have been accomplished—more than upon the difference between the K of the final and that of the initial situation.

As for assumption (*c*), we may note that Keynes's negative conclusions about the flexibility of r can only be strengthened if, as we shall argue in the text, changes in r provide no mechanism for equalizing " demand " and " supply " of capital (saving).

[3] Unless we suppose that the system for the production of each commodity changes " continuously " with r, K can assume, at any level of r where two systems co-exist, any value between the extremes set by the two systems.

capital ΔK representing the net savings which the community intended to make during the period. The form of the relation between r and K implies that such a new situation cannot always be found: however high r^* is, and however small ΔK, there may well not exist any lower rate of interest $\bar{r}$ at which $\bar{K} = K^* + \Delta K$. Or, to find a situation with an amount $\bar{K}$ of capital just larger than K^*, we may need a fall of r so drastic (cf. Fig. 6 above) as to make it clear that, in this case too, it is impossible to determine r by the supply and demand of " capital " (saving).[1]

This is not all. We saw (p. 423) that, in traditional theory, our assumption (*a*)—of a persisting equality between the quantity of labour employed and the supply of it—found its justification in the idea of a demand function for labour. But the fact that, given the quantity of labour employed, K may rise as r rises, implies that the labour employed with a constant K must *fall* with the corresponding fall of w. Thus—even if, by assumption (*d*), we grant that, in the face of changes in systems of production and relative outputs, we can speak of a constancy of capital and take that to mean constancy of K—there is no reason to suppose a tendency to equality between the demand and supply for labour. Assumption (*a*) is then unwarranted: the failure of a demand and supply analysis, which we first saw from the viewpoint of the capital market, has its mirror-image in the labour market.

Analogous results would have been reached had we imagined an initial rise in consumption (i.e. a tendency to negative net saving); or an initial change in the " demand " conditions for capital and labour (i.e. a change in the relation between r and K due to changes in consumer tastes or in the methods of production available).

Thus, after following in the footsteps of traditional theory and attempting an analysis of distribution in terms of " demand " and " supply ", we are forced to the conclusion that a change, however small, in the " supply " or " demand " conditions of labour or capital (saving) may result in drastic changes of r and w. That analysis would even force us to admit that r may fall to zero or rise to its maximum, and hence w rise to its maximum or to fall to zero, without bringing to equality the quantities supplied and demanded of the two factors.

Now, no such instability of an economy's wage- and interest-rates has ever been observed. The natural conclusion is that, in order to explain distribution, we must rely on forces other than " supply " and " demand ". The traditional theory of distribution was built, and accepted, in the belief that a fall of r—an increase in w—would always raise the proportion of " capital " to labour in the economy: the theory becomes implausible once it is admitted that this principle is not always valid.[2]

The idea that demand and supply for factors of production determine distribution has become so deeply ingrained in economic thought that it is almost viewed as an immediate

[1] This conclusion would not be affected if we chose to measure capital in the economy by means of the chain-index method proposed by Champernowne [4] and supported by Swan ([21] pp. 348 ff). It is beyond our scope to discuss this measure of capital or the claim that it permits us to consider as the increase of capital brought about by net saving " not the change in the value of the stock [in terms of consumption goods], but rather the value of the change ", ([21] pp. 349 and 356) (cf. however p. 425 above on " physical increments " of the capital stock). It is sufficient to remark here that when measured in these terms the amount of capital per worker may fall together with r (though it cannot do so in the immediate proximity of $r = 0$). In similar cases, Champernowne asserts " the only way that investment could remain positive . . . would be for food-wages to leap up and the rate of interest to leap down to levels where capital equipment . . . [giving a higher ratio of capital to labour] became competitive " ([4] p. 118).

