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EXISTENCE OF AN EQUILIBRIUM FOR A COMPETITIVE ECONOMY

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A. Wald has presented a model of production and a model of exchange and proofs of the existence of an equilibrium for each of them. Here proofs of the existence of an equilibrium are given for an *integrated* model of production, exchange and consumption. In addition the assumptions made on the technologies of producers and the tastes of consumers are significantly weaker than Wald's. Finally a simplification of the structure of the proofs has been made possible through use of the concept of an abstract economy, a generalization of that of a game.

INTRODUCTION

L. WALRAS [**24**] first formulated the state of the economic system at any point of time as the solution of a system of simultaneous equations representing the demand for goods by consumers, the supply of goods by producers, and the equilibrium condition that supply equal demand on every market. It was assumed that each consumer acts so as to maximize his utility, each producer acts so as to maximize his profit, and perfect competition prevails, in the sense that each producer and consumer regards the prices paid and received as independent of his own choices. Walras did not, however, give any conclusive arguments to show that the equations, as given, have a solution.

The investigation of the existence of solutions is of interest both for descriptive and for normative economics. Descriptively, the view that the competitive model is a reasonably accurate description of reality, at least for certain purposes, presupposes that the equations describing the model are consistent with each other. Hence, one check on the empirical usefulness of the model is the prescription of the conditions under which the equations of competitive equilibrium have a solution.

Perhaps as important is the relation between the existence of solutions to a competitive equilibrium and the problems of normative or welfare economics. It is well known that, under suitable assumptions on the preferences of consumers and the production possibilities of producers, the allocation of resources in a competitive equilibrium is optimal in the sense of Pareto (no redistribution of goods or productive resources can improve the position of one individual without making at least one other individual worse off), and conversely every Pareto-optimal allocation of resources can be realized by a competitive equilibrium (see for example Arrow [**1**], Debreu [**4**] and the references given there). From the

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point of view of normative economics the problem of existence of an equilibrium for a competitive system is therefore also basic.

To study this question, it is first necessary to specify more carefully than is generally done the precise assumptions of a competitive economy. The main results of this paper are two theorems stating very general conditions under which a competitive equilibrium will exist. Loosely speaking, the first theorem asserts that if every individual has initially some positive quantity of every commodity available for sale, then a competitive equilibrium will exist. The second theorem asserts the existence of competitive equilibrium if there are some types of labor with the following two properties: (1) each individual can supply some positive amount of at least one such type of labor; and (2) each such type of labor has a positive usefulness in the production of desired commodities. The conditions of the second theorem, particularly, may be expected to be satisfied in a wide variety of actual situations, though not, for example, if there is insufficient substitutability in the structure of production.

The assumptions made below are, in several respects, weaker and closer to economic reality than A. Wald's [**23**]. Unlike his models, ours presents an integrated system of production and consumption which takes account of the circular flow of income. The proof of existence is also simpler than his. Neither the uniqueness nor the stability of the competitive solution is investigated in this paper. The latter study would require specification of the dynamics of a competitive market as well as the definition of equilibrium.

Mathematical techniques are set-theoretical. A central concept is that of an abstract economy, a generalization of the concept of a game.

The last section contains a detailed historical note.

1. STATEMENT OF THE FIRST EXISTENCE THEOREM FOR A COMPETITIVE EQUILIBRIUM

1.0. In this section, a model of a competitive economy will be described, and certain assumptions will be made concerning the production and consumption units in the economy. The notion of equilibrium for such an economy will be defined, and a theorem stated about the existence of this equilibrium.

1.1. We suppose there are a finite number of distinct commodities (including all kinds of services). Each commodity may be bought or sold for delivery at one of a finite number of distinct locations and one of a finite number of future time points. For the present purposes, the same commodity at two different locations or two different points of time will be regarded as two different commodities. Hence, there are altogether a finite number of commodities (when the concept is used in the extended sense of including spatial and temporal specifications). Let the number of commodities be l; the letter h, which runs from 1 to l, will designate different commodities.

1.2.0. The commodities, or at least some of them, are produced in *production units* (e.g., firms). The number of production units will be assumed to be a finite number n; different production units will be designated by the letter j. Certain basic assumptions will be made about the technological nature of the production process; before stating them, a few elements of vector and set notation will be given.

1.2.1. $x \geqq y$ means $x_h \geqq y_h$ for each component h;

$x \geqslant y$ means $x \geqq y$ but not $x = y$;

$x > y$ means $x_h > y_h$ for each component h.

R^l is the Euclidean space of l dimensions, i.e., the set of all vectors with l components.

0 is the vector all of whose components are 0.

$\{x \mid \ \}$, where the blank is filled in by some statement involving x, means the set of all x's for which that statement is true.

$\Omega = \{x \mid x \in R^l,\ x \geqq 0\}$.

For any set of vectors A, let $-A = \{x \mid -x \in A\}$.

For any sets of vectors A_ι ($\iota = 1, \cdots, \nu$), let

$$\sum_{\iota=1}^{\nu} A_\iota = \left\{x \mid x = \sum_{\iota=1}^{\nu} x_\iota \text{ for some } x_1, \cdots, x_\nu, \text{ where } x_\iota \in A_\iota\right\}.$$

1.2.2. For each production unit j, there is a set Y_j of possible production plans. An element y_j of Y_j is a vector in R^l, the hth component of which, y_{hj}, designates the output of commodity h according to that plan. Inputs are treated as negative components. Let $Y = \sum_{j=1}^{n} Y_j$; then the elements of Y represent all possible input-output schedules for the production sector as a whole. The following assumptions about the sets Y_j will be made:

I.a. *Y_j is a closed convex subset of R^l containing* 0 ($j = 1, \cdots, n$).

I.b. $Y \cap \Omega = 0$.

I.c. $Y \cap (-Y) = 0$.

Assumption I.a. implies non-increasing returns to scale, for if $y_j \in Y_j$ and $0 \leqq \lambda \leqq 1$, then $\lambda y_j = \lambda y_j + (1 - \lambda)0 \in Y_j$, since $0 \in Y_j$ and Y_j is convex. If we assumed in addition the additivity of production possibility vectors, Y_j would be a convex cone, i.e., constant returns to scale would prevail. If, however, we assume that among the factors used by a firm are some which are not transferable in the market and so do not appear in the list of commodities, the production possibility vectors, if we consider only the components which correspond to marketable commodities, will not satisfy the additivity axiom.[2] The closure of Y_j merely says that if vectors arbitrarily close to y_j are in Y_j, then so is y_j. Naturally, $0 \in Y_j$, since a production unit can always go out of existence. It is to be noted that the list of production units should include not

[2] The existence of factors private to the firm is the standard justification in economic theory for diminishing returns to scale. See, e.g., the discussion of "free rationed goods" by Professor Hart [**9**], pp. 4, 38; also, Hicks [**10**], pp. 82–83; Samuelson [**18**], pp. 84.

only actually existing ones but those that might enter the market under suitable price conditions.

I.b. says that one cannot have an aggregate production possibility vector with a positive component unless at least one component is negative. I.e., it is impossible to have any output unless there is some input.

I.c. asserts the impossibility of two production possibility vectors which exactly cancel each other, in the sense that the outputs of one are exactly the inputs of the other. The simplest justification for I.c. is to note that some type of labor is necessary for any production activity, while labor cannot be produced by production-units. If $y \in Y$, and $y \neq 0$, then $y_h < 0$ for some h corresponding to a type of labor, so that $-y_h > 0$, (here, y_h is the hth component of the vector y). Since labor cannot be produced, $-y$ cannot belong to Y.[3]

Since commodities are differentiated according to time as well as physical characteristics, investment plans which involve future planned purchases and sales are included in the model of production used here.

1.2.3. The preceding assumptions have related to the *technological* aspects of production. Under the usual assumptions of perfect competition, the *economic* motivation for production is the maximization of profits taking prices as given. One property of the competitive equilibrium must certainly be

1. *y_j^* maximizes $p^* \cdot y_j$ over the set Y_j, for each j.*

Here, the asterisks denote equilibrium values, and p^* denotes the equilibrium price vector.[4] The above condition is the first of a series which, taken together, define the notion of *competitive equilibrium.*

1.3.0. Analogously to production, we assume the existence of a number of *consumption units*, typically families or individuals but including also institutional consumers. The number of consumption units is m; different consumption units will be designated by the letter i. For any consumption unit i, the vector in R^l representing its consumption will be designated by x_i. The hth component, x_{hi}, represents the quantity of the hth commodity consumed by the ith individual. For any commodity, other than a labor service supplied by the individual, the rate of consumption is necessarily non-negative. For labor services, the amount supplied may be regarded as the negative of the rate of "consumption," so that $x_{hi} \leqq 0$ if h denotes a labor service. Let $\mathfrak{L}$ denote the set of commodities which are labor services. For any $h \in \mathfrak{L}$, we may suppose there is some upper limit to the amount supplied, i.e., a lower limit to x_{hi}, since, for example, he cannot supply more than 24 hours of labor in a day.

II. *The set of consumption vectors X_i available to individual i $(= 1, \cdots, m)$ is a closed convex subset of R^l which is bounded from below; i.e., there is a vector ξ_i such that $\xi_i \leqq x_i$ for all $x_i \in X_i$.*

[3] The assumptions about production used here are a generalization of the "linear programming" assumptions. The present set is closely related to that given by Professor Koopmans [**12**]. In particular, I.b. is Koopmans' "Impossibility of the Land of Cockaigne," I.c. is "Irreversibility"; see [**12**], pp. 48–50.

[4] For any two vectors u, v, the notation $u \cdot v$ denotes their inner product, i.e., $\sum_h u_h v_h$. Since y_{hj} is positive for outputs, negative for inputs, $p^* \cdot y_j$ denotes the profit from the production plan y_j at prices p^*.

The set X_i includes all consumption vectors among which the individual could conceivably choose if there were no budgetary restraints. Impossible combinations of commodities, such as the supplying of several types of labor to a total amount of more than 24 hours a day or the consumption of a bundle of commodities insufficient to maintain life, are regarded as excluded from X_i.

