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# NONNEGATIVE SQUARE MATRICES[1]

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## 1. INTRODUCTION

SQUARE MATRICES, all of whose elements are nonnegative, have played an important role in the probabilistic theory of finite Markov chains (See **[6]** and the references there given) and, more recently, in the study of linear models in economics **[2]** to **[5]**, **[10]** to **[12]**, **[15]** to **[20]**, and **[24]**.

The properties of such matrices were first investigated by Perron **[22]**, **[23]**, and then very thoroughly by Frobenius **[7]**, **[8]**, **[9]**. Lately Wielandt **[26]** has given notably more simple proofs for the results of Frobenius.

In Section 2 we study nonnegative indecomposable matrices from a different point of view (that of the Brouwer fixed point theorem); a concise proof of their basic properties is thus obtained. In Section 3 properties of a general nonnegative square matrix $A$ are derived from those of nonnegative indecomposable matrices. In Section 4 theorems about the matrix $sI - A$ are proved; they cover in a unified manner a number of results recurringly used in economics. In Section 5 a systematic study of the convergence of $A^p$ when $p$ tends to infinity ($A$ is a general complex matrix) is linked to combinatorial properties of nonnegative square matrices.

Unless otherwise specified, all matrices considered will have *real* elements. We define for $A = (a_{ij})$, $B = (b_{ij})$:

$$A \leqq B \text{ if } a_{ij} \leqq b_{ij} \text{ for all } i, j,$$

$$A \ngeq B \text{ if } A \leqq B \text{ and } A \neq B, \qquad \leq$$

$$A < B \text{ if } a_{ij} < b_{ij} \text{ for all } ij.$$

Primed letters denote transposes.

When $A$ is an $n \cdot n$ matrix, $A_T = TAT^{-1}$ denotes the transform of $A$ by the nonsingular $n \cdot n$ matrix $T$.

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## 2. NONNEGATIVE INDECOMPOSABLE MATRICES

An $n \cdot n$ matrix $A$ ($n \geqq 2$) is said to be *indecomposable* if for no permutation matrix[2] $\Pi$ does $A_\pi = \Pi A \Pi^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ where $A_{11}$, $A_{22}$ are square.

THEOREM I: *Let $A \geqq 0$ be indecomposable. Then*

1. *$A$ has a characteristic root $r > 0$ such that*
2. *to $r$ can be associated an eigen-vector $x_0 > 0$;*
3. *if $\alpha$ is any characteristic root of $A$, $|\alpha| \leqq r$;*
4. *$r$ increases when any element of $A$ increases;*
5. *$r$ is a simple root.*

PROOF: 1. (a) *If $x \geqslant 0$, then $Ax \geqslant 0$.* For if $Ax = 0$, $A$ would have a column of zeros, and so would not be indecomposable.

1. (b) *$A$ has a characteristic root $r > 0$.*

Let $S = \{x \in R^n \mid x \geqslant 0, \sum x_i = 1\}$ be the fundamental simplex in the Euclidean $n$-space, $R^n$. If $x \in S$, we define $T(x) = [1/\rho(x)]Ax$ where $\rho(x) > 0$ is so determined that $T(x) \in S$ [by (1.a) such a $\rho$ exists for every $x \in S$]. Clearly $T(x)$ is a continuous transformation of $S$ into itself, so, by the Brouwer fixed-point theorem (see for example [**14**]), there is an $x_0 \in S$ with $x_0 = T(x_0) = [1/\rho(x_0)]Ax_0$. Put $r = \rho(x_0)$.

2. $x_0 > 0$. Suppose that after applying a proper $\Pi$, $\tilde{x}_0 = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, $\xi > 0$. Partition $A_\pi$ accordingly. $A_\pi \tilde{x}_0 = r\tilde{x}_0$ yields $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} r\xi \\ 0 \end{pmatrix}$, thus $A_{21}\xi = 0$, so $A_{21} = 0$, violating the indecomposability of $A$.

If $M = (m_{ij})$ is a matrix, we henceforth denote by $M^*$ the matrix $M^* = (|m_{ij}|)$.