[2] According to Professor Hicks, the failure of the principle about capital intensity leaves us in a position which, though not satisfactory, " has parallels in other parts of economic theory " ([9], p. 154). He thus seems to suggest that the possible fall, as r falls, in the value of capital per worker does not affect traditional theory any more than do the well known anomalies of the demand for inferior goods. This seems to ignore that the case of inferior goods did not call into question the general supply-and-demand analysis of prices only because it could be plausibly argued that: (*a*) should those anomalies give rise to a multiplicity of equilibria, the equilibrium position with the highest price would be stable, while that with the lowest price would, in all likelihood be stable too; and (*b*) if the latter equilibrium were unstable, the rest of the economic system would not be affected since all we would have is that once the price has fallen below the level of that equilibrium the commodity would not be produced due to a lack of demand willing to pay the supply price. No analogous arguments have been advanced by Professor Hicks with respect to the fall of capital-intensity as r falls.

reflection of facts, and not as the result of an elaborate theory. For the same reason, it is easily forgotten how comparatively recent that theory is. In the first systematic analysis of value and distribution by the English classical economists up to Ricardo, we would look in vain for the conception that demand and supply for labour and " capital " achieve " equilibrium " as the proportions in which those " factors " are employed in the economy change with the wage and rate of profits. Thus, Ricardo saw no inconsistency between free competition and unemployment of labour. In his view lower wages could eliminate unemployment only by decreasing the growth of population or by favouring accumulation.[1]

What we find in the Classical economists is the idea that the wage is ruled by the " necessaries of the labourer and his family ". Since they regarded these " necessaries " as determined by social as much as physiological conditions, we may see them as recognizing that distribution is governed by social forces, the investigation of which falls largely outside the domain of the pure theory of value. The proper object of value theory was seen to be the study of the *relations* between the wage, the rate of profits and the system of relative prices. These relations would then provide the basis for studying the circumstances on which depends the distribution of the product between classes.

The distinction thus made by the Classical economists between the study of value and the study of the forces governing distribution goes together with a separation between the study of value and that of levels of output. Since the inception of the marginal method this separation has been thought no more tenable than that between value and distribution. But the weakness of the marginalist position should now be apparent.

The outputs of commodities and, hence, consumer choice, can influence relative prices, *either* by modifying the technical conditions of production (i.e., the set of methods available for producing each commodity), *or* by affecting the rates of wages and profits.

The first possibility arises because increases in the output of a commodity may, on the one hand, bring about an increase of the division of labour in any of its possible forms and, on the other hand, where scarce natural resources are used, may force the adoption of methods which increase the output obtained from those resources. But with regard to the changes in the division of labour due to increases in output, the traditional analysis of the firm has in fact restricted the theory of a competitive economy to those technical improvements that are " external " to the firm. At the same time, the approach in terms of outputs of single commodities has ruled out the technical improvements deriving from the economy's general growth. Consequently, the only " economies of scale " considered were those " external to the firm ", but " internal to the industry "—the class which, it has been noted, " is most seldom to be met with ".[2] There remains the case of scarce natural resources. This—as Ricardo showed—can be conveniently treated by first assuming the outputs of the commodities to be given, *then* moving on to inquire about the technical changes associated with changes in outputs, and the consequent changes in the relations between r, w, and the prices (including the prices for the use of natural resources). This method would also allow a less restricted treatment of the " economies of scale ".[3]

The second way in which consumer choice and, hence, outputs can influence relative

[1] E.g., in his Chapter " On Machinery " in the *Principles* Ricardo wrote: " the discovery and use of machinery may be attended with a diminution of gross produce; and whenever that is the case, it will be injurious to the labouring class, as some of their number will be thrown out of employment, and population will become redundant, compared with the funds that are to employ it " ([13], vol. I, p. 390).

An interesting expression of the contrast between Ricardo and later theorists can be found in Wicksell's criticism of this position of Ricardo. Wicksell holds that the decrease of " gross produce " of which Ricardo speaks, is not possible because " as soon as a number of labourers have been made superfluous by these changes, and wages have accordingly fallen, then, as Ricardo failed to see [other] methods of production . . . will become more profitable . . . and absorb the surplus of idle labourers " ([23], p. 137).

[2] Sraffa, [19], p. 186.