1.3.1. As is standard in economic theory, the choice by the consumer from a given set of alternative consumption vectors is supposed to be made in accordance with a preference scale for which there is a utility indicator function $u_i(x_i)$ such that $u_i(x_i) \geqq u_i(x_i')$ if and only if x_i is preferred or indifferent to x_i' according to individual i.

III.a. *$u_i(x_i)$ is a continuous function on X_i.*

III.b. *For any $x_i \in X_i$, there is an $x_i' \in X_i$ such that $u_i(x_i') > u_i(x_i)$.*

III.c. *If $u_i(x_i) > u_i(x_i')$ and $0 < t < 1$, then $u_i[tx_i + (1 - t)x_i'] > u_i(x_i')$.*

III.a. is, of course, a standard assumption in consumers' demand theory. It is usually regarded as a self-evident corollary of the assumption that choices are made in accordance with an ordering, but this is not accurate. Actually, for X_i a subset of a Euclidean space (as is ordinarily taken for granted), the existence of a continuous utility indicator is equivalent to the following assumption: for all x_i', the sets $\{x_i \mid x_i \in X_i$ and x_i' preferred or indifferent to $x_i\}$ and $\{x_i \mid x_i \in X_i$ and x_i preferred or indifferent to $x_i'\}$ are closed (in X_i); see Debreu [**6**]. The assumption amounts to a continuity assumption on the preference relation.

III.b. assumes that there is no point of saturation, no consumption vector which the individual would prefer to all others. It should be noted that this assumption can be weakened to state merely that no consumption vector attainable with the present technological and resource limitations is a point of saturation. Formally, the revised assumption would read,

III′.b. *for any $x_i \in \hat{X}_i$, there is an $x_i' \in X_i$ such that $u_i(x_i') > u_i(x_i)$,* where $\hat{X}_i$ has the meaning given it in 3.3.0. below.

III.c. corresponds to the usual assumption that the indifference surfaces are convex in the sense that the set $\{x_i \mid x_i \in X_i$ and $u_i(x_i) \geqq \alpha\}$ is a convex set for any fixed real number α.

The last statement, which asserts the *quasi-concavity* of the function $u_i(x_i)$ is indeed implied by III.c. (but is obviously weaker). For suppose x^1 and x^2 are such that $u_i(x^n) \geqq \alpha$ $(n = 1, 2)$ and $0 < t < 1$. Let $x^3 = tx^1 + (1 - t)x^2$. Without loss of generality, we may suppose that $u_i(x^1) \geqq u_i(x^2)$. If the strict inequality holds, then $u_i(x^3) > u_i(x^2) \geqq \alpha$, by III.c. Suppose now $u_i(x^1) = u_i(x^2)$, and suppose $u_i(x^3) < u_i(x^2)$. Then, from III.a., we can find x^4, a strict convex combination of x^3 and x^1, such that $u_i(x^3) < u_i(x^4) < u_i(x^1) = u_i(x^2)$. The point x^3 can be expressed as a strict convex combination of x^4 and x^2; since $u_i(x^4) < u_i(x^2)$, it follows from III.c. that $u_i(x^3) > u_i(x^4)$, which contradicts the inequality just stated. Hence, the supposition that $u_i(x^3) < u_i(x^2)$ is false, so that $u_i(x^3) \geqq u_i(x^2) \geqq \alpha$.

Actually, it is customary in consumers' demand theory to make a slightly stronger assumption than the quasi-concavity of $u_i(x_i)$, namely, that $u_i(x_i)$ is *strictly* quasi-concave, by which is meant that if $u_i(x_i) \geqq u_i(x_i')$ and $0 < t < 1$, then $u_i[tx_i + (1 - t)x_i'] > u_i(x_i')$. This is equivalent to saying that the indifference surfaces do not contain any line segments, which again is equivalent to the assumption that for all sets of prices and incomes, the demand functions, which give the coordinates of the consumption vector which maximizes utility for a given set of prices and income, are single-valued. Clearly, strict quasi-concavity is a *stronger* assumption than III.c.[5]

1.3.2. We also assume that the ith consumption unit is endowed with a vector ζ_i of initial holdings of the different types of commodities available and a contractual claim to the share α_{ij} of the profit of the jth production unit for each j.

IV.a. $\zeta_i \in R^l$; *for some* $x_i \in X_i$, $x_i < \zeta_i$;

IV.b. *for all* i, j, $\alpha_{ij} \geqq 0$; *for all* j, $\sum_{i=1}^{m} \alpha_{ij} = 1$.

The component ζ_{hi} denotes the amount of commodity h held initially by individual i. We may extend this to include all debts payable in terms of commodity h, debts owed to individual i being added to ζ_{hi} and debts owed by him being deducted. Thus, for $h \in \mathfrak{L}$, ζ_{hi} would differ from 0 only by the amount of debts payable in terms of that particular labor service. (It is not necessary that the debts cancel out for the economy as a whole; thus debts to or from foreigners may be included, provided they are payable in some commodity.)

The second half of IV.a. asserts in effect that every individual could consume out of his initial stock in some feasible way and still have a positive amount of *each* commodity available for trading in the market.[6] This assumption is clearly unrealistic. However, the necessity of this assumption or some parallel one for the validity of the existence theorem points up an important principle; to have equilibrium, it is necessary that each individual possess some asset or be capable of supplying some labor service which commands a positive price at equilibrium. In IV.a, this is guaranteed by insisting that an individual be capable of supplying something of each commodity; at least one will be valuable (in the sense of having a price greater than zero) at equilibrium since there will be at least one positive price at equilibrium, as guaranteed by the assumptions about the nature of the price system made in 1.4 below. A much weaker assumption of the same type is made in Theorem II.

1.3.3. The basic economic motivation in the choice of a consumption vector is that of maximizing utility among all consumption vectors which satisfy the budget restraint, i.e., whose cost at market prices does not exceed the individual's income. His income, in turn, can be regarded as having three components: wages, receipts from sales of initially-held stocks of commodities and

[5] The remarks in the text show that strict quasi-concavity implies III.c., while III.c. implies quasi-concavity. To show that strict quasi-concavity is actually a stronger assumption than III.c., we need only exhibit a utility function satisfying III.c. but not strictly quasi-concave. The function $u_i(x_i) = \sum_{h=1}^{l} x_{hi}$ has these properties.

[6] This assumption plays the same role as the one made by Professor von Neumann in his study of a dynamic model of production [16] that each commodity enters into every production process either as an input or as an output.

claims expressible in terms of them, and dividends from the profits of production units. This economic principle must certainly hold for equilibrium values of prices and of the profits of the production units.

2. x_i^* *maximizes* $u_i(x_i)$ *over the set* $\{x_i \mid x_i \epsilon X_i, p^* \cdot x_i \leqq p^* \cdot \zeta_i + \sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^*\}$.

This, like Condition 1 in 1.2.3., is a condition of a competitive equilibrium. Because of the definition of labor services supplied as negative components of x_i, $p^* \cdot x_i$ represents the excess of expenditures on commodities over wage income. The term $p^* \cdot \zeta_i$ represents the receipts from the sale of initially-held commodities. The term $\sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^*$ denotes the revenue of consumption unit i from dividends.

1.4.0. It remains to discuss the system of prices and the meaning of an equilibrium on any market.

3. $p^* \epsilon P = \{p \mid p \epsilon R^l, p \geqq 0, \sum_{h=1}^{l} p_h = 1\}$.

Condition 3 basically expresses the requirement that prices be nonnegative and not all zero. Without any loss of generality, we may normalize the vector p^* by requiring that the sum of its coordinates be 1, since all relations are homogeneous (of the first order) in p.

1.4.1. Conditions 1 and 2 are the conditions for the equilibrium of the production and consumption units, respectively, for given p^*. Hence, the supply and demand for all commodities is determined as a function of p (not necessarily single-valued) if we vary p and at the same time instruct each production and consumption unit to behave as if the announced value of p were the equilibrium value. The market for any commodity is usually considered to be in equilibrium when the supply for that commodity equals the demand; however, we have to consider the possibility that at a zero price, supply will exceed demand. This is the classical case of a free good.

Let

$$x = \sum_i x_i,\ y = \sum_j y_j,\ \zeta = \sum_i \zeta_i,\ z = x - y - \zeta.$$

The vector z has as its components the excess of demand over supply (including both produced and initially-available supply) for the various commodities.

4. $z^* \leqq 0$, $p^* \cdot z^* = 0$.

Condition 4 expresses the discussion of the preceding paragraph. We have broadly the dynamic picture of the classical "law of supply and demand"; see, e.g., [**18**], p. 263. That is, the price of a commodity rises if demand exceeds supply, falls if supply exceeds demand. Equilibrium is therefore incompatible with excess demand on any market, since price would simply rise; hence the first part of Condition 4 for equilibrium is justified. An excess of supply over demand drives price down, but, in view of Condition 3, no price can be driven below 0. Hence, $z_h^* < 0$ for some commodity h is possible, but only if $p_h^* = 0$.

Since $p_h^* \geqq 0$ for all h and $z_h^* \leqq 0$ for all h, $p^* \cdot z^* = \sum_h p_h^* z_h^*$ is a sum of non-positive terms. This sum can be zero if and only if $p_h^* z_h^* = 0$ for all h, i.e., either $z_h^* = 0$ or $z_h^* < 0$ and $p_h^* = 0$. Condition 4, therefore, sums up precisely the equilibrium conditions that are desired.[7]

1.4.2. In the preceding paragraph, it was implicitly assumed that for a commodity with a positive price the entire initial stock held by a consumption unit was available as a supply on the market along with amounts supplied by production and consumption units as a result of profit- and utility-maximization respectively (in this context, consumption by a consumption unit out of his own stocks counts both as supply on the market and as demand to the same numerical amount).