3–4. *If $0 \leqq B \leqq A$, and if $\beta$ is a characteristic root of $B$, then $|\beta| \leqslant r$.* Moreover, $|\beta| = r$ *implies $B = A$.*

$A'$ is indecomposable and therefore has a characteristic root $r_1 > 0$ with an eigen-vector $x_1 > 0$: $A'x_1 = r_1 x_1$. Moreover $\beta y = By$. Taking absolute values and using the triangle inequality, we obtain

(i) $|\beta| y^* \leqq By^* \leqq Ay^*$. So
(ii) $|\beta| x_1' y^* \leqq x_1' A y^* = r_1 x_1' y^*$.
Since $x_1 > 0$, $x_1' y^* > 0$, thus $|\beta| \leqq r_1$.

Putting $B = A$ one obtains $|\alpha| \leqq r_1$. In particular $r \leqq r_1$ and since, similarly, $r_1 \leqq r$, $r_1$ is equal to $r$.

[2] A permutation matrix is obtained by permuting the columns of an identity matrix. $\Pi A \Pi^{-1}$ is obtained by performing the same permutation on the rows and on the columns of $A$.

Going back to the comparison of $B$ and $A$ and assuming that $|\beta| = r$ one gets from (i) and (ii)

$$ry^* = By^* = Ay^*.$$

From $ry^* = Ay^*$, application of 2 gives $y^* > 0$. Thus $By^* = Ay^*$ together with $B \leqq A$ yields $B = A$.

5.(a) *If $B$ is a principal submatrix of $A$ and $\beta$ a characteristic root of $B$, $|\beta| < r$.*

$\beta$ is also a characteristic root of the $n \cdot n$ matrix $\bar{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$. Since $A$ is indecomposable, $\bar{B} \lneqq A_\pi$ for a proper $\Pi$ and $|\beta| < r$ (by 3–4).

5.(b) *$r$ is a simple root of $\Phi(t) = \det(tI - A) = 0$.*

$\Phi'(r)$ is the sum of the principal $(n-1)\cdot(n-1)$ minors of $\det(rI - A)$. Let $A_i$ be one of the principal $(n-1)\cdot(n-1)$ submatrices of $A$. By 5(a) $\det(tI - A_i)$ cannot vanish for $t \geqq r$, whence $\det(rI - A_i) > 0$ and $\Phi'(r) > 0$.[3]

With a proof practically identical to that of 3–4, one obtains the more general result:

*If $B$ is a complex matrix such that $B^* \leqq A$, $A$ indecomposable, and if $\beta$ is a characteristic root of $B$, then $|\beta| \leqq r$. Moreover $|\beta| = r$ implies $B^* = A$.*

*More precisely if $\beta = re^{i\varphi}$, $B = e^{i\varphi}DAD^{-1}$ where $D$ is a diagonal matrix such that $D^* = I$.* A proof of this last fact is given in ([**26**] p. 646 lines 4–11).

From this can be derived

THEOREM II: *Let $A \geqq 0$ be indecomposable. If the characteristic equation $\det(tI - A) = 0$ has altogether $k$ roots of absolute value $r$, the set of $n$ roots (with their orders of multiplicity) is invariant under a rotation about the origin through an angle of $2\pi/k$, but not under rotations through smaller angles. Moreover there is a permutation matrix $\Pi$ such that*

$$\text{(1)} \qquad \Pi A \Pi^{-1} = \begin{bmatrix} 0 & A_{12} & 0 & \cdot & 0 \\ 0 & 0 & A_{23} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdot & 0 \end{bmatrix}$$

*with square submatrices on the diagonal.*

[3] As an immediate consequence of 4 one obtains:

$$\mathrm{Min}_i \textstyle\sum_j a_{ij} \leqq r \leqq \mathrm{Max}_i \sum_j a_{ij}$$

*and one equality holds only if all row sums are equal (then they both hold).*

This is proved by increasing (resp. decreasing) some elements of $A$ so as to

Again the reader is referred to the excellent proof of Wielandt [**26**, p. 646–647].

If $k = 1$, the indecomposable matrix $A \geqslant 0$ is said to be *primitive.*

## 3. NONNEGATIVE SQUARE MATRICES

If $A$ is an $n \cdot n$ matrix, there clearly exists a permutation matrix $\Pi$ such that

$$\Pi A \Pi^{-1} = \begin{bmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_H \end{bmatrix}$$

where the $A_h$ are square submatrices on the diagonal and every $A_h$ is either indecomposable or a $1 \cdot 1$ matrix.