[3] This method is apparently the one Mr Sraffa points to, when in the Preface to *Production of Commodities by Means of Commodities* he writes " no changes in output . . . are considered, so that no question arises as to the variation or constancy of returns " and adds: " this standpoint, which is that of the old classical economists from Adam Smith to Ricardo, has been submerged and forgotten since the advent of the marginal method " ([20], p. v).

prices is by affecting the relative scarcity of labour and capital, and thus the wage and rate of interest, given the supply of the two factors and the state of technical knowledge. This link between prices and outputs is one and the same thing as the explanation of distribution by demand and supply of factors of production: and it becomes untenable once that explanation is abandoned.

Thus, the separation of the pure theory of value from the study of the circumstances governing changes in the outputs of commodities, does not seem to meet any essential difficulty. On the contrary, it may open the way for a more satisfactory treatment of the relations between outputs and the technical conditions of production. Moreover, by freeing the theory of value from the assumption of consumers' tastes given from outside the economic system, this separation may favour a better understanding of consumption, and its dependence on the rest of the system.

With this, the theory of value will lose the all-embracing quality it assumed with the marginal method. But what will be lost in scope will certainly be gained in consistency and, we may hope, in fruitfulness.

POSTSCRIPT

A mathematical appendix has been omitted for reasons of space. In sections 1-6 of that appendix a demonstration is given of the propositions set out at p. 419 of the text on the positivity of prices and the properties of the relation between r and w. Then in sections 7 and 8 a proof is given of the statements on p. 420 on the relative profitability of alternative systems of production. A copy of this appendix is available to students in the Marshall Library at Cambridge.

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APPENDIX

The purpose of this Appendix is to show by a selection of numerical examples how far the relation between the rate of interest and the value of capital per worker in the production of a commodity can differ from what traditional theory postulates. For simplicity of calculation, we shall refer to the two-commodity systems of production discussed in Sections I and II of the paper. We shall indicate by k_i the value, expressed in terms of the product, of capital per worker in the integrated production of a commodity A_i (cf. p. 409 in the text); by w_i, the wage expressed in the same terms; and by q_i, the level of the net physical product per worker.

We shall begin by considering the family of systems giving the wage-curves and " envelope " of Fig. 3 (p. 413). To give a more complete idea of the freedom with which k_i may vary as r changes, we shall then show how families of systems may be found giving *any* relation between r and k_i within an area defined purely by the shape of the envelope (cf. the area *STQO* in Fig. 9 below).

I

We assume circulating capital, i.e. $d = 1$; and choose as the unit of any capital-good C the quantity of it requiring a labour-year for its direct production, so that $l_c = 1$ in each system. Let the remaining three coefficients be defined by the following functions

$$l_a = \frac{30 + 11u^{\frac{11}{10}} + u^{\frac{11}{5}} - 27e^{-2u}}{6 + u^{\frac{11}{10}}},$$

$$c_a = \frac{27e^{-2u}}{(6 + u^{\frac{11}{10}})^2}, \qquad \ldots \text{(i)}$$

$$c_c = \frac{5 + u^{\frac{11}{10}}}{6 + u^{\frac{11}{10}}},$$

where $u \geqq 0$ is the variable parameter, and e is the base of natural logarithms. The family of systems defined by the functions (i) is such that, as l_a increases (i.e. u increases), c_a decreases and c_c increases (cf. Table I).

TABLE 1

Parameter u	Production of a Unit of the Consumer Good		Production of a Unit of the Capital Good	
	Labour l_a	Physical Capital c_a	Labour l_c	Physical Capital c_c
0·000	0·500	0·750	1	0·833
0·250	2·584	0·424	1	0·839
0·500	3·930	0·237	1	0·845
0·750	4·834	0·133	1	0·851
1·000	5·478	0·075	1	0·857
1·250	5·974	0·042	1	0·863
1·505	6·391	0·023	1	0·868