This becomes evident upon noting that each individual spends his entire *potential* income because of the absence of saturation (and since the model covers his entire economic life). More precisely, III.b. shows that there exists an x_i' such that

$$u_i(x_i') > u_i(x_i^*),$$

where x_i^* is the equilibrium value of x_i. Let t be an arbitrarily small positive number; by III.c., $u_i[tx_i' + (1 - t)\, x_i^*] > u_i(x_i^*)$. That is, in every neighborhood of x_i^*, there is a point of X_i preferred to x_i^*. From Condition 2,

$$p^* \cdot x_i^* \leqq p^* \cdot \zeta_i + \sum_j \alpha_{ij} p^* \cdot y_j^*.$$

Suppose the strict inequality held. Then we could choose a point of X_i for which the inequality still held and which was preferred to x_i^*, a contradiction of Condition 2.

$$p^* \cdot x_i^* = p^* \cdot \zeta_i + \sum_j \alpha_{ij} p^* \cdot y_j^*. \tag{1}$$

To achieve his equilibrium consumption plan, x_i^*, individual i must actually receive the total income given on the right-hand side. He cannot therefore withhold any initial holdings of commodity h from the market if $p_h^* > 0$.

1.5.0. DEFINITION: *A set of vectors* $(x_1^*, \cdots, x_m^*, y_1^*, \cdots, y_n^*, p^*)$ *is said to be a competitive equilibrium if it satisfies Conditions* 1–4.

1.5.1. THEOREM I. *For any economic system satisfying Assumptions I–IV, there is a competitive equilibrium.*

2. A LEMMA ON ABSTRACT ECONOMIES

2.0. In this section, the concept of an *abstract economy*, a generalization of that of a *game*, will be introduced, and a definition of equilibrium given. A lemma giv-

[7] The view that some commodities might be free goods because supply always exceeded demand goes back to the origins of marginal utility theory; see Menger [**13**], pp. 98–100. The critical importance of rephrasing the equilibrium condition for prices in the form of Condition 4 for the problem of the existence of a solution to the Walrasian equilibrium equations was first perceived by Schlesinger [**19**].

ing conditions for the existence of equilibrium of an abstract economy will be stated. The lemma is central in the proofs of the theorems stated in this paper.

2.1. Let there be ν subsets of R^l, $\mathfrak{A}_\iota(\iota = 1, \cdots, \nu)$. Let $\mathfrak{A} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_\nu$, i.e., $\mathfrak{A}$ is the set of ordered ν-tuples $a = (a_1, \cdots, a_\nu)$, where $a_\iota \in \mathfrak{A}_\iota$ for $\iota = 1, \cdots, \nu$. For each ι, suppose there is a real function f_ι defined over $\mathfrak{A}$. Let $\bar{\mathfrak{A}}_\iota = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \cdots \times \mathfrak{A}_{\iota-1} \times \mathfrak{A}_{\iota+1} \times \cdots \times \mathfrak{A}_\nu$, i.e., the set of ordered $(\nu - 1)$-tuples $\bar{a}_\iota = (a_1, \cdots, a_{\iota-1}, a_{\iota+1}, \cdots, a\nu)$, where $a_{\iota'} \in \mathfrak{A}_{\iota'}$ for each $\iota' \neq \iota$. Let $A_\iota(\bar{a}_\iota)$ be a function defining for each point $\bar{a}_\iota \in \bar{\mathfrak{A}}_\iota$ a subset of $\mathfrak{A}_\iota$. Then the sequence $[\mathfrak{A}_1, \cdots, \mathfrak{A}_\nu, f_1, \cdots, f_\nu, A_1(\bar{a}_1), \cdots, A_\nu(\bar{a}_\nu)]$ will be termed an *abstract economy.*

2.2. To motivate the preceding definition, consider first the special case where the functions $A_\iota(\bar{a}_\iota)$ are in fact constants, i.e., $A_\iota(\bar{a}_\iota)$ is a fixed subset of $\mathfrak{A}_\iota$, independent of $\bar{a}_\iota$; for simplicity, suppose that $A_\iota(\bar{a}_\iota) = \mathfrak{A}_\iota$. Then the following interpretation may be given: there are ν individuals; the ιth can choose any element $a_\iota \in \mathfrak{A}_\iota$; after the choices are made, the ιth individual receives an amount $f_\iota(a)$, where $a = (a_1, \cdots, a_\nu)$. In this case, obviously, the abstract economy reduces to a game.

In a game, the pay-off to each player depends upon the strategies chosen by all, but the domain from which strategies are to be chosen is given to each player independently of the strategies chosen by other players. An abstract economy, then, may be characterized as a generalization of a game in which the choice of an action by one agent affects both the pay-off and the domain of actions of other agents.

The need for this generalization in the development of an abstract model of the economic system arises from the special position of the consumer. His "actions" can be regarded as alternative consumption vectors; but these are restricted by the budget restraint that the cost of the goods chosen at current prices not exceed his income. But the prices and possibly some or all of the components of his income are determined by choices made by other agents. Hence, for a consumer, who is one agent in the economic system, the function $A_\iota(\bar{a}_\iota)$ must not be regarded as a constant.

2.3. In [**14**], Professor Nash has formally introduced the notion of an *equilibrium point* for a game.[8] The definition can easily be extended to an abstract economy (see Debreu [**5**], p. 888.)

DEFINITION: *a^* is an equilibrium point of $[\mathfrak{A}_1, \cdots, \mathfrak{A}_\nu, f_1, \cdots, f_\nu, A_1(\bar{a}_1), \cdots, A_\nu(\bar{a}_\nu)]$ if, for all $\iota = 1, \cdots, \nu$, $a_\iota^* \in A_\iota(\bar{a}_\iota^*)$ and $f_\iota(\bar{a}_\iota^*, a_\iota^*) = max_{a_\iota \in A_\iota(\bar{a}_\iota^*)} f_\iota(\bar{a}_\iota^*, a_\iota)$.*

Thus an equilibrium point is characterized by the property that each individual is maximizing the pay-off to him, given the actions of the other agents, over the set of actions permitted him in view of the other agents' actions.

2.4. We repeat here some definitions from [**5**], pp. 888–889.

The ***graph*** of $A_\iota(\bar{a}_\iota)$ is the set $\{a \mid a_\iota \in A_\iota(\bar{a}_\iota)\}$. This clearly generalizes to the multi-valued functions $A_\iota(\bar{a}_\iota)$ the ordinary definition of the graph of a function.

[8] Actually, the concept had been formulated by Cournot [**3**] in the special case of an oligopolistic economy, see pp. 80–81.

The function $A_\iota(\bar{a}_\iota)$ is said to be *continuous* at $\bar{a}_\iota^0$ if for every $a_\iota^0 \in A_\iota(\bar{a}_\iota^0)$ and every sequence $\{\bar{a}_\iota^n\}$ converging to $\bar{a}_\iota^0$, there is a sequence $\{a_\iota^n\}$ converging to a_ι^0 such that $a_\iota^n \in A_\iota(\bar{a}_\iota^n)$ for all n. Again, if $A_\iota(\bar{a}_\iota)$ were a single-valued function, this definition would coincide with the ordinary definition of continuity.

2.5. LEMMA: *If, for each ι, $\mathfrak{A}_\iota$ is compact and convex, $f_\iota(\bar{a}_\iota, a_\iota)$ is continuous on $\mathfrak{A}$ and quasi-concave*[9] *in a_ι for every $\bar{a}_\iota$, $A_\iota(\bar{a}_\iota)$ is a continuous function whose graph is a closed set, and, for every $\bar{a}_\iota$, the set $A_\iota(\bar{a}_\iota)$ is convex and non-empty, then the abstract economy $[\mathfrak{A}_1, \cdots, \mathfrak{A}_\nu, f_1, \cdots, f_\nu, A_1(\bar{a}_1), \cdots, A_\nu(\bar{a}_\nu)]$ has an equilibrium point.*

This lemma generalizes Nash's theorem on the existence of equilibrium points for games [**14**]. It is a special case of the Theorem in [**5**], when taken in conjunction with the Remark on p. 889.[10]

3. PROOF OF THEOREM I

3.1.0. We will here define an abstract economy whose equilibrium points will have all the properties of a competitive equilibrium. There will be $m + n + 1$ participants, the m consumption units, the n production units, and a fictitious participant who chooses prices, and who may be termed the *market participant.*

For any consumption unit i, let $\bar{x}_i$ denote a point in $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_m \times Y_1 \times \cdots \times Y_n \times P$, i.e., $\bar{x}_i$ has as components $x_{i'}(i' \neq i)$, $y_j(j = 1, \cdots, n)$, p. Define

$$A_i(\bar{x}_i) = \left\{x_i \,\middle|\, x_i \in X_i,\ p\cdot x_i \leqq p\cdot\zeta_i + \max\left[0, \sum_{j=1}^{n} \alpha_{ij}\, p\cdot y_j\right]\right\}.$$

We will then study the abstract economy $E = [X_1, \cdots, X_m, Y_1, \cdots, Y_n, P, u_1(x_1), \cdots, u_m(x_m), p\cdot y_1, \cdots, p\cdot y_n, p\cdot z, A_1(\bar{x}_1), \cdots, A_m(\bar{x}_m), Y_1, \cdots, Y_n, P]$. That is, each of the first m participants, the consumption units, chooses a vector x_i from X_i, subject to the restriction that $x_i \in A_i(\bar{x}_i)$, and receives a pay-off $u_i(x_i)$; the jth out of the next n participants, the production units, chooses a vector y_j from Y_j (unrestricted by the actions of other participants), and receives a pay-off $p\cdot y_j$; and the last agent, the market participant, chooses p from P (again the choice is unaffected by the choices of other participants), and receives $p\cdot z$. Here, z is defined as in 1.4.1. in terms of $x_i(i = 1, \cdots, m)$ and $y_j(j = 1, \cdots, n)$. The domains X_i, Y_j, P have been defined in 1.3.0., 1.2.2., 1.4.0., respectively.

3.1.1. Only two of the component elements of the abstract economy E call for special comment. One is the pay-off function of the market participant. Note that z is determined by x_i and y_j. Suppose the market participant does

[9] For the definition of a quasi-concave function, see 1.3.1. above.