The properties of $A$ will therefore be easily derived from those of the $A_h$. For example $\det(tI - A) = \prod_{h=1}^{H} \det(tI - A_h)$ and Theorem I gives

THEOREM I*: *If $A \geqq 0$ is a square matrix, then*

1. *$A$ has a characteristic root $r \geqslant 0$ such that*
2. *to $r$ can be associated an eigen-vector $x_0 \geqslant 0$;*
3. *if $\alpha$ is any characteristic root of $A$, $| \alpha | \leqq r$;*
4. *$r$ does not decrease when an element of $A$ increases.*

Let $r_h$ be the maximal nonnegative characteristic root of $A_h$, we take $r = \operatorname{Max}_h r_h$; 1–3–4 are then immediate. To prove 2 we consider a sequence $A_\iota$ of $n \cdot n$ matrices converging to $A$ such that for all $\iota$ $A_\iota > 0$. Let $r_\iota$ be the maximal positive characteristic root of $A_\iota$, $x_\iota > 0$ its associated eigen-vector so chosen that $x_\iota \in S$, the fundamental simplex of $R^n$. Clearly $r_\iota$ tends to $r$. Let us then select $x_0 \in S$ a limit point of the set $(x_\iota)$; thus there is a subsequence $x_{\iota'}$ converging to $x_0 \geqslant 0$ and for every $\iota'$, $A_{\iota'} x_{\iota'} = r_{\iota'} x_{\iota'}$, therefore $Ax_0 = rx_0$.

Statement 5 of Theorem I no longer holds, but 5.(a) becomes:

*If $B$ is a principal submatrix of $A$ and $\beta$ a characteristic root of $B$, $| \beta | \leqq r$.*

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make all row sums equal to

$$\operatorname{Max}_i \textstyle\sum_j a_{ij} \quad (\text{resp. } \operatorname{Min}_i \sum_j a_{ij}).$$

A similar result naturally holds for column sums.

The proof, almost identical, now rests on 4 of Theorem I*.[4]

## 3. PROPERTIES OF $sI - A$ FOR $s > r$

*In this section $A \geqq 0$ is an $n \cdot n$ matrix, and $r$ is its maximal nonnegative characteristic root.*

LEMMA*: *If for an $x > 0$, $Ax \leqq sx$ (resp. $\geqq$), then $r \leqq s$ (resp. $\geqq$). If for an $x \geqq 0$, $Ax < sx$ (resp. $>$), then $r < s$ (resp. $>$).*

The proofs of the four statements being practically identical, we present only the first one. Let $x_0 \geqslant 0$ be a characteristic vector of $A'$ associated with $r$ (2 of Theorem I*): $A'x_0 = rx_0$. $Ax \leqq sx$ with $x > 0$, therefore $x_0' Ax \leqq sx_0' x$ i.e., $rx_0' x \leqq sx_0' x$ and, since $x_0' x > 0$, $r \leqq s$.

We now derive two theorems (III* and III) from the study of the equation

$$(sI - A)x = y \tag{2}$$

THEOREM III*: $(sI - A)^{-1} \geqq 0$ *if and only if* $s > r$.

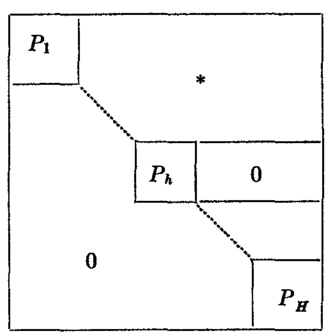
*Sufficiency.* Since $s > r$, (2) has a unique solution $x = (sI - A)^{-1}y$ for every $y$; we show that $y \geqq 0$ implies $x \geqq 0$.

If $x$ had negative components (2) could be given the form [by proper (identical) permutations of the rows and columns and partition]

$$\begin{bmatrix} sI - A_1 & -A_{12} \\ -A_{21} & sI - A_2 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = y$$

[4] A *stochastic $n \cdot n$ matrix $P$* is defined by $p_{ij} \geqslant 0$ for all $i$, $j$ and $\sum_j p_{ij} = 1$ for all $i$. Clearly 1 is a characteristic root of $P$ (take an eigen-vector with all components equal). 1 is therefore a root of some of the indecomposable matrices $P_1$, $P_2, \cdots, P_H$. Suppose that 1 is a root of $P_h$, it follows from footnote (3) that all row sums of $P_h$ are equal to 1, i.e.,

$\Pi P \Pi^{-1} =$




This remark makes many properties of stochastic matrices (the subject of the theory of finite Markov chains; see [6] and its references) ready consequences of the results of this article.

where $x_1 > 0$, $x_2 \geqq 0$, $y \geqq 0$. Therefore $-(sI - A_1)x_1 - A_{12}x_2 \geqq 0$, i.e., $-(sI - A_1)x_1 \geqq 0$ i.e., $A_1 x_1 \geqq s x_1$. From the Lemma* the maximal nonnegative characteristic root of $A_1$, $r_1 \geqq s$, a contradiction to the fact that $r \geqq r_1$ (see end of Section 3) and $s > r$.