TABLE II

Rate of Interest % r	System in Use u	Wage w_l	Net Phys. Prod. per Worker q_l	Value of Capital per Worker k_l
00·0	0·000	0·200	0·200	1·080
2·6	0·250	0·175	0·192	0·635
4·1	0·500	0·169	0·183	0·393
6·1	0·750	0·159	0·175	0·257
8·3	1·000	0·151	0·167	0·184
10·5	1·250	0·144	0·159	0·148
12·9	1·505	0·129	0·152	0·179
14·4	1·250	0·105	0·159	0·379
15·1	1·000	0·083	0·167	0·552
15·9	0·750	0·061	0·175	0·715
16·9	0·500	0·041	0·183	0·850
17·5	0·250	0·026	0·192	0·947
20·0	0·000	0·000	0·200	1·000

Substituting the expressions for the five coefficients in function (2) of the text (p. 408), we obtain the equation of the family of wage-curves,

$$w = \frac{1 - (5 + u^{\frac{11}{10}})r}{(5 + u^{\frac{11}{10}}) + \{27e^{-2u} - (5 + u^{\frac{11}{10}})^2\}r}.$$

The system defined by $u = 0$ gives a wage-curve convex to the origin and having $W = R = 0{\cdot}20$, shown in Fig. 7*a*. As u increases, both W and R, given by $1/(5 + u^{\frac{11}{10}})$, decrease. At the same time, the ratio of C to labour decreases in the A industry and increases in the C industry; as a result the wage-curves become progressively less convex to the origin. Thus, the system for $u \cong 0{\cdot}034$ gives a straight line wage-curve, while the wage-curves corresponding to still higher values of u are concave.

The fall of W and R on the one hand, and the increasingly concave shape of the wage-curves on the other, are such that each cuts the wage-curves corresponding to any lower u