[10] To see this, we need only remark that a compact convex set is necessarily a contractible polyhedron (the definition of a contractible polyhedron is given in [**5**] pp. 887–888), that the compactness of the graph of $A_\iota(\bar{a}_\iota)$ follows from its closure, as assumed here, and the compactness and hence boundedness of $\mathfrak{A}$ which contains the graph of $A_\iota(\bar{a}_\iota)$, and that the set $\{a_\iota \mid a_\iota \in A_\iota(\bar{a}_\iota), f_\iota(\bar{a}_\iota, a_\iota) = \max_{a'_\iota \in A_\iota(\bar{a}_\iota)} f_\iota(\bar{a}_\iota, a'_\iota)\}$ is, for any given $\bar{a}_\iota$, a convex and therefore contractible set when $f_\iota(\bar{a}_\iota, a_\iota)$ is quasi-concave in a_ι.

not maximize instantaneously but, taking other participants' choices as given, adjusts his choice of prices so as to increase his pay-off. For given z, $p \cdot z$ is a linear function of p; it can be increased by increasing p_h for those commodities for which $z_h > 0$, decreasing p_h if $z_h < 0$ (provided p_h is not already 0). But this is precisely the classical "law of supply and demand" (see 1.4.1. above), and so the motivation of the market participant corresponds to one of the elements of a competitive equilibrium. This intuitive comment is not, however, the justification for this *particular* choice of a market pay-off, that justification will be found in 3.2.[11]

3.1.2. In the definition of $A_i(\bar{x}_i)$, the expression $\sum_{j=1}^{n} \alpha_{ij} p \cdot y_j$ is replaced by $\max\,[0, \sum_{j=1}^{n} \alpha_{ij} p \cdot y_j]$. For arbitrary choices of p and y_j (within their respective domains, P and Y_j), it is possible that $\{x_i \mid x_i \in X_i, px_i \leqq p \cdot \zeta_i + \sum_{j=1}^{n} \alpha_{ij} p \cdot y_j\}$ is empty. To avoid this difficulty, we make the replacement indicated. Since, for some $x_i' \in X_i$, $\zeta_i \geqq x_i'$ (by Assumption IV.a. 1.3.2. above), $p \cdot \zeta_i \geqq p \cdot x_i'$, and

$$p \cdot \zeta_i + \max\left[0, \sum_{j=1}^{n} \alpha_{ij} p \cdot y_j\right] \geqq p \cdot \zeta_i \geqq p \cdot x_i',$$

so that $A_i(\bar{x}_i)$ is non-empty.

Of course, it is necessary to show that the substitution makes no difference *at equilibrium*. By definition of E-equilibrium (see 2.3. above), y_j^* maximizes $p^* \cdot y_j$ subject to the condition that $y_j \in Y_j$ (here asterisks denote E-equilibrium values). By Assumption I.a (see 1.2.2. above), $0 \in Y_j$; hence, in particular

(1) $$p^* \cdot y_j^* \geqq p^* \cdot 0 = 0.$$

By Assumption IV.b.; $\sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^* \geqq 0$, and $\max\,[0, \sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^*] = \sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^*$. Therefore,

$$A_i(\bar{x}_i^*) = \left\{x_i \mid x_i \in X_i,\, p^* \cdot x_i \leqq p^* \cdot \zeta_i + \sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^*\right\}.$$

From the definition of an equilibrium point for an abstract economy and the pay-off for a consumption unit,

(2) Condition 2 is satisfied at an equilibrium point of the abstract economy E.

3.2. Before establishing the existence of an equilibrium point for E, it will be shown that such an equilibrium point is also a competitive equilibrium in the sense of 1.5.0. It has already been shown that Condition 2 is satisfied, while Conditions 1 and 3 follow immediately from the definition of an equilibrium point and the pay-offs specified.

In 1.4.2., it was shown that equation (1) of that section followed from Condition 2, which we have already shown to hold here, and Assumptions III.b. and III.c. Sum over i, and recall that, from IV.b. $\sum_{i=1}^{m} \alpha_{ij} = 1$. Then, from the definition of z

(1) $$p^* \cdot z^* = 0.$$

[11] A concept similar to that of the present market pay-off is found in Debreu [4] sections 11, 12.

Let δ^h be the vector in which every component is 0, except the hth, which is 1. Then $\delta^h \epsilon P$ (see Condition 3, 1.4.0.). Hence, by definition of an equilibrium point,

$$0 = p^* \cdot z^* \geqq \delta^h \cdot z^* = z_h^*,$$

or,

$$z^* \leqq 0. \tag{2}$$

(1) and (2) together assert Condition 4. It has been shown that any equilibrium point of E satisfies Conditions 1–4 and hence is a competitive equilibrium. The converse is obviously also true.

3.3.0. Unfortunately, the Lemma stated in 2.5 is not directly applicable to E, since the action spaces are not compact.

Let

$$\hat{X}_i = \{x_i \mid x_i \epsilon X_i, \text{ there exist } x_{i'} \epsilon X_{i'} \text{ for each } i' \neq i \text{ and } y_j \epsilon Y_j \text{ for each } j \text{ such that } z \leqq 0\},$$

$$\hat{Y}_j = \{y_j \mid y_j \epsilon Y_j, \text{ there exist } x_i \epsilon X_i \text{ for each } i, y_{j'} \epsilon Y_{j'} \text{ for each } j' \neq j \text{ such that } z \leqq 0\}.$$

$\hat{X}_i$ is the set of consumption vectors available to individual i if he had complete control of the economy but had to take account of resource limitations. $\hat{Y}_j$ has a similar interpretation. We wish to prove that these sets are all bounded. It is clear that an E equilibrium x_i^* must belong to $\hat{X}_i$ and that an E equilibrium y_j^* must belong to $\hat{Y}_j$.

3.3.1. Suppose $\hat{Y}_1$ is unbounded. Then there exist sequences y_j^k, x_i^k such that

$$\lim_{k\to\infty} \left| y_1^k \right| = \infty, \qquad \sum_{j=1}^n y_j^k \geqq \sum_{i=1}^m x_i^k - \zeta, \qquad y_j^k \epsilon Y_j, \qquad x_i^k \epsilon X_i. \tag{1}$$

Let

$$\xi = \sum_{i=1}^m \xi_i .$$

Then, from Assumption II, $\sum_{i=1}^m x_i^k \geqq \xi$, so that

$$\sum_{j=1}^n y_j^k \geqq \xi - \zeta. \tag{2}$$

Let $\mu^k = \max_j \mid y_j^k \mid$; for k sufficiently large, $\mu^k \geqq 1$. From Assumption I.a., $(1/\mu^k)y_j^k + (1 - 1/\mu^k)0 \epsilon Y_j$. From (1) and (2),

$$\sum_{j=1}^n (y_j^k/\mu^k) \geqq (\xi - \zeta)/\mu^k; \qquad y_j^k/\mu^k \epsilon Y_j \text{ for } k \text{ sufficiently large}; \tag{3}$$

$$\lim_{k\to\infty} \mu^k = \infty; \qquad \mid y_j^k/\mu^k \mid \leqq 1.$$

From the last statement, a subsequence $\{k_q\}$ can be chosen so that for every j

(4) $$\lim_{q\to\infty} y_j^{k_q}/\mu^{k_q} = y_j^o.$$

From (3), (4), and the closure of Y_j (see Assumption I.a.),

(5) $$\sum_{j=1}^{n} y_j^o \geqq 0, \qquad \text{and} \qquad y_j^o \in Y_j.$$

From (5), $\sum_{j=1}^{n} y_j^o \in Y$. From Assumption I.b., $\sum_{j=1}^{n} y_j^o = 0$, or, for any given j',

(6) $$\sum_{j\neq j'} y_j^o = -y_{j'}^o.$$

Since $0 \in Y_j$ for all j, both the left-hand side and $y_{j'}^o$ belong to Y. The right hand side therefore belongs to both Y and $-Y$; by I.c., $y_{j'}^o = 0$ for any j'. From (4), then, the equality $| y_j^{k_q} | = \mu^{k_q}$, can hold for at most finitely many q for fixed j. But this is a contradiction since, from the definition of μ^{k_q}, the equality must hold for at least one j for each q, and hence for infinitely many q for some j. It has therefore been shown that $\hat{Y}_1$ is bounded, and, by the same argument,

(7) $\hat{Y}_j$ is bounded for all j.

3.3.2. Let $x_i \in \hat{X}_i$. By definition,

(1) $$\xi_i \leqq x_i \leqq \sum_{j=1}^{n} y_j - \sum_{i'\neq i} x_{i'} + \zeta, \qquad (x_{i'} \in X_{i'}, y_j \in Y_j)$$

By definition, again, it follows that $y_j \in \hat{Y}_j$ for all j; also $x_{i'} \geqq \xi_{i'}$.

$$\xi_i \leqq x_i \leqq \sum_{j=1}^{n} y_j - \sum_{i'\neq i} \xi_{i'} + \zeta, \qquad (y_j \in \hat{Y}_j).$$

From (7) in 3.3.1., the right-hand side is bounded.

(2) $\hat{X}_i$ is bounded for all i.

3.3.3. We can therefore choose a positive real number c so that the cube $C = \{x | \, | x_h | \leqq c \text{ for all } h\}$ contains in its *interior* all $\hat{X}_i$ and all $\hat{Y}_j$. Let $\tilde{X}_i = X_i \cap C$, $\tilde{Y}_j = Y_j \cap C$.

3.3.4. Now introduce a new abstract economy $\tilde{E}$, identical with E in 3.1., except that X_i is replaced by $\tilde{X}_i$ and Y_j by $\tilde{Y}_j$ everywhere. Let $\tilde{A}_i(\bar{x}_i)$ be the resultant modification of $A_i(\bar{x}_i)$ (See 3.1.0.). It will now be verified that all the conditions of the Lemma are satisfied for this new abstract economy.