*Necessity.* Since $(sI - A)^{-1} \geqq 0$, to a $y > 0$ corresponds an $x \geqq 0$. Therefore from $sx - Ax = y$ follows $Ax < sx$ and, by the Lemma*, $r < s$.

If $A$ is indecomposable these results can be sharpened to the

LEMMA: *Let $A$ be indecomposable.*
*If for an $x \geqslant 0$, $Ax \leqq sx$ (resp. $\geqq$), then $r \leqq s$ (resp. $\geqq$).*
*If for an $x \geqq 0$, $Ax \leqslant sx$ (resp. $\geqslant$), then $r < s$ (resp. $>$).*

The proofs, practically identical to those of the Lemma*, use a *positive* characteristic vector of $A'$ associated with $r$. One of these statements indeed has already been proved in 3–4 of Theorem I.

THEOREM III: *Let $A$ be indecomposable. $(sI - A)^{-1} > 0$ if and only if $s > r$.*

*Sufficiency.* We show that $y \geqslant 0$ implies $x > 0$. It is already known (from the proof of sufficiency of Theorem III*) that $x \geqslant 0$. If $x$ had zero components, (2) could be given the form

$$\begin{bmatrix} sI - A_1 & -A_{12} \\ -A_{21} & sI - A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y$$

where $x_1 = 0$, $x_2 > 0$, $y \geqslant 0$. Therefore $-A_{12}x_2 \geqq 0$, and, since $x_2 > 0$, $A_{12} = 0$ violating the indecomposability of $A$.

The *Necessity* has already been proved since $(sI - A)^{-1} > 0$ implies $(sI - A)^{-1} \geqq 0$.[5]

THEOREM IV: *The principal minors of $sI - A$ of orders $1, \cdots, n$ are all positive if and only if $s > r$.*

*Sufficiency.* $\det(tI - A)$ cannot vanish for $t > r$, thus $\det(sI - A) > 0$ for $s > r$. Similarly, the maximal nonnegative characteristic root of a principal submatrix of $A$ is not larger than $r$ (see end of Section 3); it is therefore smaller than $s$, and the corresponding minor of $sI - A$ is positive.

*Necessity.* The derivative of order $m(<n)$ of $\det(tI - A)$ with respect to $t$, for $t = s$, is a sum of principal minors of order $(n - m)\cdot(n - m)$

[5] It is worth [12] emphasizing a result obtained in the proof of necessity of Theorem III*.

*Remark.* Let $A \geqq 0$ (resp. $A \geqq 0$ indecomposable) be a square matrix. If for a $y > 0$ (resp. $y \geqslant 0$), $x \geqq 0$, then $(sI - A)^{-1} \geqq 0$ [resp. $(sI - A)^{-1} > 0$].

The proof for indecomposable matrices uses the Lemma instead of the Lemma*.

of $sI - A$ and thus is positive. As its derivatives of all orders (0, 1, $\cdots$, $n - 1$, $n$) are positive for $t = s$, the polynomial $\det(tI - A)$ can vanish for no $t \geqq s$ i.e., $s > r$.[6, 7]

Since a square matrix with nonpositive (resp. negative) off-diagonal elements can always be given the form $sI - A$ where $A \geqq 0$ (resp. $>0$), many of the results of Arrow [**2**], Bray [**3**], Chipman [**4**], [**5**], Georgescu-Roegen [**10**], Goodwin [**11**], Hawkins and Simon [**12**], Metzler [**15**] to [**18**], Morishima[8] [**19**], Mosak [**20**], Solow [**24**] are contained in the above.

## 5. CONVERGENCE[9] OF $A^p$

THEOREM V: *Let $A$ be a $n \cdot n$ complex matrix. The sequence $A$, $A^2$, $\cdots$, $A^p$, $\cdots$ of its powers converges if and only if*

[6] Georgescu-Roegen [**10**] stated a result whose counterpart here would be the following theorem (stronger than IV): *The $n$ northwest principal minors of $sI - A$ of orders* 1, $\cdots$, *$n$ are all positive if and only if $s > r$.*

[7] We give a last property useful in economics [**17**], [**18**].