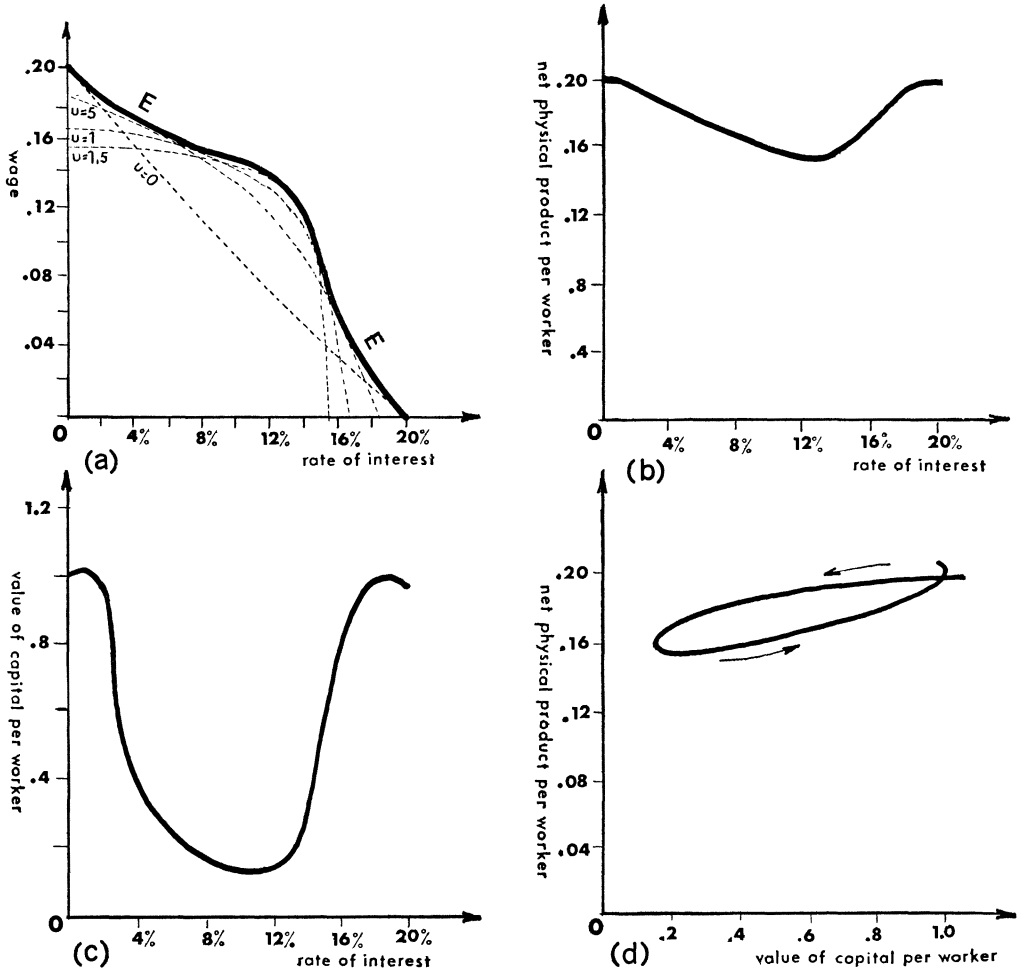

FIGURE 7

Relations between r, w, q_i, k_i with the family of systems whose input coefficients are defined by functions (i).

twice, and shows a middle section above all of them (see Fig. 7*a*). This is so up to the point where $u = 1{\cdot}505$. As u rises further, the wage-curves recede towards the origin and the corresponding systems are less profitable than others at *all* possible levels of r.

As is clear from Fig. 7*a*, the family of wage-curves for $0 \leqq u \leqq 1{\cdot}505$ admits an envelope E. Each wage-curve touches E twice (with the exception of that for $u = 1{\cdot}505$ which touches E only once), and each system will be in use at *two* levels of r. Thus, as r rises from zero to the maximum of 20 per cent admissible in the economy, we shall run through the whole series of systems twice: first (up to r = 13 per cent) in the order of increasing u, and then in the opposite order (Table II). Accordingly q_i, the net physical product per worker in the integrated industry will first fall as r rises, and then rise, as shown by Fig. 7*b*, (cf. also Table II). The same is true for k_i (see Fig. 7*c*).

The relation between k_i and q_i is shown by the curve of Fig. 7*d*, whose loop-like shape requires some explanation. As r rises from zero, we enter the diagram from the right and move in the direction of the arrows: q_i and k_i decrease together. But, as we reach $r \cong 12$ per cent, while q_i continues to decrease, k_i starts rising and we move down and to the right. Then, as r increases beyond 13 per cent, q_i also begins to rise and we move along the lower part of the loop. Thus, for the same q_i—indicating that the same system is in use—we have two levels of k_i due to the *two* levels of r at which the same physical capital is evaluated. So long as the systems are those in use for the middle range of r (i.e. the systems having a proportion of C to labour higher in the C than in the A industry), k_i is higher at the second (higher) rate of interest. The contrary is true for the systems in use in the two extreme ranges of r. We have the same q_i for the same k_i only when the system in use is that corresponding to $u \cong 0{\cdot}034$, giving $q_i \cong 1{\cdot}99$: the system in which the ratio of C to labour is the same in the A and in the C industries. In correspondence to that system the two parts of the loop intersect.

II

Suppose a decreasing rectangular hyperbola with asymptotes parallel to the r and w axes, which intersects both positive semi-axes: for short, we shall call *segment* the part of this hyperbola included between the two intersections. It can be shown that when the " segment " is *concave* to the origin, it can always be interpreted as the wage-curve for a two commodity system of the kind used for our numerical examples: i.e., the equation of the curve is always compatible with positive values of all the four co-efficients l_a, $(c_a l_c)$, c_c, d.[1]

Let us now take the parabola defined by the equation

$$w = -20r^2 - r + 1$$

and call E the part of that curve included between the points of intersection (0, 1) and (0·2, 0) with the positive semi-axes (Fig. 8).

Consider the possible families of " segments " having E as the envelope enclosing the family from above: an infinite number of such families can be conceived. Since E is concave to the origin, any family must include only concave " segments ", and is therefore a possible family of wage-curves. Accordingly, E can be conceived as the relation between r and w for any among an infinite number of possible families of systems of production.