From II and I.a., X_i and Y_j are closed convex sets; the set C is a compact convex set; therefore, $\tilde{X}_i$ and $\tilde{Y}_j$ are compact convex sets. P is obviously compact and convex.

For a consumption unit, the continuity and quasi-concavity of $u_i(x_i)$ are assured by III.a. and III.c. (see the discussion in 1.3.1.). For a production unit or the market participant, the continuity is trivial, and the quasi-concavity holds for any linear function.

For a production unit or the market participant, Y_j or P is a constant and therefore trivially continuous; the closure of the graph is simply the closure of $\tilde{\mathfrak{A}} = \tilde{X}_1 \times \cdots \times \tilde{X}_m \times \tilde{Y}_1 \times \cdots \times \tilde{Y}_n \times P$. The sets Y_j, P are certainly convex and non-empty.

For a consumption unit, the set $\tilde{A}_i(\bar{x}_i)$ is defined by a linear inequality in x_i (3.1.0.) and hence is certainly convex. For each i, let x_i' have the property $x_i' \leqq \zeta_i$, $x_i' \in X_i$ (see Assumption IV.a.); set $y_j' = 0$. Since $\sum_{i=1}^m x_i' - \sum_{j=1}^n y_j' - \zeta \leqq 0$, $x_i' \in \hat{X}_i$ for each i, by definition, and hence $x_i' \in C$. It was shown in 3.1.2. that $x_i' \in A_i(\bar{x}_i)$ for all $\bar{x}_i$; since $\tilde{A}_i(\bar{x}_i) = [A_i(\bar{x}_i)] \cap C$, $\tilde{A}_i(\bar{x}_i)$ contains x_i' and therefore is non-null.

Since the budget restraint is a weak inequality between two continuous functions of a, it is obvious that the graph of $\tilde{A}_i(\bar{x}_i)$ is closed.

3.3.5. It remains only to show that $\tilde{A}_i(\bar{x}_i)$ is continuous.

REMARK: *If* $p \cdot \zeta_i > \min_{x_i \in \tilde{X}_i} p \cdot x_i$, *then* $\tilde{A}_i(\bar{x}_i)$ *is continuous at the point* $\bar{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m, y_1, \cdots, y_n, p)$.

PROOF: Let $r_i = p \cdot \zeta_i + \max\,[0, \sum_{j=1}^n \alpha_{ij} p \cdot y_j]$. When $\bar{x}_i^k$ converges to $\bar{x}_i$, $\lim_{k\to\infty} p^k = p$, $\lim_{k\to\infty} r_i^k = r_i$. Consider a point $x_i \in \tilde{A}_i(\bar{x}_i)$; then,

$$(1) \qquad x_i \in \tilde{X}_i, \qquad p \cdot x_i \leqq r_i.$$

(a) If $p \cdot x_i < r_i$, then $p^k \cdot x_i < r_i^k$ for all k sufficiently large, and $x_i \in \tilde{A}_i(\bar{x}_i^k)$. Then we need only choose $x_i^k = x_i$ for all k sufficiently large. (See the definition of continuity in 2.4).

(b) If $p \cdot x_i = r_i$, choose x_i', by hypothesis, so that $x_i' \in \tilde{X}_i$, $p \cdot x_i' < p \cdot \zeta_i \leqq r_i$. For k sufficiently large, $p^k \cdot x_i' < r_i^k$. Define $x_i(\lambda) = \lambda x_i + (1 - \lambda) x_i'$, and consider the set of values of λ for which $0 \leqq \lambda \leqq 1$, $x_i(\lambda) \in \tilde{A}_i(\bar{x}_i^k)$. Since $\tilde{X}_i$ is convex, $x_i(\lambda) \in \tilde{X}_i$. Then one must have

$$p^k \cdot [\lambda x_i + (1 - \lambda) x_i'] \leqq r_i^k,$$

or

$$\lambda \leqq (r_i^k - p^k \cdot x_i')/(p^k \cdot x_i - p^k \cdot x_i'),$$

if we note that the denominator is positive for k sufficiently large, since $p \cdot x_i = r_i > p \cdot x_i'$. The largest value of λ satisfying the above conditions is, then

$$\lambda^k = \min\,[1, (r_i^k - p^k \cdot x_i')/(p^k \cdot x_i - p^k \cdot x_i')].$$

For k sufficiently large, $\lambda^k > 0$. Then $x_i(\lambda^k) \in \tilde{A}_i(\bar{x}_i^k)$ for all k sufficiently large. But also

$$\lim_{k\to\infty} r_i^k = r_i = \lim_{k\to\infty} p^k \cdot x_i, \qquad \text{so that} \quad \lim_{k\to\infty} \lambda^k = 1, \qquad \text{and} \quad \lim_{k\to\infty} x_i(\lambda^k) = x_i.$$

The continuity of $\tilde{A}_i(\bar{x}_i)$ is therefore established.

If Assumption IV.a holds, then the condition of the Remark is trivially satisfied for any $p \in P$, and $y_j \in \tilde{Y}_j$ $(j = 1, \cdots, n)$.

3.4.0. The existence of an equilibrium point $(x_1^*, \cdots, x_m^*, y_1^*, \cdots, y_n^*, p^*)$ for the abstract economy $\tilde{E}$ has, therefore, been demonstrated. It will now be shown that this point is also an equilibrium point for the abstract economy E described in 3.1. The converse is obvious; therefore a competitive equilibrium is *equivalent* to an $\tilde{E}$ equilibrium. (See end of 3.2.).

3.4.1. From Assumption I.a. and the definition of C (3.3.3.) it follows that $0 \in \tilde{Y}_j$ for each j. So that, as in 3.1.2.,

$$\max\left[0, \sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^*\right] = \sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^*.$$

From the definition of $\tilde{A}_i(\bar{x}_i)$,

$$p^* \cdot x_i^* \leqq p^* \cdot \zeta_i + \sum_{j=1}^{n} \alpha_{ij} p^* \cdot y_j^*.$$

Sum over i; then $p^* \cdot x^* \leqq p^* . \zeta + p^* \cdot y^*$, or $p^* \cdot z^* \leqq 0$. For fixed z^*, p^* maximizes $p \cdot z^*$ for $p \in P$; by an argument similar to that used in 3.2., this implies that

(1) $$z^* \leqq 0.$$

From (1) and the definitions in 3.3.0., $x_i^* \in \hat{X}_i$, $y_j^* \in \hat{Y}_j$ for all i and j, and, by 3.3.3., x_i^* and y_j^* are *interior* points of C.

Suppose, for some $x_i' \in A_i(\bar{x}_i^*)$, $u_i(x_i') > u_i(x_i^*)$. By III.c., $u_i[tx_i' + (1 - t)x_i^*] > u_i(x_i^*)$ if $0 < t < 1$. But for t sufficiently small, $tx_i' + (1 - t)x_i^*$ belongs to C. Since $tx_i' + (1 - t)x_i^* \in A_i(\bar{x}_i^*)$, by the convexity of the latter set, $tx_i' + (1 - t)x_i^* \in \tilde{A}_i(\bar{x}_i^*)$, for t small enough, which contradicts the definition of x_i^* as an equilibrium value for $\tilde{E}$.

(2) $$x_i^* \text{ maximizes } u_i(x_i) \text{ for } x_i \in A_i(\bar{x}_i^*).$$

Suppose, for some $y_j' \in Y_j$, $p^* \cdot y_j' > p^* \cdot y_j^*$. Then, $p^* \cdot [ty_j' + (1 - t)y_j^*] > p^* \cdot y_j^*$ for $0 < t < 1$. As in the preceding paragraph, the convex combination belongs to $\tilde{Y}_j$ for t sufficiently small, a contradiction to the equilibrium character of y_j^* for $\tilde{E}$.

(3) $$y_j^* \text{ maximizes } p^* \cdot y_j \text{ for } y_j \in Y_j.$$

That p^* maximizes $p \cdot z^*$ for $p \in P$ is directly implied by the definition of equilibrium point for $\tilde{E}$, since the domain of p is the same in both abstract economies.

It has been shown, therefore, that the point $(x_1^*, \cdots, x_m^*, y_1^*, \cdots, y_n^*, p^*)$ is also an equilibrium point for E; as shown in 3.2., it is, therefore, a competitive equilibrium. Theorem I has thus been proved.

4. STATEMENT OF THE SECOND EXISTENCE THEOREM FOR A COMPETITIVE EQUILIBRIUM

4.0. As noted in 1.3.2., Assumption IVa, which states in effect that a consumption unit has initially a positive amount of every commodity available for trading, is clearly unrealistic, and a weakening is very desirable. Theorem II accomplishes this goal, though at the cost of making certain additional assumptions in different directions and complicating the proof. Assumptions I–III are retained. The remaining assumptions for Theorem II are given in the following paragraphs of this section.

4.1. Assumption IV.a. is replaced by the following:

IV′.a. $\zeta_i \in R^l$; *for some* $x_i \in X_i$, $x_i \leqq \zeta_i$ *and, for at least one* $h \in \mathcal{P}$, $x_{hi} < \zeta_{hi}$.

The set $\mathcal{P}$ is defined more closely in 4.4 below; briefly, it consists of all types of labor that are always productive. IV.′a. is a weakening of IV.a.; it is now only supposed that the individual is capable of supplying at least one type of productive labor. IV.′a. and IV.b. together will be denoted by IV′.

4.2. Let $X = \sum_{i=1}^{m} X_i$.

V. *There exist* $x \in X$ *and* $y \in Y$ *such that* $x < y + \zeta$.

V asserts that it is possible to arrange the economic system by choice of production and consumption vectors so that an excess supply of all commodities can be achieved.

4.3. As in 3.2., δ^h will be the positive unit vector of the hth axis in R^l. For any $\lambda > 0$, $x_i + \lambda\delta^h$ represents an increase λ in the amount of the hth commodity over x_i, with all other commodities remaining unchanged in consumption.