*Theorem. Let $A > 0$ be a square matrix and let $C_{ij}$ be the cofactor of the $i^{\text{th}}$ row, $j^{\text{th}}$ column element of $sI - A$. If $s > \sum_j a_{ij}$ for all $i$, then $i \neq j$ implies $C_{ii} > C_{ij}$.*

Let us define the matrix $B = (b_{pq})$ as follows:

$$b_{pq} = a_{pq} \text{ if } p \neq i;\ b_{iq} = 0 \text{ if } i \neq q \neq j;\ b_{ii} = s/2 = b_{ij}.$$

$B$ is indecomposable, moreover $\sum_q b_{iq} = s$, $\sum_q b_{pq} < s$ for $p \neq i$. Therefore (see footnote 3) the maximal positive characteristic root of $B$, $r(B) < s$. Thus $\det(sI - B) > 0$; a development according to the $i^{\text{th}}$ row yields:

$$s/2C_{ii} - s/2C_{ij} > 0.$$

[8] Morishima studies square matrices $A$ such that for a permutation matrix $\Pi$,

$$\Pi A \Pi^{-1} = A_\pi = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \geqq 0$ and $A_{22} \geqq 0$ are square, $A_{12} \leqq 0$, $A_{21} \leqq 0$. The relation

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{bmatrix} = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}$$

shows how properties of $A_\pi$ can be immediately derived from those of the *nonnegative* matrix

$$A_\pi^S = \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}$$

In particular $A_\pi$ and $A_\pi^S$ have the same characteristic roots.

[9] The Cesaro convergence of $A^p$ i.e., the convergence of $\dfrac{1}{p}(A + A^2 + \cdots + A^p)$ can be studied in exactly the same fashion.

1. *each characteristic root $\alpha$ of $A$ satisfies either $|\alpha| < 1$ or $\alpha = 1$;*
2. *when the second case occurs the order of multiplicity of the root 1 equals the dimension of the eigen-vector space associated with that root.*

There is a nonsingular complex matrix $T$ such that

$$A_T = TAT^{-1} = \begin{bmatrix} J_1 & & & & & \\ & \cdot & & 0 & & \\ & & \cdot & & & \\ & & & J_i & & \\ & 0 & & & \cdot & \\ & & & & & \cdot \\ & & & & & & J_q \end{bmatrix} \text{ where}$$

$$J_i = \begin{bmatrix} \alpha_\iota & 1 & & & & & \\ & \cdot & \cdot & & & 0 & \\ & & \cdot & 1 & & & \\ & & & \alpha_\iota & 1 & & \\ & 0 & & & \cdot & \cdot & \\ & & & & & \cdot & 1 \\ & & & & & & \alpha_\iota \end{bmatrix}$$

is a square matrix on the diagonal and $\alpha_\iota$ a characteristic root of $A$. To every root $\alpha_\iota$ corresponds at least one $J_i$ (for this reduction of $A$ to its Jordan canonical form see for example [25]).

Since

$$TA^pT^{-1} = \begin{bmatrix} J_1^p & & & & & \\ & \cdot & & & & \\ & & \cdot & & 0 & \\ & & & J_i^p & & \\ & 0 & & & \cdot & \\ & & & & & \cdot \\ & & & & & & J_q^p \end{bmatrix},$$

$A^p$ converges if and only if every one of the $J_i^p$ converges. Let us therefore study one of them; for this purpose we drop the subscripts $i$ and $\iota$.

$J$ is a $k \cdot k$ matrix of the form $J = \alpha I + M$ where $M = (m_{st})$: $m_{st} = 1$ if $t = s + 1$, $m_{st} = 0$ otherwise.

$$J^p = \alpha^p I + \binom{p}{1}\alpha^{p-1}M + \cdots \binom{p}{k-1}\alpha^{p-k+1}M^{k-1}.$$

It is easily seen that for $M^h$, $m_{st}^{(h)} = 1$ if $t = s + h$ and $m_{st}^{(h)} = 0$ otherwise. Thus $M^h = 0$ if $h \geqq k$; also the nonzero elements of $M^h$ and $M^{h'}$

$(h \neq h')$ never occur in the same place so $J^p$ converges if and only if every term of the right-hand sum does.