[1] The equation of the hyperbola assumed in the text is

$$w = \frac{1-xr}{y-zr},$$

where x, y, z are positive and such that $(xy-z)>0$. On the other hand, the equation of a wage-curve (see function (2), p. 408 above) can be written as

$$w = \frac{1-\dfrac{c_c}{1-dc_c}r}{\dfrac{l_a+(l_c c_a - l_a c_c)}{1-dc_c}+\dfrac{l_c c_a - l_a c_c}{1-dc_c}r}.$$

Equating one by one the coefficients of the two functions we obtain the following three equations

$$\frac{c_c}{1-dc_c}=x; \quad \frac{l_a+(c_a-l_a c_c)}{1-dc_c}=y; \quad \frac{c_a-l_a c_c}{1-dc_c}=z;$$

which give

$$l_a=\frac{y+dz}{1+dx}; \quad c_c=\frac{x}{1+dx}; \quad l_c c_a=\frac{xy-z}{1+dx};$$

where l_a, c_c and $(l_c c_a)$ are positive because x, y, z, $(xy-z)$ and d are positive.

We may now define each family j of such systems in terms of the function $k_i^{(j)} = k_i^{(j)}(r)$, relating the rate of interest and the value of capital per worker in the family. To see the conditions which $k_i^{(j)}$ must satisfy, let us consider in Fig. 8 the wage-curve QPS of the family j, tangent to E at the point P for $r = r^*$, where $0 < r^* \leqq 0{\cdot}2$: the " slope " of the straight line QP measures $k_i^{(j)}(r^*)$ (see proposition (iii) of p. 418 in the text). If E is to enclose all wage-curves of the family j from above, that " slope " must be lower than the " slope " of the straight line drawn to P from T, a slope given by $(1-w^*)/r^* = 1+20r^*$. Further, $k_i^{(j)}(0)$—the slope for $r = 0$ of the wage-curve tangent to E at point T—must equal the " slope " of E at T, which is 1. Thus, the function $k_i^{(j)}(r)$, defining a possible family of systems having E as the envelope, must satisfy the conditions:

$$k_i^{(j)}(0) = 1; \; \textit{and} \; k_i^{(j)}(r) < 1+20r, \text{ for } 0 < r \leqq 0{\cdot}2. \qquad \ldots\text{(ii)}$$

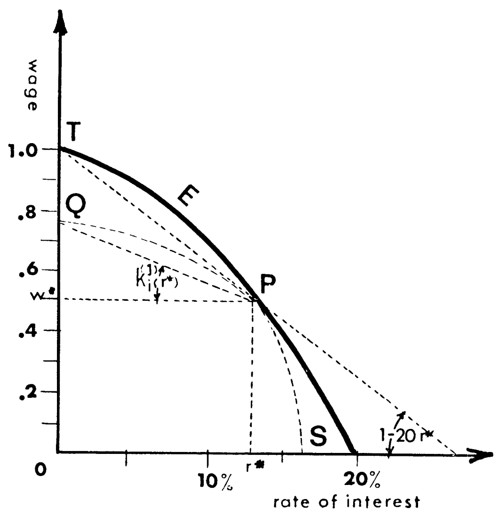

FIGURE 8

E is the envelope of the family of systems j defined by the function $k_i^{(j)}(r)$.

To any function $k_i(r)$ satisfying these conditions there corresponds a family of wage-curves having E as its envelope.[1] It follows that families of systems can be conceived giving as the relation between r and k_i any curve which, in Fig. 9, has Q as its point for $r = 0$, and then keeps within the area $STQO$—an area depending upon the curve E of our example.[2]

We have chosen to illustrate this freedom of the relation between r and k_i by considering the families of systems defined by the following three functions, represented in Fig. 9:[2]

$$k_i^{(1)} = 1,$$

$$k_i^{(2)} = 1+10r,$$

$$k_i^{(3)} = 1+18r-90r^2.$$

[1] Conditions (ii) above are sufficient to exclude that a wage-curve tangent to E would also intersect it. By equating the function of the hyperbola giving the wage-curve with that of the parabola giving E, we obtain an equation of the third degree in r. Of its three possible roots, two are accounted for by the point of tangency, while the third root can be shown to be negative.