DEFINITION: Let $\mathfrak{D}$ be the set of commodities such that if $i = 1, \cdots, m$, $x_i \in X_i$, $h \in \mathfrak{D}$, then there exists $\lambda > 0$ such that $x_i + \lambda\delta^h \in X_i$ and

$$u_i(x_i + \lambda\delta^h) > u_i(x_i).$$

$\mathfrak{D}$ is the set of commodities which are always desired by every consumer.

VI. *The set* $\mathfrak{D}$ *is not empty.*

Assumption VI is a stronger form of III.b. as given in 1.3.1. In the same manner as noted there, VI can be weakened to assert that the set $\mathfrak{D}'$ of commodities desired for all consumption vectors compatible with existing resource and technological conditions is not empty. Formally we could introduce the

DEFINITION: Let $\mathfrak{D}'$ be the set of commodities such that if $i = 1, \cdots, m$, $x_i \in \hat{X}_i$, $h \in \mathfrak{D}'$, then there exists $\lambda > 0$ such that $x_i + \lambda\delta^h \in X_i$ and

$$u_i(x_i + \lambda\delta^h) > u_i(x_i).$$

VI can then be replaced by:

VI′. *The set* $\mathfrak{D}'$ *is not empty.*

4.4. DEFINITION: Let $\mathcal{P}$ be the set of commodities such that if $y \in Y$, $h \in \mathcal{P}$, then (a) $y_h \leqq 0$ and (b) for some $y' \in Y$ and all $h' \neq h$, $y'_{h'} \geqq y_{h'}$, while for at least one $h'' \in \mathfrak{D}$, $y'_{h''} > y_{h''}$.

VII. *The set* $\mathcal{P}$ *is not empty.*

Assumption VII plays a key role in the following proof. We interpret the set $\mathcal{P}$ as consisting of some types of labor. Part (a) simply asserts that no labor service, at least of those included in $\mathcal{P}$, can be produced by a production unit. Part (b) asserts that, if no restriction is imposed on the amount (consumed) of some one type of "productive" labor, then it is possible to increase the output of at least one "desired" commodity (a commodity in $\mathfrak{D}$) without decreasing the output or increasing the input of any commodity other than the type of productive labor under consideration.

A case where VII might not hold is an economic system with fixed technological coefficients where production requiring a given type of labor also requires, directly or indirectly, some complementary factors. It is easy to see intuitively in this case how an equilibrium may be impossible. Given the amount of complementary resources initially available,[12] there will be a maximum to the quantity of labor that can be employed in the sense that no further increase in the labor force will increase the output of any commodity. Now, as is well known, the supply of labor may vary either way as real wages vary (see Robbins [**17**]) and broadly speaking is rather inelastic with respect to real wages. In particular, as real wages tend to zero, the supply will not necessarily become zero; on the contrary, as real incomes decrease, the necessity of satisfying more and more pressing needs may even work in the direction of increasing the willingness to work despite the increasingly less favorable terms offered. It is, therefore, quite possible that for any positive level of real wages, the supply of labor will exceed the maximum employable and hence *a fortiori* the demand by firms. Thus, there can be no equilibrium at positive levels of real wages. At zero real wages, on the contrary, demand will indeed be positive but of course supply of labor will be zero, so that again there will be no equilibrium. The critical point in the argument is the discontinuity of the supply curve for labor as real wages go to zero.

Assumption VII rules out any situation of limitational factors in which the marginal productivity of all types of labor in terms of desired commodities is zero. In conjunction with IV′.a., on the one hand, and VI, on the other, it insures that any individual possesses the ability to supply a commodity which has at least derived value.

It may be remarked that Assumption VII is satisfied if there is a productive process turning a form of labor into a desired commodity without the need of complementary commodities. Domestic service or other personal services may fall in this category.[13]

Let $\hat{Y} = \{y \mid y \in Y$, there exists $x_i \in X_i$ for all i such that $z \leqq 0\}$. It may be remarked that VII can be effectively weakened (in the same way that VI could be weakened to VI′) to

VII′. *The set* $\mathcal{P}'$ *is not empty,* where

DEFINITION: Let $\mathcal{P}'$ be the set of commodities such that if $h \in \mathcal{P}'$ and

(a) $y \in Y$, then $y_h \leqq 0$,

(b) $y \in \hat{Y}$, then for some $y' \in Y$ and all $h' \neq h$, $y'_{h'} \geqq y_{h'}$, while for at least one $h'' \in \mathfrak{D}$, $y'_{h''} > y_{h''}$.

Note that III.b., VI and VII can *simultaneously* be weakened to III′.b., VI′, and VII′.

4.5. THEOREM II. *For an economic system satisfying Assumptions I–III, IV′, and V–VII, there is a competitive equilibrium.*

[12] These complementary resources may be land, raw materials critical in certain industrial processes, or initial capital equipment.

[13] The possibility of disequilibrium and therefore unemployment through failure of Assumption VII to hold corresponds to so-called "structural unemployment."

5. PROOF OF THEOREM II

5.0. Let π be the number of elements of $\mathcal{P}$. For any ε, $0 < \varepsilon \leqq 1/2\pi$, define

$$P^\varepsilon = \{p \mid p \in P,\ p_h \geqq \varepsilon \text{ for all } h \in \mathcal{P}\}.$$

From IV′a, we can choose $x_i \in X_i$ so that $x_{hi} \leqq \zeta_{hi}$ for all h, $x_{h'i} < \zeta_{h'i}$ for some $h' \in \mathcal{P}$. For any $p \in P^\varepsilon$,

$$p\cdot(\zeta_i - x_i) = \sum_h p_h(\zeta_{hi} - x_{hi}) \geqq p_{h'}(\zeta_{h'i} - x_{h'i}) > 0$$

or

(1) for some $x_i \in X_i$, $p\cdot x_i < p\cdot\zeta_i$.

5.1.0. The basic method of proof of Theorem II will be similar to that of Theorem I. We seek to show that an equilibrium point for the abstract economy E, defined in 3.1.0, exists. As already shown in 3.2, such an equilibrium point would define a competitive equilibrium. First, the economy E is replaced by the economy E^ε $[X_1, \cdots, X_m,\ Y_1, \cdots, Y_n,\ P^\varepsilon,\ u_1(x_1), \cdots, u_m(x_m),\ p\cdot y_1, \cdots, p\cdot y_n,\ p\cdot z, A_1(\bar{x}_1), \cdots, A_m(\bar{x}_m), Y_1, \cdots, Y_m, P^\varepsilon]$. Clearly, E^ε is the same as E, except that the price domain has been contracted to P^ε. The existence of an equilibrium point for E^ε for each ε will first be shown; then, it will be shown that for some ε, an equilibrium point of E^ε is also an equilibrium point of E.[14]

To show the existence of an equilibrium point for E^ε, the same technique will be used as in proving the existence of an equilibrium point for E in Theorem I. The argument is that the equilibrium point, if it exists at all, must lie in a certain bounded domain. Hence, if we alter the abstract economy E^ε by intersecting the action domains with a suitably chosen hypercube, we will not disturb the equilibrium points, if any; but the Lemma of 2.5. will now be applicable, and the existence of an equilibrium point shown (see 3.3. above).

5.1.1. This section will be purely heuristic, designed to motivate the choice of the hypercube mentioned in the previous paragraph. Suppose an equilibrium point $[x_1^*, \cdots, x_m^*, y_1^*, \cdots, y_n^*, p^*]$ exists for the abstract economy E^ε. Since $x_i^* \in A_i(\bar{x}_i^*)$ for all i, by definition (see 3.1.0.),

$$p^*\cdot x_i^* \leqq p^*\cdot\zeta_i + \sum_{j=1}^n \alpha_{ij}\, p^*\cdot y_j^*,$$

(see also 3.1.2.) If we sum over i and recall that $\sum_{i=1}^m \alpha_{ij} = 1$,

$$p^*\cdot\left(\sum_{i=1}^m x_i^* - \sum_{i=1}^m \zeta_i - \sum_{j=1}^n y_j^*\right) \leqq 0,$$

or

$$p^*\cdot z^* \leqq 0.$$

[14] The introduction of E^ε is made necessary by the following fact: (1) of 5.0 may not hold for all $p \in P$ and the condition of the *Remark* in 3.3.5., may not be satisfied for all $p \in P$.

Since p^* maximizes $p \cdot z^*$ for $p \in P^\varepsilon$, by definition of equilibrium, $p \cdot z^* \leqq 0$ for all $p \in P^\varepsilon$, or,

(1) $$p_{h'} z^*_{h'} \leqq \sum_{h \neq h'} p_h(\zeta_h - x^*_h + y^*_h), \textit{ for any } h'.$$

Note that, since $y^*_h \leqq 0$ for $h \in \mathcal{P}$, by (a) of the first Definition in 4.4

(2) $$x^*_h - y^*_h \geqq x^*_h \geqq \xi_h, \text{ for } h \in \mathcal{P},$$

by II. ζ and ξ are defined in 1.4.1. and 3.3.1., respectively.