The first term shows that necessarily either $|\alpha| < 1$ or $\alpha = 1$.

If $|\alpha| < 1$, every term tends to zero and $J^p$ converges.

If $\alpha = 1$ no term other than the first one converges and necessarily $k = 1$ i.e., $J = [1]$; clearly $J^p$ converges in this case.

We wish, however, to obtain for this necessary and sufficient condition of convergence an expression independent of a reduction to Jordan canonical form.

Consider then an arbitrary $n \cdot n$ complex matrix $A$ and let $\mathcal{I}$ be the set of $i$ for which $J_i$ corresponds to the root 1. The equation $A_T x = x$, in which $x$ is partitioned in the same way as $A_T$, yields $J_i x_i = x_i$ for all $i$, i.e.,

if $i \notin \mathcal{I}$, $\quad x_i = 0$

if $i \in \mathcal{I}$, $\quad$ all components of $x_i$ but the first one equal zero.

Thus the dimension of the eigen-vector space associated with the root 1 equals the number of elements of $\mathcal{I}$. This number, in turn, equals the order of multiplicity of the root 1 if and only if $J_i = [1]$ for all $i \in \mathcal{I}$.

The above theorem and method of proof were first given by Oldenburger **[21]**.

We now assume that the limit $C$ exists and give its expression. If 1 is not a characteristic root of $A$, $C = 0$. Let therefore 1 be a root of $A$ of order $\mu$. Thus $x$ (resp. $y$), an eigen-vector of $A$ (resp. $A'$) associated with the root 1, has the form $x = X\xi$ (resp. $y = Y\eta$) where $X$ (resp. $Y$) is a $n \cdot \mu$ matrix of rank $\mu$ and $\xi$ (resp. $\eta$) is a $\mu \cdot 1$ matrix. For an arbitrary $x$ the relation $AA^p x = A^{p+1} x$ gives in the limit $ACx = Cx$ i.e., $Cx = X\xi(x)$. To determine $\xi(x)$ we remark that $Y' = Y'A$ i.e., by iteration $Y' = Y'A^p$, and therefore $Y' = Y'C$; thus $Y'x = Y'Cx = Y'X\xi(x)$. $Y'X$ is a nonsingular[10] $\mu \cdot \mu$ matrix i.e., $\xi(x) = (Y'X)^{-1}Y'x$. Finally for all $x$, $Cx = X(Y'X)^{-1}Y'x$ i.e., $C = X(Y'X)^{-1}Y'$.

COROLLARY: *Let $A \geqq 0$ be indecomposable and 1 be its maximal positive characteristic root. The sequence $A^p$ converges if and only if $A$ is primitive.*

The necessity is obvious. The sufficiency follows from the fact that 1 is a simple root.

Let then $x_0 > 0$ (resp. $y_0 > 0$) be an eigen-vector of $A$ (resp. $A'$)

[10] $X_T = TX$ (resp. $Y'_T = Y'T^{-1}$) plays for $A_T$ the same role as $X$ (resp. $Y'$) does for $A$. Moreover $Y'X = Y'_T X_T$. The right-hand matrix is nonsingular for the form taken by the Jordan matrix $A_T$ in the convergence case implies that the eigen-vector space $U$ generated by $X_T$ is identical with the eigen-vector space $V$ generated by $Y_T$. Thus $Y'_T X_T \xi = 0$ implies $X_T \xi = 0$ (there is no vector different from zero in $U$ perpendicular to $V$ i.e., to $U$) therefore $\xi = 0$ since the rank of $X_T$ is $\mu$.

associated with the root 1, the limit $C$ of $A^p$ has the simple expression $C = x_0 y_0'/y_0' x_0$.

Clearly $C > 0$, thus if the indecomposable matrix $A \geqq 0$ is primitive, there is a positive integer $m$ such that $A^p > 0$ when $p \geqq m$. The converse is an immediate consequence of the decomposition (1) of Theorem II.[11]

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[11] This characterization of a primitive matrix, due to Frobenius, is typical of the purely combinatorial properties of the nonnegative square matrix $A$ (used for example in the theory of communication networks): the smallest $m$ satisfying the above condition is independent of the values of the nonzero elements of $A$ as long as they stay positive.

The development of combinatorial techniques adapted to the treatment of such properties is the subject of [**13**].

[12] In these references, we have tried to cover the economic literature with reasonable completeness. No such attempt has been made for the mathematical literature of which only a few essential papers have been quoted.

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