[2] See opposite page.

As in the preceding example we suppose $d = 1$ and $l_c = 1$. The co-efficients l_a, c_a, c_c are functions of the parameter u, chosen so that the system defined by any value u^* of u is adopted at the level $r = u^*$ of r.

In what follows, we give for the three families of systems: (*a*) tables with the numerical values of the coefficients for some of the systems; and with the values of r, w, q_i and k_i in the situation where those systems are in use; (*b*) diagrams of the relation between net production and value of capital per worker (it should be noted how the curves differ from those rising and concave from below generally assumed).

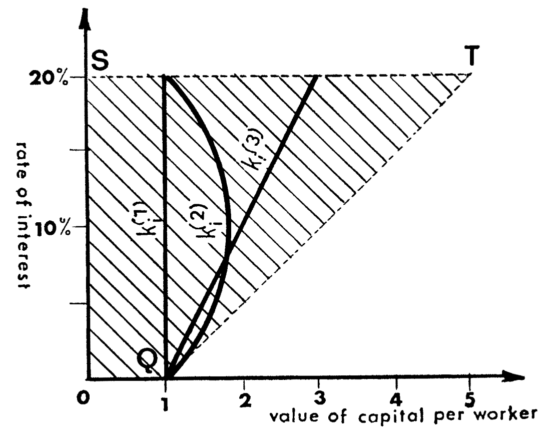

FIGURE 9

Any curve $k_i^{(j)}$ which starts from *Q* and stays within the area *STQO* is a possible relation between the value of capital per worker k_i and the rate of interest.

TABLE III

The case $k_i^{(1)} = 1$

Systems of Production						In use at		Value of Capital per Worker k_i
Param. u	Input-Coefficients (*)				Net Phys. Prod. per Worker q_i			
	l_a	c_a	l_c	c_c		r	w	
0	0·976	0·0006	1	0·976	1	0	1	1
0·05	1·027	0·0017	1	0·932	0·95	0·05	0·9	1
0·10	1·216	0·0037	1	0·892	0·80	0·10	0·7	1
0·15	1·750	0·0097	1	0·857	0·55	0·15	0·4	1
0·20	4·537	0·0772	1	0·833	0·20	0·20	0	1

2 If we considered the possibility that *two* systems be in use at the same level of r (cf. n., p. 411 above) the relation between r and k_i would be given by an area. An example is provided by the shaded area in Figure *a*: as r falls from its maximum of 20 per cent down to zero, the possible values of k_i narrow down to a single intermediate value *OB*.

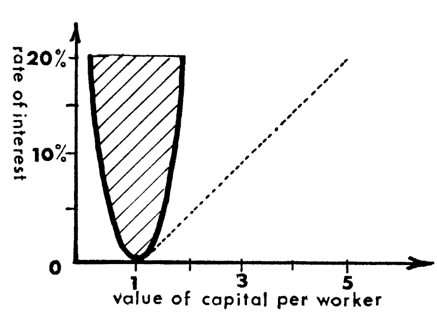

FIGURE *a*

The shaded area is a possible relation between k_i and r.

TABLE IV

The case $k_i^{(2)} = 1+10r$

Systems of Production						In use at		Value of Capital per Worker k_i
Param. u	Input-Coefficients (*)				Net Phys. Prod. per Worker q_i			
	l_a	c_a	l_c	c_c		r	w	
0	0·968	0·0010	1	0·969	1	0	1	1
0·05	0·959	0·0006	1	0·915	0·975	0·05	0·9	1·5
0·10	0·986	0·0159	1	0·873	0·9	0·10	0·7	2
0·15	2·978	0·0359	1	0·845	0·775	0·15	0·4	2·5
0·20	1·204	0·0772	1	0·834	0·6	0·20	0	3