For any given h', define p as follows: $p_h = \varepsilon$ for $h \in \mathcal{P}$ and $h \neq h'$; $p_h = 0$ for $h \notin \mathcal{P}$ and $h \neq h'$; $p_{h'} = 1 - \sum_{h \neq h'} p_h$. Then, if $h' \in \mathcal{P}$, $p_{h'} = 1 - (\pi - 1)\varepsilon$ (which is indeed $\geqq \varepsilon$ if $\varepsilon \leqq 1/2\pi$); if $h' \notin \mathcal{P}$, $p_{h'} = 1 - \pi\varepsilon$. From (1) and (2),

(3) $$\begin{cases} \text{if } h' \in \mathcal{P}, [1 - (\pi - 1)\varepsilon] z^*_{h'} \leqq \varepsilon \sum_{\substack{h \in \mathcal{P} \\ h \neq h'}} (\zeta_h - \xi_h), \\ \text{if } h' \notin \mathcal{P}, (1 - \pi\varepsilon) z^*_{h'} \leqq \varepsilon \sum_{h \in \mathcal{P}} (\zeta_h - \xi_h). \end{cases}$$

If $0 < \varepsilon \leqq (1/2\pi)$, then certainly $1 - \pi\varepsilon > 0$, $1 - (\pi - 1)\varepsilon > 0$, and necessarily

$$\varepsilon/[1 - (\pi - 1)\varepsilon] < \varepsilon/(1 - \pi\varepsilon).$$

Finally, for any h, $\zeta_h - \xi_h \geqq 0$ from IV′a and II. If we divide through the first inequality in (3) by $[1 - (\pi - 1)\varepsilon]$,

(4) $$z^*_{h'} \leqq \{\varepsilon/[1 - (\pi - 1)\varepsilon] \sum_{\substack{h \in \mathcal{P} \\ h \neq h'}} (\zeta_h - \xi_h) \leqq [\varepsilon/(1 - \pi\varepsilon)] \sum_{h \in \mathcal{P}} (\zeta_h - \xi_h), \text{ for } h' \in \mathcal{P}.$$

The same inequality between the extreme items holds for $h' \notin \mathcal{P}$, as can be seen by dividing through in the second inequality in (3) by $(1 - \pi\varepsilon)$. But if $\varepsilon \leqq (1/2\pi)$, then we see in turn that $2\pi\varepsilon \leqq 1$, $\pi\varepsilon \leqq 1 - \pi\varepsilon$, and, by division by $\pi(1 - \pi\varepsilon)$, $\varepsilon/(1 - \pi\varepsilon) \leqq 1/\pi$. From (4),

$$z^*_h \leqq (1/\pi) \sum_{h \in \mathcal{P}} (\zeta_h - \xi_h).$$

Let $\zeta'_h = \zeta_h + (1/\pi) \sum_{h \in \mathcal{P}} (\zeta_h - \xi_h)$, with ζ' being the vector whose components are $\zeta_1, \cdots, \zeta_l$; then

(5) $$x^* - y^* \leqq \zeta'.$$

The equilibrium point then will lie in a region defined by (5) and the conditions $x^*_i \in X_i$, $y^*_j \in Y_j$, $p^* \in P^\varepsilon$. These are exactly the same as the requirements for E in the proof of Theorem I, except that ζ has been replaced by ζ', and P by P^ε.

5.2.0. The proof proper will now be resumed. Define

$\hat{X}'_i = \{x_i \mid x_i \in X_i$, and there exist $x_{i'} \in X_{i'}$ for all $i' \neq i$, $y_j \in Y_j$ for all j such that $x - y \leqq \zeta'\}$,

$\hat{Y}'_j = \{y_j \mid y_j \in Y_j$, and there exist $x_i \in X_i$ for all i, $y_{j'} \in Y_{j'}$ for all $j' \neq j$ such that $x - y \leqq \zeta'\}$.

These definitions are identical with those of $\hat{X}_i$, $\hat{Y}_j$ in 3.3.0., except that ζ has been replaced by ζ'. The arguments of 3.3.0.–.3. may therefore be repeated exactly. We can choose a positive real number c' so that the cube

$$C' = \{x \mid |x_h| \leqq c' \text{ for all } h\}$$

contains in its *interior* all $\hat{X}'_i$ and all $\hat{Y}'_j$. Let $\tilde{X}'_i = X_i \cap C'$, $\tilde{Y}'_j = Y_j \cap C'$.

5.2.1. Let $\tilde{E}^\varepsilon$ be an abstract economy identical with E^ε in 5.1.0., except that X_i is replaced by $\tilde{X}'_i$ and Y_j by $\tilde{Y}'_j$ everywhere. Let $\tilde{A}'_i(\bar{x}_i)$ be the resultant modification of $A_i(\bar{x}_i)$. It is easy to see that the argument of 3.3.4. remains completely applicable in showing that all the requirements of the Lemma are satisfied other than the continuity of $\tilde{A}'_i(\bar{x}_i)$. The last follows immediately from the Remark of 3.3.5., and (1) in 5.0., since $x_i \in \hat{X}'_i$ and hence to $\tilde{X}'_i$. Hence, $\tilde{E}^\varepsilon$ has an equilibrium point $[x_1^*, \cdots, x_m^*, y_1^*, \cdots, y_n^*, p^*]$ for each ε, $0 < \varepsilon \leqq (1/2\pi)$. We show now that an equilibrium point of $\tilde{E}^\varepsilon$ is an equilibrium point of E^ε (the converse is obvious).

5.2.2. Since $0 \in \tilde{Y}'_j$,

(1) $$p^* \cdot y_j^* \geqq p^* \cdot 0 = 0,$$

so that $\sum_{j=1}^m \alpha_{ij} p^* \cdot y_j^* \geqq 0$, and, as in 5.1.1., $p^* \cdot z^* \leqq 0$, from which it can be concluded that, as in equation (5), section 5.1.1., $x^* - y^* \leqq \zeta'$. From the definitions of $\hat{X}'_i$, $\hat{Y}'_j$ in 5.2.0., $x_i^* \in \hat{X}'_i$, $y_j^* \in \hat{Y}'_j$ for all i and j; hence, as shown in that section,

(2) x_i^*, y_j^* are interior points of C'.

From the definition of an equilibrium point, x_i^* maximizes $u_i(x_i)$ for $x_i \in \tilde{A}'_i(\bar{x}_i^*)$. From (2), it follows exactly as in 3.4.1., that

(3) x_i^* maximizes $u_i(x_i)$ for $x_i \in A_i(\bar{x}_i^*)$.

In the same way,

(4) y_j^* maximizes $p^* \cdot y_j$ for $y_j \in Y_j$.

From the definition of equilibrium for $\tilde{E}^\varepsilon$,

(5) p^* maximizes $p \cdot z^*$ for $p \in P^\varepsilon$.

5.3.0. Suppose that, for some ε, $0 < \varepsilon \leqq 1/2\pi$,

(1) $$p_h^* > \varepsilon \text{ for all } h \in \mathcal{P}.$$

Let p be any element of P, $p' = tp + (1 - t)p^*$, where $0 < t \leqq 1$. Suppose $p \cdot z^* > p^* \cdot z^*$; then $p' \cdot z^* > p^* \cdot z^*$. But, from (1), $p' \in P^\varepsilon$ for t sufficiently small, which contradicts (5) of the preceding paragraph. Thus, if (1) holds for some ε, $p \cdot z^* \leqq p^* \cdot z^*$ for all $p \in P$, i.e., p^* maximizes $p \cdot z^*$ for $p \in P$ (and not merely $p \in P^\varepsilon$). In conjunction with (3) and (4) of the preceding paragraph, this shows that the abstract economy E has an equilibrium point and therefore, as shown in 3.2.,

(2) If (1) holds, there is a competitive equilibrium.

5.3.1. It will therefore now be assumed that (1) of 5.3.0. does not hold for any $\varepsilon > 0$. Then, for each ε, $0 < \varepsilon \leqq \frac{1}{2}\pi$,

(1) $$p_h^* = \varepsilon \text{ for at least one } h \in \mathcal{P}.$$

For all ε, $p^* \in P$, $x_i^* \in C'$, $y_j^* \in C'$ (see 5.2.2 (2)). P and C' are compact sets; a set of converging sequences can therefore be chosen so that

(2) $\lim_{k \to \infty} \varepsilon_k = 0$, $(x_1^k, \cdots, x_m^k, y_1^k, \cdots, y_n^k, p^k)$ is an equilibrium point for E^{ε_k}, $\lim_{k \to \infty} x_i^k = x_i^0$, $\lim_{k \to \infty} y_j^k = y_j^0$, $\lim_{k \to \infty} p^k = p^0$.

Since the sets X_i, Y_j, P are closed, $x_i^0 \in X_i$, $y_j^0 \in Y_j$, $p^0 \in P$. From (1), there must be at least one $h \in \mathcal{P}$ for which $p_h^k = \varepsilon_k$ for infinitely many k, and hence by (2) $p_h^0 = 0$ for that h. For convenience, let $h = 1$.

(3) $$p_1^0 = 0, 1 \in \mathcal{P}.$$

As shown in 3.2., statement (3) of 5.2.2., which is Condition 2, implies equation (1) of 3.2., namely, $p^k \cdot z^k = 0$. Let k approach ∞; by (2),

(4) $$p^0 \cdot z^0 = 0.$$

For any fixed y_j, statement (4) of 5.2.2. tells us that $p^k \cdot y_j^k \geqq p^k \cdot y_j$. Let k approach ∞; then $p^0 \cdot y_j^0 \geqq p^0 \cdot y_j$.

(5) $$y_j^0 \text{ maximizes } p^0 \cdot y_j \text{ for } y_j \in Y_j.$$

5.3.2. Choose any $x_i \in X_i$ such that $u_i(x_i) > u_i(x_i^0)$. For k sufficiently large, $u_i(x_i) > u_i(x_i^k)$, from 5.3.1. (2) and the continuity of u_i. This is not compatible with the statement that $x_i \in A_i(\bar{x}_i^k)$, by 5.2.2. (3), so that $p^k \cdot x_i > p^k \cdot x_i^k$. Let k approach ∞.

(1) If $x_i \in X_i$ and $u_i(x_i) > u_i(x_i^0)$, then $p^0 \cdot x_i \geqq p^0 \cdot x_i^0$.

5.3.3. This section is a digression which may be of some interest for general techniques in the theory of the consumer. It can easily be shown that from 5.3.2. (1)

(1) $$x_i^0 \text{ minimizes } p^0 \cdot x_i \text{ on } \{x_i \mid x_i \in X_i,\ u_i(x_i) \geqq u_i(x_i^0)\}$$

and that p^0 maximizes $p \cdot z^0$ for $p \in P$. In conjunction with 5.3.1. (5), it is then shown that all the conditions for a competitive equilibrium are satisfied, except that utility-maximization by a consumption unit under a budget restraint has been replaced by minimization of cost for a given utility level (compare (1) with Condition 2). The duality between cost-minimization and utility-maximization is indeed valid almost everywhere, i.e., in the interior of P, where all prices are positive, but not everywhere.