TABLE V

The case for $k_i^{(3)} = 1+18r-90r^2$

Systems of Production						In use at		Value of Capital per Worker k_i
Param. u	Input-Coefficients (*)				Net Phys. Prod. per Worker q_i			
	l_a	c_a	l_c	c_c		r	w	
0	0·958	0·0017	1	0·958	1	0	1	1
0·05	0·928	0·0083	1	0·907	0·984	0·05	0·9	1·675
0·10	1·010	0·0142	1	0·875	0·890	0·10	0·7	1·900
0·15	1·394	0·0211	1	0·850	0·651	0·15	0·4	1·675
0·20	4·537	0·0772	1	0·833	0·200	0·20	0	1

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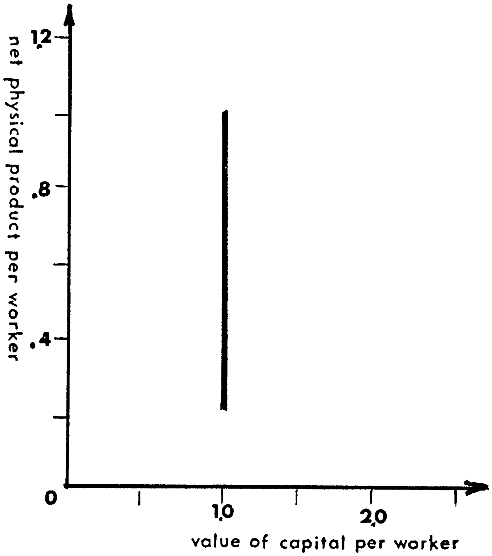
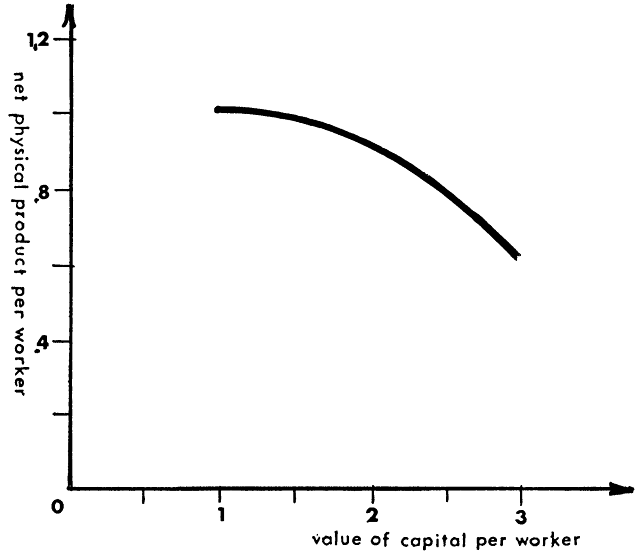


Case for $k_t^{(1)}$



Case for $k_t^{(2)}$

Case for $k_t^{(3)}$

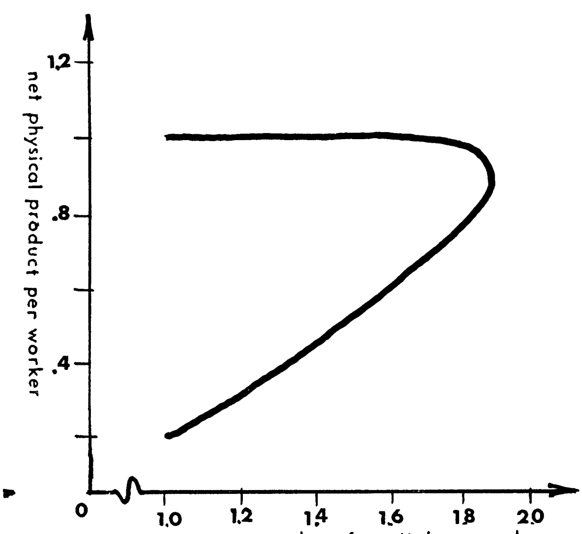


FIGURE 10

Relations between q_t and k_t for the three families of systems of the example.

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