From the viewpoint of welfare economics, it is the principle that the consumption vector chosen should be the one which achieves the given utility at least cost which is primary, and the principle of maximizing utility at a given cost only relevant when the two give identical results.[15] For a descriptive theory of behavior under perfect competition, on the other hand, it is, of course, the concept of utility maximization which is primary. To the extent that the duality is valid, the principle of cost minimization leads to much simpler derivations, for example, of Slutzky's relations. Actually, minimization of cost for a given utility is essentially minimization of a linear function when the argument is limited to a convex set; mathematically, the problem is identical with that of maximizing profits subject to the transformation conditions, so that the theories of the consumer and the firm become identical.[16] However, the failure of the duality to hold in all cases shows that there are delicate questions for which the principle of utility maximization cannot be replaced by that of cost minimization.

5.3.4. From 5.3.1. (3), $1 \in \mathcal{P}$. By (*b*) of the first Definition in 4.4, there exists $y' \in Y$ such that

(1) $$y'_h \geqq y^0_h \text{ for all } h \neq 1; \qquad y'_{h'} > y^0_{h'} \text{ for some } h' \in \mathfrak{D}.$$

Here, $y^0 = \sum_{j=1}^{n} y^0_j$. From 5.3.1. (5), $p^0 \cdot y'_j \leqq p^0 \cdot y^0_j$ for all j. Summing over j then gives

(2) $$p^0 \cdot y' \leqq p^0 \cdot y^0.$$

With the aid of (1) and 5.3.1. (3),

$$p^0 \cdot (y' - y^0) = \sum_h p^0_h (y'_h - y^0_h) = \sum_{h \neq 1} p^0_h (y'_h - y^0_h) \geqq p^0_{h'} (y'_{h'} - y^0_{h'}).$$

Since $y'_{h'} - y^0_{h'} > 0$, (2) requires that $p^0_{h'} = 0$.

(3) $$p^0_{h'} = 0 \text{ for at least one } h' \in \mathfrak{D}.$$

Let $x_i \in X_i$, $x_i(t) = tx_i + (1 - t)x^0_i$, where $0 < t \leqq 1$. From the first Definition in 4.3, there exists

(4) $$\lambda > 0, u_i(x^0_i + \lambda\delta^{h'}) > u_i(x^0_i).$$

Since $x_i(t) + \lambda\delta^{h'}$ approaches $x^0_i + \lambda\delta^{h'}$ as t approaches 0, it follows from (4) that,

(5) $$u_i[x_i(t) + \lambda\delta^{h'}] > u_i(x^0_i) \text{ for } t \text{ sufficiently small.}$$

[15] See Arrow [**1**], Lemma 4, p. 513; a brief discussion of the conditions for the duality to be valid is given in Lemma 5, pp. 513–4. See also Debreu [**4**], Friedman [**8**].

[16] Professors Knight [**11**] and Friedman [**7**] (esp. pp. 469–474) have therefore gone so far as to argue that it is always better to draw up demand functions as of a given real income (i.e., utility) instead of a given money income.

From (5) and 5.3.2. (1), $p^0 \cdot [x_i(t) + \lambda\delta^{h'}] \geqq p^0 \cdot x_i^0$. But, from (3),

$$p^0 \cdot (\lambda\delta^{h'}) = \lambda p_{h'}^0 = 0.$$

Since $p^0 \cdot [x_i(t) + \lambda\delta^{h'}] = tp^0 \cdot x_i + (1 - t)p^0 \cdot x_i^0 + p^0 \cdot (\lambda\delta^{h'})$, it follows easily that $p^0 \cdot x_i \geqq p^0 \cdot x_i^0$.

(6) $$x_i^0 \text{ minimizes } p^0 \cdot x_i \text{ over } X_i.$$

Let X be defined as in 4.2. Since $p^0 \cdot x = \sum_{i=1}^m p^0 \cdot x_i$, it follows immediately from (6) that,

(7) $$x^0 \text{ minimizes } p^0 \cdot x \text{ over } X.$$

5.3.5. In accordance with Assumption V, choose $x \epsilon X$, $y \epsilon Y$ so that $x < y + \zeta$. Then, with the aid of 5.3.4. (7), $p^0 \cdot (y + \zeta) > p^0 \cdot x \geqq p^0 \cdot x^0$, or

(1) $$p^0 \cdot y > p^0 \cdot (x^0 - \zeta).$$

From 5.3.1. (4),

(2) $$p^0 \cdot (x^0 - y^0 - \zeta) = 0.$$

This, combined with (1), gives

(3) $$p^0 \cdot y > p^0 \cdot y^0.$$

But this implies that, for some j, $p^0 \cdot y_j > p^0 \cdot y_j^0$, while $y_j \epsilon Y_j$, a contradiction to 5.3.1. (5). Thus, the assumption made at the beginning of 5.3.1., that for every $\varepsilon > 0$, $p_h^* = \varepsilon$ for at least one $h \epsilon \mathcal{P}$, has led to a contradiction and must be false. Statement 5.3.0. (1) must then be valid, and by statement (2) in the same paragraph, Theorem II has been proved.

5.3.6. The following theorem, slightly more general than theorem II, can easily be proved in a way practically identical to the above.

Assumption IV′a. is replaced by

IV″a. $\zeta_i \epsilon R^l$; *for some* $x_i \epsilon X_i$, $x_i \leqq \zeta_i$ *and, for at least one* $h \epsilon \mathfrak{D} \cup \mathcal{P}$, $x_{hi} < \zeta_{hi}$. IV″a. and IVb. together are denoted by IV″.

THEOREM II′. *For an economic system satisfying Assumptions I–III, IV″, V, and VI there is a competitive equilibrium.*

6. HISTORICAL NOTE

The earliest discussion of the existence of competitive equilibrium centered around the version presented by Cassel [**2**]. There are four basic principles of his system: (1) demand for each final good is a function of the prices of all final goods; (2) zero profits for all producers; (3) fixed technical coefficients relating use of primary resources to output of final commodities; and (4) equality of supply and demand on each market. Let x_i be the demand for final commodity i, p_i the price of final commodity i, a_{ij} the amount of primary resource j used in the production of one unit of commodity i, q_j the price of resource j and r_j the amount of resource j available initially. Then Cassel's system may be written,

$$(1) \qquad x_i = f_i(p_1, \cdots, p_m)$$

$$(2) \qquad \sum_j a_{ij} q_j = p_i \qquad \text{for all } i,$$

$$(3) \qquad \sum_i a_{ij} x_i = r_j \qquad \text{for all } j.$$

Professor Neisser [**15**] remarked that the Casselian system might have negative values of prices or quantities as solutions. ([**15**], pp. 424–425). Negative quantities are clearly meaningless and, at least, in the case of labor and capital, negative prices cannot be regarded as acceptable solutions since the supply at those prices will be zero. Neisser also observed that even some variability in the technical coefficients might not be sufficient to remove the inconsistency. (p. 448–453).

Stackelberg [**20**] pointed out that if there were fewer commodities than resources, the equations (3) would constitute a set of linear equations with more equations than unknowns and therefore possess, in general, no solution. He correctly noted that the economic meaning of this inconsistency was that some of the equations in (3) would become inequalities, with the corresponding resources becoming free goods. He argued that this meant the loss of a certain number of equations and hence the indeterminacy of the rest of the system. For this reason, he held that the assumption of fixed coefficients could not be maintained and the possibility of substitution in production must be admitted. This reasoning is incorrect; the loss of the equations (3) which are replaced by inequalities is exactly balanced by the addition of an equal number of equations stating that the prices of the corresponding resources must be zero.

Indeed, this suggestion had already been made by Professor Zeuthen [**25**] (see pages 2–3, 6), though not in connection with the existence of solutions. He argued that the resources which appeared in the Casselian system were properly only the scarce resources; but it could not be regarded as known *a priori* which resources are free and which are not. Hence equations (3) should be rewritten as inequalities,

$$\sum_i a_{ij} x_i \leqq r_j,$$

with the additional statement that if the strict inequality holds for any j, then the corresponding price $q_j = 0$.

Schlesinger [**19**] took up Zeuthen's modification and suggested that it might resolve the difficulties found by Neisser and Stackelberg. It was in this form that the problem was investigated by Wald [**21**, **22**] under various specialized assumptions. These studies are summarized and commented on in [**23**].

From a strictly mathematical point of view the first theorem proved by Wald [**23**] p. 372–373 neither contains nor is contained in our results. In the assumptions concerning the productive system, the present paper is much more general since Wald assumes fixed proportions among the inputs and the single output of every process. On the demand side, he makes assumptions concerning the demand functions instead of deriving them, as we do, from a utility maximization assumption. It is on this point that no direct comparison is possible. The assumptions made by Wald are somewhat specialized ([**23**], p. 373, assumptions 4, 5 and 6). One of them, interestingly enough, is the same as Samuelson's postulate ([**18**], pp. 108–111), but applied to the collective demand functions rather than to individual ones. Wald gives a heuristic argument for this assumption which is based essentially on utility-maximization grounds. In the same model, he also assumes that the demand functions are independent of the distribution of income, depending solely on the total. In effect, then, he assumes a single consumption unit.

In his second theorem, [**23**], pp. 382–383, about the pure exchange case, he assumes utility maximization but postulates that the marginal utility of each commodity depends on that commodity alone and is a strictly decreasing nonnegative function of the amount of that commodity. The last clause implies both the convexity of the indifference map and nonsaturation with respect to every commodity. This theorem is a special case of our Theorem II′, when $\mathcal{P}$ is the null set and $\mathfrak{D}$ contains all commodities (See 5.3.6).

Wald gives an example, under the pure exchange case, where competitive equilibrium does not exist ([**23**], pp. 389–391). In this case, each individual has an initial stock of only one commodity, so that Theorem I is not applicable.

At the same time only one commodity is always desired by all, but two of the three consumers have a null initial stock of that commodity. Hence Theorem II′ is not applicable.

It may be added that Wald has also investigated the uniqueness of the solutions; this has not been done here.

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